MH 408: TOPICS IN ANALYSIS: TEST 2

Show work for full credit. A scientific, non-graphing, calculator is allowed. You are also allowed one page of notes on regular sized paper front and back. At least 100pts to earn here.

(1.) (10pts) Use limit laws and/or the appropriate theorem to show that

$$\lim_{n\to\infty}\left(\frac{n+\sin(n^4+4n+17)}{3n^2+7}\right)=0.$$

(2.) (10pts) Prove $a_n = \cos\left(\frac{n\pi}{3}\right)$ is a divergence sequence.

(3.) (10pts) Prove $a_n = \frac{n\sin(4n^2+2)}{n+3}$ has a convergent subsequence.

(4.) (10pts) Prove $\lim_{x\to 2}\left(\frac{x+13}{7-x}\right) = 3$ by direct argument based on the $\varepsilon-\delta$ definition of the limit.

(5.) (10pts) Let $f(x) = 2x^2+3$. Prove f is continuous on $\mathbb{R}$ by direct argument from the $\varepsilon-\delta$-definition of continuity.

(6.) (10pts) Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be functions which are continuous at $x_o \in D$. Prove $f+g$ is continuous at $x_o \in D$ by a by direct argument from the $\varepsilon-\delta$-definition of continuity.

(7.) (10pts) Suppose A and B are compact subsets of $\mathbb{R}$. Prove $A\cap B$ is compact.

(8.) (10pts) Fix $a, b, c \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3+ax^2+bx+c$. Prove there exists $x_o \in \mathbb{R}$ for which $f(x_o) = 0$. You may assume it is known that f is continuous.

(9.) (10pts) Let $f : [a, b] \to [2a, 2b]$ be a continuous function. Prove $f(x) = 2x$ has a solution on $[a, b]$.

(10.) (10pts) Suppose f, g are continuous functions on $\mathbb{R}$ and $f(x) = g(x)$ for all $x \in \mathbb{R}-\mathbb{Q}$. Prove $f(x) = g(x)$ for all $x \in \mathbb{R}$. You may assume it is known that for each $x \in \mathbb{Q}$ there exists a sequence $\{x_n\}$ such that $x_n \in \mathbb{R}-\mathbb{Q}$ for all $n \in \mathbb{N}$ and $x_n \to x$.

(11.) (10pts) Choose to answer **one** of the following:

(a.) (10pts) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is nowhere continuous.

(b.) (10pts) Give an example of a continuous function whose nonempty domain has no limit points.

SOLUTION TO TEST 2 : TOPICS IN ANALYSIS

$\boxed{P1}$ Observe $-1 \leq \sin(n^4 + 4n + 17) \leq 1$

implies $n-1 \leq n + \sin(n^4+4n+17) \leq n+1$

then divide by $3n^2 + 7 > 0$ to obtain

$$\frac{n-1}{3n^2+7} \leq \frac{n+\sin(n^4+4n+17)}{3n^2+7} \leq \frac{n+1}{3n^2+7}$$

However, $\dfrac{n \pm 1}{3n^2+7} = \dfrac{\frac{1}{n} \pm \frac{1}{n^2}}{3 + 7/n^2} \longrightarrow \dfrac{0 \pm 0}{3+0} = 0$ as $n \to \infty$.

Thus $\displaystyle\lim_{n\to\infty}\left(\frac{n+\sin(n^4+4n+17)}{3n^2+7}\right) = 0$ by Squeeze Th$\stackrel{\underline{m}}{}$. //

$\boxed{P2}$ Consider $a_n = \cos\left(\frac{n\pi}{3}\right)$. We seek to show $\{a_n\}$ divergent by providing two subsequences whose limit differs. Notice

$$a_{6k} = \cos\left(\frac{6k\pi}{3}\right) = \cos(2\pi k) = 1$$

$$a_{6k+3} = \cos\left(\frac{(6k+3)\pi}{3}\right) = \cos(2\pi k + \pi) = \cos(\pi) = -1$$

Thus $a_{6k} \to 1$ and $a_{6k+3} \to -1$ hence $\{a_n\}$ diverges.

$\boxed{P3}$ Prove $a_n = \dfrac{n\sin(4n^2+2)}{n+3}$ has a convergent subsequence.

We intend to use Bolzano-Weierstrauss, it suffices to show the sequence is bounded. Notice,

$$\left|\frac{n\sin(4n^2+2)}{n+3}\right| \leq \frac{n}{n+3} < \frac{n+3}{n+3} = 1.$$

Thus $\{a_n\}$ is bounded and hence by Bolzano Weierstrauss $\{a_n\}$ has a convergent subsequence.

$\boxed{P4}$ Let $\varepsilon > 0$ and choose $\delta = \min(1, \varepsilon)$. Suppose $x \in \mathbb{R}$ and $0 < |x-2| < \delta$ then $|x-2| < 1$ thus $-1 < x-2 < 1$ consequently $-6 < x-7 < -4$ and so $4 < |x-7| < 6$.

Consider,

$$\left|\frac{x+13}{7-x} - 3\right| = \left|\frac{x+13-3(7-x)}{7-x}\right|$$

$$= \left|\frac{4x-8}{7-x}\right|$$

$$= \frac{4|x-2|}{|x-7|} < \frac{4|x-2|}{4} = |x-2| < \delta \leq \varepsilon.$$

Therefore, $\lim_{x\to 2}\left(\frac{x+13}{7-x}\right) = 3.$ //

$\boxed{P5}$ Let $f(x) = 2x^2+3$. Suppose $\varepsilon > 0$ and let $x_0 \in \mathbb{R}$ choose $\delta = \min\{1, \frac{\varepsilon}{2M}\}$ where $M = \max\{|2x_0-1|, |1+2x_0|\}$. Suppose $x \in \mathbb{R}$ and $|x-x_0| < \delta$ then note $|x-x_0| < 1$ hence $-1 < x-x_0 < 1 \Rightarrow 2x_0-1 < x+x_0 < 2x_0+1$ thus $|x+x_0| < M$. Furthermore,

$$|f(x)-f(x_0)| = |2x^2+3-(2x_0^2+3)|$$

$$= 2|x^2-x_0^2|$$

$$= 2|x-x_0|\cdot|x+x_0|$$

$$< 2\delta M \leq 2\left(\frac{\varepsilon}{2M}\right)M = \varepsilon.$$

Thus $\lim_{x\to x_0}(f(x)) = f(x_0)$ $\forall x_0 \in \mathbb{R}$ $\therefore$ f is continuous. //

$\boxed{P6}$ Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are functions continuous at $x_0 \in D$. Let $\varepsilon > 0$ and choose $\delta_f, \delta_g > 0$ s.t. $|x - x_0| < \delta_f \Rightarrow |f(x) - f(x_0)| < \varepsilon/2$ and $|x - x_0| < \delta_g \Rightarrow |g(x) - g(x_0)| < \varepsilon/2$. Choose $\delta = \min(\delta_f, \delta_g)$ and suppose $|x - x_0| < \delta$ then

$$
\begin{aligned}
|(f+g)(x) - (f+g)(x_0)| &= |f(x) - f(x_0) + g(x) - g(x_0)| \\
&\le |f(x) - f(x_0)| + |g(x) - g(x_0)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus $f+g$ is continuous at x_0. //

$\boxed{P7}$ Suppose $A, B \subseteq \mathbb{R}$ and A & B are compact. Recall a subset of $\mathbb{R}$ is compact iff it is closed and bounded. Note $A \cap B$ is closed. ~~Let M_A bounds A; $|x| < M_A$ $\forall x \in A$. Likewise let M_B bound B; $|x| < M_B$ $\forall x \in B$. If $M = \max\{M_A$~~ (no need...)

If M_A bounds A then M_A bounds $A \cap B \subseteq A$ as $x \in A \cap B \Rightarrow x \in A$ thus $|x| < M_A$. Therefore, $A \cap B$ is closed and bounded $\therefore A \cap B$ compact. //

P8 Fix $a, b, c \in \mathbb{R}$. Let $f(x) = x^3 + ax^2 + bx + c$.

For $x_1 << 0$ we have x_1^3 dominates $f(x_1)$ and as $x_1 < 0$ it follows $f(x_1) \approx x_1^3 < 0$. Likewise for $x_2 >> 0$ we have $f(x_2) \approx x_2^3 > 0$. Thus, $\exists \tilde{c}$ with $f(\tilde{c}) = 0$.

Remark: sorry, I should have given more information about a, b, c. Perhaps $0 < a < b < c$ would make an explicit solution less cumbersome.

P9 Let $f : [a, b] \longrightarrow [2a, 2b]$ be continuous function. Prove $f(x) = 2x$ has a solution on $[a, b]$

(this was a modified version of P80)

As $f : [a, b] \longrightarrow [2a, 2b]$ is given, for each $x \in [a, b]$ we have $f(x) \in [2a, 2b]$ which means $2a \leq f(x) \leq 2b$. Consider $g(x) = 2x$ for $x \in [a, b]$ has $g(a) = 2a$ and $g(b) = 2b$. Let $h(x) = f(x) - g(x)$. notice h is continuous on $[a, b]$ and $h(a) = f(a) - 2a \geq 0$ whereas $h(b) = f(b) - 2b \leq 0$. If equality is attained at $h(a) = 0$ or $h(b) = 0$ then we're done since $h(a) = 0 \Rightarrow f(a) = 2a$ and $h(b) = 0 \Rightarrow f(b) = 2b$. Thus suppose $h(a) > 0$ and $h(b) < 0$ and apply IVT on $[a, b]$ to select $c \in (a, b)$ for which $h(c) = 0$ thus $f(2c) = 2c$.

In short, $f(x) = 2x$ has a sol$^{\underline{n}}$ on $[a, b]$.

(we could continue and prove this sol$^{\underline{n}}$ was unique as in P80 but, I didn't ask for that here)

$\boxed{P10}$ Suppose f & g are continuous on $\mathbb{R}$ and $f(x) = g(x)$ $\forall x \in \overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q}$. Prove $f(x) = g(x)$ $\forall x \in \mathbb{R}$. You are given for each $x \in \mathbb{Q}$ there exists sequence $\{x_n\}$ such that $x_n \in \mathbb{R} - \mathbb{Q}$ $\forall n \in \mathbb{N}$ and $x_n \to x$

Let $h(x) = f(x) - g(x)$ and observe h is continuous on $\mathbb{R}$. Let $x \in \mathbb{Q}$ and $\{x_n\}$ a sequence in $\mathbb{R} - \mathbb{Q}$ with $x_n \to x$ as $n \to \infty$. By continuity of h we have

Remark: this was based on P76

$$
\begin{aligned}
h(x) &= \lim_{n\to\infty} (h(x_n)) \\
&= \lim_{n\to\infty} (f(x_n) - g(x_n)) \\
&= \lim_{n\to\infty} (0) \\
&= 0.
\end{aligned}
$$

But this gives $f(x) = g(x)$ for $x \in \mathbb{Q}$ and we're given $f(x) = g(x)$ for $x \in \mathbb{R} - \mathbb{Q}$ hence $f(x) = g(x)$ $\forall x \in \mathbb{R}$.

$\boxed{P11}$

(a.) the Dirichlet Function $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$ is nowhere continuous on $\mathbb{R}$.

(b.) any sequence in $\mathbb{R}$ will suffice. Each point in the domain of $a : \mathbb{N} \to \mathbb{R}$ is isolated hence for $n_o \in \mathbb{N}$ and $\varepsilon > 0$ pick $\delta = 1/2$ then $|n - n_o| < \delta = \frac{1}{2} \Rightarrow n = n_o$ hence $|a(n) - a(n_o)| = |a(n_o) - a(n_o)| = 0 < \varepsilon$.