Mission 5 Solution

$\boxed{P81}$ Exercise 4.1.1

Th^{m} 4.1.3 a/ Assume f & g are differentiable at a, then

$$\lim_{h\to 0}\left[\frac{(f+g)(a+h)-(f+g)(a)}{h}\right]=\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]+\lim_{h\to 0}\left[\frac{g(a+h)-g(a)}{h}\right]$$

$$\therefore\ (f+g)'(a)=f'(a)+g'(a).//$$

Th^{m} 4.1.3 b/ Assume $f'(a)=\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]$ exists. Then

$$cf'(a)=c\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]=\lim_{h\to 0}\left[\frac{cf(a+h)-cf(a)}{h}\right]$$

Thus $(cf)'(a)=cf'(a).//$

$\boxed{P82}$ Ex 4.1.3/ Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=\begin{cases}x^2 & \text{if } x>0\\ 0 & \text{if } x\le 0\end{cases}$

(a.) $\lim_{h\to 0^+}\left[\frac{f(h)-f(0)}{h}\right]=\lim_{h\to 0^+}\left[\frac{h^2-0}{h}\right]=\lim_{h\to 0^+}(h)=0$

$\lim_{h\to 0^-}\left[\frac{f(h)-f(0)}{h}\right]=\lim_{h\to 0^-}\left[\frac{0-0}{h}\right]=0 \ \therefore\ \lim_{h\to 0}\left[\frac{f(h)-f(0)}{h}\right]=0.$

Therefore, $f'(0)=0$. For $x\neq 0$ we may use existing results, $x<0$, $\frac{d}{dx}(x^2)=2x$ and $x>0$, $\frac{d}{dx}(0)=0$

hence $f'(x)=\begin{cases}2x & \text{if } x>0\\ 0 & \text{if } x\le 0.\end{cases}$

(b.) $f''(x)=\begin{cases}2 & \text{if } x>0\\ 0 & \text{if } x\le 0\end{cases}$ and $f''(0)$ d.n.e.

Since $\lim_{h\to 0^+}\left(\frac{f'(h)-f'(0)}{h}\right)=\lim_{h\to 0^+}\left(\frac{2h}{h}\right)=2$

whereas $\lim_{h\to 0^-}\left(\frac{f'(h)-f'(0)}{h}\right)=\lim_{h\to 0^-}\left(\frac{0-0}{0}\right)=0$

hence f' is <u>not</u> differentiable. However, f' is continuous as $\lim_{x\to 0}(f'(x))=0$ from studying $x\to 0^+$ and $x\to 0^-$ separately and all other nonzero pts. are pts of continuity for f'.

P83 Ex 4.1.4

$$f(x) = \begin{cases} x^\alpha & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases} \Rightarrow f'(x) = \begin{cases} \alpha x^{\alpha-1} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $f'(0)$ requires further study.

Consider,

$$\lim_{h\to 0^+}\left(\frac{f(h)-f(0)}{h}\right) = \lim_{h\to 0^+}\left(\frac{h^\alpha}{h}\right) = \lim_{h\to 0^+}\left(h^{\alpha-1}\right) = \begin{cases} 0 & \text{if } \alpha > 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha < 1 \end{cases}$$

On the other hand $\lim_{h\to 0^-}\left(\frac{f(h)-f(0)}{h}\right) = \lim_{h\to 0^-}\left(\frac{0-0}{h}\right) = 0 \quad \forall \alpha.$

Hence $\boxed{f \text{ is differentiable with } f'(x) = \begin{cases} \alpha x^{\alpha-1} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}}$

(b.) $\boxed{\text{provided } \alpha > 1.}$ Notice, $\lim_{x\to 0^-}(f(x)) = 0$ and provided $\alpha > 0$,

$$\lim_{x\to 0^+}\left(x^\alpha\right) = 0 \Rightarrow \lim_{x\to 0}(f(x)) = 0$$

Thus $\boxed{f \text{ continuous for } \alpha > 0}$ (a.) (sorry out of order)

P84 on TEST 3 ☺, but sol^n very similar to

P85 From the Definition of the derivative, we calculate,

$$\frac{d}{dx}[\sin(x)] = \lim_{h\to 0}\left[\frac{\sin(x+h)-\sin(x)}{h}\right]$$

adding angles formula for sine.

$$= \lim_{h\to 0}\left[\frac{\sin x\cos(h) + \cos x\sin(h) - \sin x}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin(x)\left[\frac{\cos(h)-1}{h}\right] + \cos(x)\frac{\sin(h)}{h}\right]$$

since we're given $*$ & $**$ exist

$$= \sin x \underbrace{\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right]}_{* = 0} + \cos(x)\underbrace{\lim_{h\to 0}\left[\frac{\sin h}{h}\right]}_{** = 1}$$

$$= \cos(x). //$$

P86 Let $f(x) = x\sqrt{x^2}$ for $x \in \mathbb{R}$.

Then for $x \neq 0$, $\frac{d}{dx}\left(x\sqrt{x^2}\right) = \frac{dx}{dx}\sqrt{x^2} + x\frac{d}{dx}\left(\sqrt{x^2}\right)$ and so,

$$\frac{d}{dx}\left(x\sqrt{x^2}\right) = \sqrt{x^2} + x\frac{1}{2\sqrt{x^2}}(2x) = \sqrt{x^2} + \frac{x^2}{\sqrt{x^2}} = \frac{\left(\sqrt{x^2}\right)^2 + x^2}{\sqrt{x^2}}$$

that is $\frac{d}{dx}\left(x\sqrt{x^2}\right) = \frac{2x^2}{\sqrt{x^2}}$. In other words, since

$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ we find for $x \neq 0$,

$$\frac{d}{dx}\left(x|x|\right) = \frac{2x^2}{|x|} = \frac{2|x|^2}{|x|} = 2|x| = \begin{cases} -2x : x < 0 \\ 2x : x > 0 \end{cases}$$

The derivative at $x = 0$,

$$\lim_{h \to 0^+}\left(\frac{f(h) - f(0)}{h}\right) = \lim_{h \to 0^+}\left(\frac{h|h|}{h}\right) = \lim_{h \to 0^+}\left(|h|\right) = 0$$

$$\lim_{h \to 0^-}\left(\frac{f(h) - f(0)}{h}\right) = \lim_{h \to 0^-}\left(\frac{h|h|}{h}\right) = \lim_{h \to 0^-}\left(|h|\right) = 0$$

Oh, didn't need to break into left/right limits. In fact,
$f'(0) = 0$ is clear from $\lim_{h \to 0}\left(\frac{f(h) - f(0)}{h}\right) = \lim_{h \to 0}\left(\frac{h|h|}{h}\right) = \lim_{h \to 0}(|h|) = 0$.

In summary, $f'(x) = \begin{cases} -2x & : x < 0 \\ 2x & : x \geq 0 \end{cases}$ or $\boxed{f'(x) = 2|x|}$

$f''(0)$ does not exists since $\underbrace{\lim_{h \to 0^+}\left(\frac{f'(h) - f'(0)}{h}\right)}_{2.} \neq \underbrace{\lim_{h \to 0^-}\left(\frac{f'(h) - f'(0)}{h}\right)}_{-2.}$

thus $f''(x)$ does not exist
for all $x \in \text{dom}(f')$.
That is enough to declare f' not differentiable.

(it does seem kinda mean, after all $f''(x) = \pm 2$ for all other $x \neq 0$)

$\boxed{P87}$ Ex 4.1.11

I'll let you look at the sol^{n} in the text.
This is the quintessential example of a function which is differentiable but not continuously differentiable.

$\boxed{P88}$ Ex 4.1.12 Suppose f is differentiable at $x_o \in (a,b)$ and $c \in \mathbb{R}$,

(a.) Notice, $\lim_{n\to\infty}\left[n\left(f\left(x_o+\frac{1}{n}\right)-f(x_o)\right)\right] = \lim_{n\to\infty}\left[\frac{f(x_o+\frac{1}{n})-f(x_o)}{\frac{1}{n}}\right]$

We wish to show $\lim_{n\to\infty}\left[\frac{f(x_o+\frac{1}{n})-f(x_o)}{\frac{1}{n}}\right] = f'(x_o)$.

and we know $f'(x_o) = \lim_{H\to 0}\left[\frac{f(x_o+H)-f(x_o)}{H}\right]$ exists.

Let $\varepsilon > 0$ and choose $\delta > 0$ such that $0 < |H| < \delta$ implies $\left|\frac{f(x_o+H)-f(x_o)}{H} - f'(x_o)\right| < \varepsilon$.

Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$ then $n > N$ and $\frac{1}{n} < \frac{1}{N} < \delta$ thus $\left|\frac{f(x_o+\frac{1}{n})-f(x_o)}{\frac{1}{n}} - f'(x_o)\right| < \varepsilon$

and so we've shown $\lim_{n\to\infty}\left[\frac{f(x_o+\frac{1}{n})-f(x_o)}{\frac{1}{n}}\right] = f'(x_o)$. //

(b.) Given $\lim_{h\to 0}\left(\frac{f(x_o+h)-f(x_o)}{h}\right) = f'(x_o)$. Assume $c \neq 0$ for brevity.

Notice $cf'(x_o) = c\lim_{h\to 0}\left[\frac{f(x_o+h)-f(x_o)}{h}\right] = \lim_{h\to 0}\left[\frac{f(x_o+h)-f(x_o)}{h/c}\right]$

If $h\to 0$ then $H = h/c \to 0$ hence the limit above is the same as $H\to 0$ with $h = cH$ thus, $cf'(x_o) = \lim_{H\to 0}\left[\frac{f(x_o+cH)-f(x_o)}{H}\right]$. //

$\boxed{P89}$ Exercise 4.1.13 Let f be differentiable at $x_0 \in (a,b)$ and c constant.

Find $\lim_{h \to 0} \left[\frac{f(x_0+ch) - f(x_0-ch)}{h} \right] := ☺$

$$☺ = \lim_{h\to 0} \left[\frac{f(x_0+ch) - f(x_0) + f(x_0) - f(x_0-ch)}{h} \right]$$

$$= \lim_{h\to 0} \left[\frac{f(x_0+ch) - f(x_0)}{h} \right] - \lim_{h\to 0} \left[\frac{f(x_0-ch) - f(x_0)}{h} \right]$$

$$= c f'(x_0) - (-c f'(x_0)) \quad (\text{by } \boxed{P88})$$

$$= \underline{2 c f'(x_0)}.$$

$\boxed{P90}$ Ex 4.2.1 Let f & g be differentiable at x_0. Suppose $f(x_0) = g(x_0)$ and $f(x) \leq g(x)$ $\forall x \in \mathbb{R}$. Prove $f'(x_0) = g'(x_0)$.

Let $h(x) = f(x) - g(x)$ then notice $f(x) \leq g(x)$ implies $f(x) - g(x) = h(x) \leq 0$. Also, $h(x_0) = f(x_0) - g(x_0) = 0$.

Thus $h(x_0) = 0 \geq h(x)$ $\forall x \in \mathbb{R}$ (absolute maximum)

Then $h'(x_0) = 0$ by Fermat's Th$^{\text{m}}$ and $\underline{h'(x) = f'(x) - g'(x)}$ for $x = x_0$.

$$\Rightarrow h'(x_0) = f'(x_0) - g'(x_0) = 0 \Rightarrow f'(x_0) = g'(x_0). //$$

$\boxed{P91}$ Ex 4.2.2 b THIS IS ON TEST 3 ☺

$\boxed{P92}$ Ex 4.2.3 : $|\sin x - \sin y| \leq |x - y|$ (sol$^{\text{n}}$ given in text.)

Sketch : $f(x) = \sin(x)$ has $f'(x) = \cos(x)$. If $a < b$ then $\exists$ $c \in (a,b)$ s.t.

$$\underbrace{\left| \frac{f(b) - f(a)}{b - a} \right|}_{\frac{|\sin(b) - \sin(a)|}{|b-a|}} = |f'(c)|$$

$$\frac{|\sin(b) - \sin(a)|}{|b-a|} = |\cos(c)| \leq 1$$

$$\Rightarrow \underline{|\sin(b) - \sin(a)| \leq |b - a|}.$$

$\boxed{P93}$ Ex 4.2.4: Let n be a positive integer and $a_k, b_k \in \mathbb{R}$ for $k = 1, 2, \ldots, n$. Prove that the equation

$$x + \sum_{k=1}^{n} \left(a_k \sin(kx) + b_k \cos(kx) \right) = 0$$

has a solution on $(-\pi, \pi)$.

Following the text's hint, define $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ by

$$f(x) = x + \sum_{k=1}^{n} \left(a_k \sin(kx) + b_k \cos(kx) \right)$$

Then integration,

$$g(x) = \frac{x^2}{2} + \sum_{k=1}^{n} \left(\frac{-a_k \cos(kx)}{k} + \frac{b_k \sin(kx)}{k} \right)$$

has $g'(x) = f(x)$ and $g(-\pi) = g(\pi)$ $\quad (\sin(k\pi) = 0)$

thus $\exists\, c \in (-\pi, \pi)$ s.t. $g'(c) = f(c) = 0$ by Rolle's Th^{m}. //

$\boxed{P94}$ Ex 4.3.3 Let f & g be diff fncts. on $\mathbb{R}$ s.t. $f(x_0) = g(x_0)$ and $f'(x) \leq g'(x)$ $\forall x \geq x_0$. Prove that $f(x) \leq g(x)$ $\forall x \geq x_0$.

Let $h(x) = g(x) - f(x)$ and note $h'(x) = g'(x) - f'(x)$. But,

$f'(x) \leq g'(x) \Rightarrow g'(x) - f'(x) = h'(x) \geq 0$ and so h is

increasing on $[x_0, \infty)$. But $h(x_0) = g(x_0) - f(x_0) = 0$

hence $h(x) \geq h(x_0)$ $\forall x \in [x_0, \infty)$. Therefore,

$$0 \leq h(x) = g(x) - f(x) \quad \forall x \in [x_0, \infty)$$

$$\therefore \quad g(x) \geq f(x) \text{ for } x \geq x_0. //$$

$\boxed{P95}$ Ex. 4.3.5 ON TEST 3 ☺.

P96 Ex 4.5.2 / Find the 5^{th} Taylor Polynomial $P_5(x)$, at $\bar{x}=0$ for $\cos(x)$. Determine an upper bound for error $|P_5(x)-\cos(x)|$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Taylor's $Th^{=}$ says $f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-\bar{x})^{n+1}$

where $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\bar{x})}{k!}(x-\bar{x})^k$. Let $f(x) = \cos(x)$

$$\begin{aligned}
f(x) &= \cos(x) & f(0) &= 1 \\
f'(x) &= -\sin(x) & f'(0) &= 0 \\
f''(x) &= -\cos(x) & f''(0) &= -1 \\
f'''(x) &= \sin(x) & f'''(0) &= 0 \\
f^{(4)}(x) &= \cos(x) & f^{(4)}(0) &= 1 \\
f^{(5)}(x) &= -\sin(x) & f^{(5)}(0) &= 0 \\
f^{(6)}(x) &= -\cos(x) & f^{(6)}(0) &= -1
\end{aligned}$$

Hence $P_5(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$ and $\frac{f^{(6)}(0)}{6!} = \frac{1}{6!}$

Notice $|P_5(x) - \cos(x)| = \left|\frac{1}{6!}(x-0)^6\right| = \frac{|x|^6}{6!} \leq \boxed{\frac{\pi^6}{2^6 \cdot 6!}}$ ← error bound!

P97 Ex. 4.5.3. ON TEST 3 ☺

P98 Let $|r| \geq 1$ then $\lim_{n\to\infty}(ar^n) \neq 0$ so $\lim_{n\to\infty}(a_n) \neq 0$ thus the series diverges by the n^{th} term test. (I'll skip the details here)

If $|r| < 1$ then notice

$$S_{n-1} = a + ar + \cdots + ar^{n-2} + ar^{n-1}$$

$$r S_{n-1} = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Thus $S_{n-1} - rS_{n-1} = a - ar^n \Rightarrow S_{n-1} = \frac{a-ar^n}{1-r}$

then $\lim_{n\to\infty}\left(\frac{a-ar^n}{1-r}\right) = \frac{a}{1-r} - \frac{a}{1-r}\underbrace{\lim_{n\to\infty}(r^n)}_{0} = \frac{a}{1-r}$.

Remark: my apologies, if I had more time I'd develop more here, I highly recommend you study series further...

$\boxed{P99}$ A sequence of functions $\{f_n\}_{n=1}^{\infty}$ where $f_n : I \to \mathbb{R}$ is said to converge to $f : I \to \mathbb{R}$ if for each $x \in I$ we have $\lim_{n\to\infty} (f_n(x)) = f(x)$. If $x \in (-1,1)$ then $|x| < 1$ thus $1 + x + x^2 + \cdots = \frac{1}{1-x}$ thus $f_n(x) = 1 + x + x^2 + \cdots + x^{n-1}$ has $\lim_{n\to\infty} (f_n(x)) = \frac{1}{1-x}$ $\forall x \in (-1,1)$. //

$\boxed{P100}$ Let $x \in \mathbb{R}$, consider $\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ observe that $f(x) = \sin(x)$ has $P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ then by Taylor's Th$^{\underline{m}}$ as $f^{(n)}(x) \in \{\pm \sin(x), \pm \cos(x)\}$ we have $|f^{(n)}(x)| \leq 1$ hence,

$$| \sin(x) - P_n(x) | \leq \frac{|x|^{n+1}}{(n+1)!}$$

Fix $x_0 \in \mathbb{R}$ then as $n \to \infty$ we find $\frac{|x_0|^{n+1}}{(n+1)!} \to 0$ thus $|\sin(x_0) - P_n(x_0)| \to 0$ thus $\lim_{n\to\infty} (P_n(x_0)) = \sin(x_0)$.

But, x_0 is arbitrary, the result follows. //