The text for this course is *Mathematical Analysis I* second edition by Beatriz Laferriere, Gerardo Laferriere and Nguyen Mau Nam. The exercises below are from this text. This homework covers the material discussed in Lectures 14, 15, 16, 17 and 18. It is due 10-26-20.

✓**Problem 61:** Exercise 2.6.3 (compact)

✓**Problem 62:** Exercise 2.6.5 (limit points and isolated points, practice on terms)

✓**Problem 63:** Exercise 2.6.6 (relative topology)

✓**Problem 64:** Exercise 3.1.1 part c

✓**Problem 65:** Exercise 3.1.1 part e

✓**Problem 66:** Exercise 3.1.2 part b

✓**Problem 67:** Exercise 3.1.3

✓**Problem 68:** Exercise 3.2.1 part a

✓**Problem 69:** Exercise 3.2.3 part b

✓**Problem 70:** Exercise 3.2.5

✓**Problem 71:** Exercise 3.2.7

✓**Problem 72:** Exercise 3.3.1 part a

✓**Problem 73:** Exercise 3.3.1 part b (oops, I did $f(x) = x^3 - 3$ the problem was easier! $f(x) = x^2 - 3$ can use my Lecture discussion etc.~)

✓**Problem 74:** Exercise 3.3.2 parts b, c and d.

✓**Problem 75:** Exercise 3.3.3

✓**Problem 76:** Exercise 3.3.11 (extension from rationals)

✓**Problem 77:** Exercise 3.4.1

✓**Problem 78:** Exercise 3.4.2

✓**Problem 79:** Exercise 3.4.4

✓**Problem 80:** Exercise 3.4.5

Solution to Mission 4

P61 Ex. 2.6.3 (compact): Prove if A and B are compact subsets of $\mathbb{R}$ then $A \cup B$ compact

Th^m 2.6.5 provides compact subsets of $\mathbb{R}$ are precisely those subsets which are closed and bounded. If $A, B \subseteq \mathbb{R}$ are compact then A, B are closed thus $A \cup B$ is closed. (recall finite union of closed sets is closed). Furthermore A, B bounded implies $\exists\, M_A, M_B > 0$ such that $|x| \leq M_A \ \forall x \in A$ and $|x| \leq M_B \ \forall x \in B$. Let $M = \max(M_A, M_B)$ then if $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $|x| \leq M_A \leq M$ and if $x \in B$ then $|x| \leq M_B \leq M$. Thus M bounds $A \cup B$ and we've shown $A \cup B$ is both closed and bounded hence by $Th^{\underline{m}}$ 2.6.5 we find $A \cup B$ is compact.

Remark: direct argument from Def$^{\underline{n}}$ of sequential compactness seems very difficult here in comparison to argument above

P62 Ex 2.6.5 / Find all limit points and isolated points,

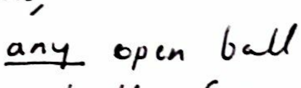
remember limit pt. of D has infinitely many pts in any open ball around the pt. whereas an isolated pt. permits a ball of some radius for which the isolated pt. is the only pt. in D in the ball.

(a.) $A = (0,1)$ ← every pt. in A is limit point, A has no isolated pts.

(b.) $B = [0,1)$ ← every pt. in B is limit point, B has no isolated pts.

(c.) $C = \mathbb{Q}$ ← every pt. in C is limit point, C has no isolated pts. (by density of rationals)

(d.) $D = \{ m + 1/n \mid m, n \in \mathbb{N} \}$



$\therefore$ $\mathbb{N} = \{1, 2, \dots\}$

X = limit points of $D = \mathbb{N} - \{1\}$

isolated pts of $D = D - X$ (all non-integer points)

P63 Ex. .2.6.6

Let $D = [0, \infty)$. Classify sets below as open, closed or neither open nor closed in D. Recall, $S \subseteq D$ is open iff $\exists$ open set $U \subseteq \mathbb{R}$ s.t. $S = D \cap U$.

(b.) $B = \mathbb{N}$ has

$$D - B = \left(\bigcup_{i=1}^{\infty} (i, i+1)\right) \cup [0, 1)$$

$$= \Bigg[\bigcup_{i=1}^{\infty} \underbrace{(i, i+1) \cap [0, \infty)}_{\text{open in } D}\Bigg] \cup \Big[\underbrace{(-1, 1) \cap [0, \infty)}_{\text{open in } D}\Big]$$

thus $D - B$ open in $D \Rightarrow$ B closed in D.

(a.) $A = (0, 1)$

Notice $x_n = \frac{1}{n} \in D \ \forall n \in \mathbb{N}$ and $x_n \to 0 \in D$.

Observe $x_n \in A \ \forall n \in \mathbb{N}$ yet $x_n \to 0 \notin A$

thus A is not closed in D by implication of Cor 2.6.10

//

$\underbrace{D - A}_{\text{not open in } D} = [1, \infty)$, $\nexists U$ open in $\mathbb{R}$ s.t. $U \cap D = [1, \infty)$

to prove this assertion, suppose towards →← $\exists U$ open s.t. $U \cap D = [1, \infty)$ then $[1, 2) \subseteq U$ but $\nexists \ \varepsilon > 0$ s.t. $B(1; \varepsilon) \subseteq [1, 2)$ since $B(1; \varepsilon) = (1 - \varepsilon, 1 + \varepsilon) \nsubseteq [1, 2)$.

– (there might be an easier way to argue (a.), this is just what occured to me now... ☺) –

(c.) $C = \mathbb{Q} \cap D$ is neither open nor closed in D.

Let $q \in \mathbb{Q} \cap D$ and let $\varepsilon > 0$ then $B(q, \varepsilon) = (q - \varepsilon, q + \varepsilon)$ contains $(q, q + \varepsilon)$ and by density of irrational #'s $\exists x \in (q, q + \varepsilon)$ where $x \notin \mathbb{Q} \Rightarrow B(q, \varepsilon) \nsubseteq \mathbb{Q} \cap D$

Therefore, $\mathbb{Q} \cap D$ is not open.

A similar argument shows $D - \mathbb{Q}$ is also not open hence C not closed.

$\boxed{P64}$ Exercise 3.1.1 part c / $\lim_{x\to 1}\left(\frac{x+3}{x+1}\right) = 2$ prove via Def^n /

Let $\varepsilon > 0$ and suppose $\delta = \min(\varepsilon, 1)$. If $x \in \mathbb{R}$ and

$0 < |x-1| < \delta \le 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 1 < x+1 < 3$

thus $|x+1| = x+1 > 1$. Consider, for $0 < |x-1| < \delta$,

$$\left|\frac{x+3}{x+1} - 2\right| = \left|\frac{x+3-2(x+1)}{x+1}\right| = \frac{|1-x|}{|x+1|} < \frac{\delta}{1} \le \varepsilon.$$

Therefore, $\lim_{x\to 1}\left(\frac{x+3}{x+1}\right) = 2$ by Def^n of limit.//

$\boxed{P65}$ Ex. 3.1.1e / $\lim_{x\to 2}(x^3) = 8$

Let $\varepsilon > 0$ and suppose $\delta = \min(\varepsilon/19, 1)$. Suppose $0 < |x-2| < \delta \le 1$

then $-1 < x-2 < 1 \Rightarrow 1 < x < 3 \Rightarrow |x| < 3$. Consider,

$$\begin{aligned} |x^3 - 8| &= |(x-2)(x^2+2x+4)| \\ &\le |x-2|\left(|x|^2 + 2|x| + 4\right) \\ &< \delta\left(9 + 2(3) + 4\right) = 19\delta \le 19\left(\tfrac{\varepsilon}{19}\right) = \varepsilon. // \end{aligned}$$

$\boxed{P66}$ Ex 3.1.2b / Show $\lim_{x\to 0}(\cos(\frac{1}{x}))$ does not exist

Observe $x_n = \frac{1}{n\pi} \longrightarrow 0$ as $n \longrightarrow \infty$

yet $\cos\left(\frac{1}{x_n}\right) = \cos\left(\frac{1}{\frac{1}{n\pi}}\right) = \cos(n\pi) = (-1)^n$ does not converge.

Thus identifying $\bar{x} = 0$ and $f(x) = \cos\left(\frac{1}{x}\right)$ for Cor. 3.1.4

we find $\lim_{x\to 0}\left(\cos\frac{1}{x}\right)$ does not exist.

P67 Ex. 3.1.3

Let $f: D \to \mathbb{R}$ and $\bar{x}$ a limit point of D. Prove that if $\lim_{x \to \bar{x}} f(x) = l$ then $\lim_{x \to \bar{x}} |f(x)| = |l|$. Also, give counterexample to converse of the claim.

Proof: Suppose $\lim_{x \to \bar{x}} f(x) = l$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that $0 < |x - \bar{x}| < \delta$ implies $|f(x) - l| < \varepsilon$. However, note $\big||f(x)| - |l|\big| \leq |f(x) - l|$ by Corolary 1.4.4 on pg. 22 hence $0 < |x - \bar{x}| < \delta \Rightarrow \big||f(x)| - |l|\big| < \varepsilon$ and we conclude by Defn of limit $\lim_{x \to \bar{x}} |f(x)| = |l|$. //

Counter-example: $\lim_{x \to 0} \left| \frac{x}{|x|} \right| = \lim_{x \to 0} \frac{|x|}{|x|} = \lim_{x \to 0} (1) = 1$

Yet $\lim_{x \to 0} \left(\frac{x}{|x|} \right)$ d.n.e. since $\lim_{x \to 0^+} \left(\frac{x}{|x|} \right) = \lim_{x \to 0^+} \left(\frac{x}{x} \right) = 1$ whereas $\lim_{x \to 0^-} \left(\frac{x}{|x|} \right) = \lim_{x \to 0^-} \left(\frac{x}{-x} \right) = -1 \neq 1$.

P68 Ex 3.2.1 part a

$$\lim_{x \to 2} \left(\frac{3x^2 - 2x + 5}{x - 3} \right) = \frac{\lim_{x \to 2} (3x^2 - 2x + 5)}{\lim_{x \to 2} (x - 3)}$$

$$= \frac{3 \lim_{x \to 2} (x^2) - 2 \lim_{x \to 2} (x) + \lim_{x \to 2} (5)}{\lim_{x \to 2} (x) - \lim_{x \to 2} (3)}$$

$$= \frac{3(4) - 2(2) + 5}{2 - 3}$$

$$= \frac{13}{-1}$$

$$= \boxed{-13}$$

Remark: You could write a lot less here.

P69 Ex 3.2.3 part 6

$$\lim_{x\to 1}\left(\frac{x^m-1}{x^n-1}\right) = \lim_{x\to 1}\left(\frac{(x-1)(x^{m-1}+x^{m-2}+\cdots+x+1)}{(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)}\right)$$

$$= \lim_{x\to 1}\left(\frac{x^{m-1}+x^{m-2}+\cdots+x+1}{x^{n-1}+x^{n-2}+\cdots+x+1}\right)$$

$$= \frac{1+1+\cdots+1+1 \leftarrow m\text{-summands}}{1+1+\cdots+1+1 \leftarrow n\text{-summands}}$$

$$= \boxed{\frac{m}{n}}$$

Remark: L'Hopital's Rule nice here, but, we don't have that to work with here. However, outside our context, use L'Hop's rule $\lim_{x\to 1}\left(\frac{x^m-1}{x^n-1}\right) \underset{(\frac{0}{0})}{\overset{L}{=}} \lim_{x\to 1}\left(\frac{mx^{m-1}}{nx^{n-1}}\right) = \frac{m}{n}$.

P70 Ex 3.2.5 / Let $f: D \to \mathbb{R}$ and let $\bar{x}$ be a limit pt. of D. Suppose $|f(x)-f(y)| \le k|x-y|$ $\forall x, y \in D-\{\bar{x}\}$ where $k \ge 0$ is a constant. Prove $\lim_{x\to\bar{x}} f(x)$ exists.

Proof: by Cauchy's criterion (Thm 3.2.2)

Suppose $|f(x)-f(y)| \le k|x-y|$ $\forall x, y \in D-\{\bar{x}\}$ where $k \ge 0$.

Let $\varepsilon > 0$, if $k=0$ then choose $\delta = 1$ and note $|f(x)-f(y)| = 0 < \varepsilon$.

Otherwise, choose $\delta = \frac{\varepsilon}{2k}$ as $k > 0$. Suppose $x, y \in D-\{\bar{x}\}$ and $0 < |x-\bar{x}| < \delta$ and $0 < |y-\bar{x}| < \delta$. Consider,

$$\begin{aligned} |f(x)-f(y)| &\le k|x-y| \\ &\le k|x-\bar{x}+\bar{x}-y| \\ &\le k(|x-\bar{x}|+|y-\bar{x}|) \\ &< 2k\delta = 2k\frac{\varepsilon}{2k} = \varepsilon. \end{aligned}$$

Thus $\lim_{x\to\bar{x}}(f(x))$ exists by Cauchy's criterion Thm. //

P71 Ex 3.2.7 Find each of the following limits (if they exist)

(a.) $\lim_{x \to 1^+} \left(\frac{x+1}{x-1} \right) = \infty$ (does not exist as real #)

(I like to think about $x \to 1^+$ as $x = 1+\delta$ for small $\delta > 0$ then $\frac{x+1}{x-1} = \frac{2+\delta}{\delta} = \frac{2}{\delta} + 1 \gg 0$ for $\delta \approx 0$.)

(b.) $\lim_{x \to 0^+} \left| x^3 \sin\left(\frac{1}{x}\right) \right| = 0$

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for $x \neq 0$

$\Rightarrow -|x|^3 \leq |x|^3 \sin\left(\frac{1}{x}\right) \leq |x|^3$ since $|x|^3 > 0$ for $x \neq 0$

thus $\lim_{x \to 0} \left(|x|^3 \sin\left(\frac{1}{x}\right) \right) = 0$ by Squeeze Thm

$\Rightarrow \lim_{x \to 0} \left| x^3 \sin\left(\frac{1}{x}\right) \right| = 0.$

(c.) $\lim_{x \to 1} (x - [x])$ where $[x]$ = greatest integer less than or equal to x

$[1+\delta] = 1$ whereas $[1-\delta] = 0$ for $0 < \delta < 1$

it follows that $\lim_{x \to 1^+} (x - [x]) = 1 - 1 = 0$

whereas $\lim_{x \to 1^-} (x - [x]) = 1 - 0 = 1$

therefore, $\lim_{x \to 1} (x - [x])$ d.n.e. since its left & right limits differ. //

$\boxed{P72}$ Ex 3.3.1 part a/ Show $f(x) = ax + b$ for $a, b \in \mathbb{R}$ is continuous on $\mathbb{R}$ via the Def$^{\underline{n}}$ 3.3.1

Proof: If $a = 0$ then $f(x) = b$ and if $\varepsilon > 0$ then $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| = |b - b| = 0 < \varepsilon$ thus f continuous at x_0 for arbitrary $x_0 \in \mathbb{R}$. If $a \neq 0$ then suppose $\varepsilon > 0$ and choose $\delta = \varepsilon/|a|$. If $|x - x_0| < \delta$ then consider,

$$\begin{aligned}|f(x) - f(x_0)| &= |ax + b - (ax_0 + b)| \\ &= |a(x - x_0)| \\ &< |a|\delta = |a|\frac{\varepsilon}{|a|} = \varepsilon.\end{aligned}$$

Thus f is continuous on $\mathbb{R}$ as x_0 was arbitrary.//

$\boxed{P73}$ Ex 3.3.1 part b/ $f(x) = x^3 - 3$ on $\mathbb{R}$

Remark: I need to find my glasses! This was $x^2 - 3$ in text!

scratch work.

Notice $f(x) - f(x_0) = x^3 - x_0^3 = (x - x_0)(x^2 + xx_0 + x_0^2)$
then $|x^2 - xx_0 + x_0^2| \leq |x|^2 + |x_0||x| + |x_0|^2 \leq M^2 + M|x_0| + |x_0|^2$
if we have $|x| \leq M$. Note, $-\delta < |x - x_0| < \delta$ if $\delta \leq 1$ then
$-1 < x - x_0 < 1 \Rightarrow x_0 - 1 < x < 1 + x_0 < 1 + |x_0|$
so $-(|x_0| + 1) < x < 1 + |x_0| \Rightarrow |x| \leq 1 + |x_0| = M$.

Let $x_0 \in \mathbb{R}$.
Let $\varepsilon > 0$ and choose $\delta = \min(1, \varepsilon/K)$ where $K = (1 + |x_0|)^2 + (1 + |x_0|)|x_0| + |x_0|^2$ that is $K = (1 + |x_0|)(1 + 2|x_0|) + |x_0|^2 = M^2 + M|x_0| + |x_0|^2$ where $M = 1 + |x_0|$.
Anyway, if $|x - x_0| < \delta \leq 1$ then $-1 < x - x_0 < 1 \Rightarrow |x| < 1 + |x_0| = M$.
Suppose $|x - x_0| < \delta$ and consider, for $f(x) = x^3 - 3$

$$\begin{aligned}|f(x) - f(x_0)| = |x^3 - x_0^3| &= |x - x_0|\,|x^2 + xx_0 + x_0^2| \\ &< \delta(|x|^2 + |x||x_0| + |x_0|^2) \\ &< \delta(M^2 + M|x_0| + |x_0|^2) \leq \frac{\varepsilon}{K}K = \varepsilon.\end{aligned}$$

Thus $f(x) = x^3 - 3$ continuous on $\mathbb{R}$.

P74 Ex 3.3.2 parts b, c & d

Determine all $x \in \mathbb{R}$ at which each fnct. is continuous.

(b.) $f(x) = \begin{cases} \left|\frac{\sin x}{x}\right| & , x \neq 0 \\ 1 & , x = 0 \end{cases}$

$\lim\limits_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$ thus $\lim\limits_{x \to 0} \left|\frac{\sin x}{x}\right| = |1| = 1. = f(0)$

and $\lim\limits_{x \to \bar{x}} \left(\left|\frac{\sin x}{x}\right|\right) = \frac{\sin \bar{x}}{\bar{x}} = f(\bar{x})$ for $\bar{x} \neq 0$

f is continuous on $\mathbb{R}$.

#

(c.) $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

If $x > 0$ then $-x \leq x \sin\left(\frac{1}{x}\right) \leq x$ (*)

If $x < 0$ then $-x \geq x \sin\left(\frac{1}{x}\right) \geq x$ (**)

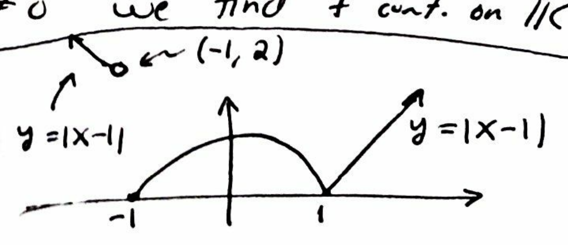
thus $x \sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0^+$ by Squeeze Th^m on (*)

and $x \sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0^-$ by Squeeze Th^m on (**)

Hence $\lim\limits_{x \to 0} \left(x \sin\left(\frac{1}{x}\right)\right) = 0 = f(0)$ and as

$\lim\limits_{x \to \bar{x}} \left(x \sin\left(\frac{1}{x}\right)\right) = \bar{x} \sin\left(\frac{1}{\bar{x}}\right)$ for $\bar{x} \neq 0$ we find f cont. on $\mathbb{R}$.

(d.) $f(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right) & \text{if } |x| \leq 1 \\ |x - 1| & \text{if } |x| > 1 \end{cases}$



It is clear f is continuous at all pts except ± 1. Considering $\bar{x} = 1$ note $\lim\limits_{x \to 1^+} (f(x)) = \lim\limits_{x \to 1^+} (|x-1|) = 0 = \lim\limits_{x \to 1^-} (f(x)) = \lim\limits_{x \to 1^-} \left(\cos\frac{\pi x}{2}\right)$.

However, $\lim\limits_{x \to -1^-} (f(x)) = \lim\limits_{x \to -1^-} |x-1| = 2 \neq 0 = \lim\limits_{x \to -1^+} \cos\left(\frac{\pi x}{2}\right)$.

P74 continued

(d.) we've shown $f(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right) & \text{if } |x| \leq 1 \\ |x-1| & \text{if } |x| \geq 1 \end{cases}$

is continuous on $(-\infty, -1) \cup (-1, \infty) = \mathbb{R} - \{-1\}$.

Remark: when we glue together continuous functions with casewise $\text{def}^{\underline{n}}$ then the glue-junctions, the edges of the cases, is where we must check for discontinuity.

P75 Ex 3.3.3 $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x^2 + a, & \text{if } x > 2 \\ ax - 1, & \text{if } x \leq 2 \end{cases}$$

Find value of a for which f is continuous

For $\bar{x} \neq 2$ we have $\lim_{x \to \bar{x}} f(x) = f(\bar{x})$. However, at $\bar{x} = 2$ we should study left vs. right limits,

$$\lim_{x \to 2^-} (f(x)) = \lim_{x \to 2^-} (x^2 + a) = 4 + a$$

$$\lim_{x \to 2^+} (f(x)) = \lim_{x \to 2^+} (ax - 1) = 2a - 1$$

We need $x \to 2^-$ and $x \to 2^+$ to match for the double-sided limit to exist. Hence solve

$$4 + a = 2a - 1 \implies \boxed{a = 5}$$

P76 Ex 3.3.11

Suppose f, g are continuous functions on $\mathbb{R}$ and $f(x) = g(x)$ $\forall x \in \mathbb{Q}$. Prove $f(x) = g(x)$ $\forall x \in \mathbb{R}$.

Let $x \in \mathbb{R} - \mathbb{Q}$ then let $x_n \in \mathbb{Q}$ $\forall n \in \mathbb{N}$ and suppose $x_n \to x$ as $n \to \infty$. Then by continuity of f, g we have $\lim\limits_{n\to\infty} (f(x_n)) = f(x)$ and $\lim\limits_{n\to\infty} (g(x_n)) = g(x)$

Therefore, $\lim\limits_{n\to\infty} (f(x_n) - g(x_n)) = \lim\limits_{n\to\infty} ((f-g)(x_n)) = f(x) - g(x) =$

Yet $f(x_n) = g(x_n)$ as r $\forall n \in \mathbb{N}$ as $x_n \in \mathbb{Q}$ thus $f(x_n) - g(x_n) = 0$ and $\lim\limits_{n\to\infty} (0) = 0 = f(x) - g(x)$.

Thus $f(x) = g(x)$ $\forall x \in \mathbb{R} - \mathbb{Q}$ and thus $f(x) = g(x)$ $\forall x \in \mathbb{R}$ as $f(x) = g(x)$ $\forall x \in \mathbb{Q}$ was given from the outset. //

Remark: I should probably prove $\exists \{x_n\} \subset \mathbb{Q}$ s.t. $x_n \to x$ as above. This claim follows from the density results for both $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$. Recall, between $x, y \in \mathbb{Q}$ $\exists r \in \mathbb{Q}$ with $x < r < y$. Likewise, for any $x, y \in \mathbb{Q}$, $\exists s \in \mathbb{R} - \mathbb{Q}$ with $x < s < y$.

(1.) fix $x \in \mathbb{R} - \mathbb{Q}$ (2.) Let $\tilde{x}_1$ be the decimal expansion of x truncated after the first decimal place. (e.g. $\pi = x \Rightarrow \tilde{x} = 3.1$ or $e = x \Rightarrow \tilde{x} = 2.7$) then $x \in (\tilde{x}_1 - 0.1, \tilde{x}_1 + 0.1)$ hence select $x_1 \in (\tilde{x} - 0.1, \tilde{x} + 0.1)$ with $x_1 \in \mathbb{Q}$ and $|x_1 - x| < 0.2$.

(3.) Let $\tilde{x}_2$ be decimal expansion of x truncated to two places ($x = \pi$, $\tilde{x}_2 = 3.14$) then $\tilde{x}_2 \in \mathbb{Q}$ and $x \in (\tilde{x}_2 - 0.01, \tilde{x}_2 + 0.01)$ with $|x_2 - x| < 0.02$

Continue in this fashion to construct $\{x_n\} \subseteq \mathbb{Q}$ with $|x_n - x| < 2(10^{-n}) = \frac{2}{10^n}$

It follows $x_n \to x$ as $n \to \infty$.

P77 Ex 3.4.1

Let $f: D \longrightarrow \mathbb{R}$ be continuous at $c \in D$ and let $\gamma \in \mathbb{R}$. Suppose $f(c) > \gamma$. Prove that there exists $\delta > 0$ such that $f(x) > \gamma$ for every $x \in B(c;\delta) \cap D$

Proof:

Suppose $f: D \longrightarrow \mathbb{R}$ is continuous at $c \in D$ and suppose $f(c) > \gamma$. Then $\varepsilon = f(c) - \gamma > 0$ hence $\exists \delta > 0$ such that $x \in D$ with $|x - c| < \delta$ implies $|f(x) - f(c)| < f(c) - \gamma$. Then note $-(f(c) - \gamma) < f(x) - f(c) < f(c) - \gamma$ hence $\gamma - f(c) < f(x) - f(c)$ and we find $\gamma < f(x)$. Therefore, $f(x) > \gamma$ for every $x \in B(c;\delta) \cap D$.

P78 Ex 3.4.2 Let f, g be continuous functs on $[a,b]$. Suppose, $f(a) < g(a)$ and $f(b) > g(b)$. Prove $\exists x_0 \in (a,b)$ s.t. $f(x_0) = g(x_0)$

Let $h(x) = f(x) - g(x)$ $\forall x \in [a,b]$ and note h is continuous on $[a,b]$. Notice $f(a) < g(a) \Rightarrow f(a) - g(a) < 0$ and $f(b) > g(b) \Rightarrow f(b) - g(b) > 0$ thus $h(a) < 0$ and $h(b) > 0$. Therefore, by Bolzano's Thm, $\exists x_0 \in (a,b)$ such that $h(x_0) = 0$. Thus, $f(x_0) - g(x_0) = 0 \Rightarrow f(x_0) = g(x_0)$. //

$\boxed{P79}$ Ex 3.4.4: Prove $x^2 - 2 = \cos(x+1)$ has at least two real solutions. We're given cosine is continuous

Let $f(x) = x^2 - 2 - \cos(x+1)$ then f is continuous on $\mathbb{R}$.

Moreover, calculate that:

① $f(-1-\pi) = (-1-\pi)^2 - 2 - \cos(-1-\pi+1)$
$= 1 + 2\pi + \pi^2 - 2 + 1$
$= \pi(\pi + 2) > 0$

② $f(-1) = (-1)^2 - 2 - \cos(-1+1)$
$= 1 - 3$
$= -2 < 0$

③ $f(\pi - 1) = (\pi-1)^2 - 2 - \cos(\pi - 1 + 1)$
$= \pi^2 - 2\pi + 1 - 2 + 1$
$= \pi(\pi - 2) > 0$

Bolzano's Thm on $[-1-\pi, -1] \Rightarrow \exists x_1 \in (-1-\pi, -1)$ such that $f(x_1) = 0$

Bolzano's Thm on $[-1, \pi - 1] \Rightarrow \exists x_2 \in (-1, \pi - 1)$ such that $f(x_2) = 0$

Thus $\exists x_1, x_2 \in \mathbb{R}$ s.t. $x^2 - 2 = \cos(x+1)$. //

$\boxed{P80}$ Ex. 3.4.5: let $f: [a,b] \longrightarrow [a,b]$ be continuous fnct,

(a.) prove $f(x) = x$ has sol^{n} on $[a,b]$

(b.) suppose further that

$$|f(x) - f(y)| < |x - y| \quad \forall x, y \in [a,b],\ x \neq y$$

Prove the eq^{n} $f(x) = x$ has unique sol^{n} on $[a,b]$

↙

P80 continued:

(a.) $f : [a,b] \rightarrow [a,b]$ a continuous fnct.

Then for $a \le x \le b$ we have $f(x) \in [a,b]$

that is $a \le f(x) \le b$. Consider $g(x) = x$ and

observe $g(a) = a \le f(a)$ and $g(b) = b \ge f(b)$

Therefore, using Ex 3.4.2 (P78) we find $\exists\, x_0 \in (a,b)$

for which $f(x_0) = g(x_0) \Rightarrow f(x_0) = x_0$.

(b.) Suppose $|f(x) - f(y)| < |x-y| \ \forall x,y \in [a,b],\ x \neq y$.

Suppose $\exists\, x_1 \in [a,b]$ s.t. $f(x_1) = x_1$ and $x_1 \neq x_0$

where $f(x_0) = x_0$. then

$$|f(x_1) - f(x_0)| < |x_1 - x_0|$$

$$\Rightarrow |x_1 - x_0| < |x_1 - x_0|$$ which is impossible.

Thus $\nexists\, x_1 \neq x_0$ for which $f(x_1) = x_1$.

The sol^{n} to $f(x) = x$ on $[a,b]$ must be unique. //