Ma 341-002: Test I, n-th order ODEs

For full credit make sure to show all work. If in doubt ask.

(1.)[15pts] Suppose that a particle of mass *m* is subject to the frictional force $F = -\beta v$ where $\beta > 0$ is some fixed constant and $v = \frac{dx}{dt}$ is the velocity. Assume that $v(0) = 7$. Assuming the motion is one-dimensional we find Newton's 2nd Law reads

$$\boxed{F = -\beta v = ma}$$

where $a = \frac{dv}{dt}$ is the acceleration. ***Find** v **as a function of time** t**.***

☻For a bonus point calculate the velocity as function of the position x. (for the bonus you may assume that the initial position is the origin; $x(0) = 0$.)

SOLUTION: Use separation of variables,

$$m\frac{dv}{dt} = \beta v \quad \Longrightarrow \quad \frac{dv}{v} = \frac{-\beta}{m}dt$$

Now integrate both sides to obtain,

$$\ln|v| = \frac{\beta t}{m} + c$$

I'll apply the initial condition $v(0) = 7$ to see that $\ln(7) = c$. Thus,

$$\ln|v| = \frac{-\beta t}{m} + \ln(7) \quad \Longrightarrow \quad |v| = e^{-\frac{\beta t}{m} + \ln(7)} = 7e^{-\frac{\beta t}{m}}$$

A moments reflection will reveal that this only has solutions when $v > 0$ thus we find

$$\boxed{v(t) = 7e^{-\frac{\beta}{m}t}}$$

I have added the (t) to emphasize the functional dependence of velocity on time in our equation. In contrast we wish to calculate $v(x)$ for the bonus point, recall the standard trick that in one-dimensional motion $a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$ so Newton's Law reads,

$$mv\frac{dv}{dx} = -\beta v \quad \Longrightarrow \quad dv = -\frac{\beta}{m}dx \quad \Longrightarrow \quad \int_{v(0)}^{v(x)} dv = \int_0^x -\frac{\beta}{m}dx$$

whence we find $v(x) - v(0) = -\frac{\beta}{m}(x - 0)$ therefore,

$$\boxed{v(x) = 7 - \frac{\beta}{m}x}$$

(2.)[15pts] Suppose we have a donut shop where the donuts are fried such that once they leave the hopper they are at a toasty 200 degrees F. If the room temperature is 70 deg. F and if it takes 3 minutes for it to cool to 150 deg. F then what is the temperature of the donut at time t assuming that Newton's Law of Cooling applies to this tasty treat ?

SOLUTION: we begin by applying Newton's Law of cooling,

$$\frac{dT}{dt} = k(T - 70)$$

If we rewrite this in prime notation we have $T' - kT = 70k$. This is a first order, constant coefficient, non-homogeneous ODE so we may apply our usual bag of tricks,

$$T' + kT = 70k \implies \lambda + k = 0 \text{ thus } \lambda = -k \implies T_h(t) = c_1 e^{-kt}.$$

Moreover it is clear that we should guess $T_p = A$, and $T_p' + kT_p = 70k$ reveals that $kA = 70k$ thus the particular solution is simply $T_p = 70$. Our general solution is simply $T = T_h + T_p$ so

$$T(t) = c_1 e^{-kt} + 70$$

We still need to apply the two conditions we were given,

$$T(0) = 200 = c_1 + 70 \implies c_1 = 130.$$
$$T(3) = 150 = 130e^{-3k} + 70 \implies \frac{80}{130} = e^{-3k} \implies k = -\frac{1}{3}\ln\left(\frac{8}{13}\right).$$

Now we just need to put it all together,

$$T = 130e^{\frac{1}{3}\ln(\frac{8}{13})t} + 70 \implies T = 130exp\left(\ln[\frac{8}{13}]^{\frac{t}{3}}\right) + 70$$

Which simplifies to

$$\boxed{T = 130\left(\sqrt[3]{\frac{8}{13}}\right)^t + 70.}$$

Of course I gave full credit for a variety of answers besides this one. I'd guess some of you prefer the following answer,

$$\boxed{T = 130e^{-0.1618t} + 70}$$

(3.)[10pts] Find the general solution to the following differential equation,

$$\boxed{x\frac{dv}{dx} = \frac{1-4v^2}{3v}}$$

SOULTION: we can solve this by separation of variables,

$$\frac{3vdv}{1-4v^2} = \frac{dx}{x} \quad \Longrightarrow \quad \int \frac{3vdv}{1-4v^2} = \ln|x| + c.$$

The integral on the r.h.s was elementary, but the l.h.s requires a little work. Not too much though it's just a u-substitution. Use $u = 1-4v^2$ so that $du = -8vdv$ thus $vdv = \frac{-1}{8}du$,

$$\int \frac{3vdv}{1-4v^2} = \int \frac{3(\frac{-1}{8}du)}{u} = -\frac{3}{8}\ln\left|1-4v^2\right|$$

Consequently,

$$\boxed{-\frac{3}{8}\ln\left|1-4v^2\right| = \ln|x| + c}$$

We could simplify this answer further but without an initial condition we'd need to be real careful with the absolute value bars on both sides.

(4.)[10pts] Find the general solution to the following differential equation,

$$\boxed{\frac{dy}{dx} + 2xy = e^{x-x^2}}$$

SOLUTION: this is an integrating factor method problem. Identify that it is already conveniently in standard form with $p = 2x$ so the integrating factor is

$$\mu = exp\left(\int(2xdx)\right) = e^{x^2}$$

Now multiply our given ODE by μ to obtain,

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = e^{x^2}e^{x-x^2} = e^x$$

Now as usual we can apply the product rule due to the insight of the method,

$$\frac{d}{dx}\left(e^{x^2}y\right) = e^x \quad \Longrightarrow \quad e^{x^2}y = \int e^x dx = e^x + c \quad \Longrightarrow \quad \boxed{y = e^{x-x^2} + ce^{-x^2}}.$$

(5.)[15pts] Solve the following constant coefficient ODEs,

a.) $y'' + 9y = 0$

b.) $y''' - y' = 0$

c.) $[\ (D-1)^3(D^2+1)((D-2)^2+9)^2\ ](y) = 0$

SOLUTION: in each case we write the auxiliary equation and solve it in order that we can apply our general results which we derived in the lecture,

(a.) $\lambda^2 + 9 = 0$ which has solutions $\lambda = \pm 3i$ thus $\alpha = 0$ and $\beta = 3$ so,

$$\boxed{y = c_1\cos(3x) + c_2\sin(3x)}$$

(b.) $\lambda^3 - \lambda = \lambda(\lambda^2 - 1) = \lambda(\lambda-1)(\lambda+1) = 0$ thus $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = -1$. Which gives us the following general solution,

$$\boxed{y = c_1 + c_2e^x + c_3e^{-x}}$$

Although we could also write $y = b_1 + b_2\cosh(x) + b_3\sinh(x)$ it is a simple exercise to show these solutions are equivalent.

c.) $[\ (D-1)^3(D^2+1)((D-2)^2+9)^2\ ](y) = 0$ tells us that the aux. equation is

$$(\lambda-1)^3(\lambda^2+1)((\lambda-2)^2+9)^2 = 0.$$

We can break this up into three cases,

$$\begin{aligned} (\lambda-1)^3 = 0 \quad &\Longrightarrow \quad \lambda = 1,\ \text{three times} \\ \lambda^2+1 = 0 \quad &\Longrightarrow \quad \lambda = \pm i \\ ((\lambda-2)^2+9)^2 = 0 \quad &\Longrightarrow \quad \lambda = 2 \pm 3i,\ \text{twice} \end{aligned}$$

So we find the following general solution,

$$\boxed{\begin{aligned} y \ &= c_1e^x + c_2xe^x + c_3x^2e^x + c_4\cos(x) + c_5\sin(x) \\ &\quad + c_6e^{2x}\cos(3x) + c_7e^{2x}\sin(3x) + c_8xe^{2x}\cos(3x) + c_9xe^{2x}\sin(3x) \end{aligned}}$$

(6.)[35pts] Solve the following non-homogeneous ODE. Please justify your particular solution via the annihilator method before you get too far into the problem.

$$\boxed{y'' - 2y' + y = e^x + e^{2x} + x^2}$$

SOLUTION: To begin lets find the homogeneous solution,

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \quad \Longrightarrow \quad y_h = c_1 e^x + c_2 x e^x.$$

This shows us that we can rewrite the given equation as $L[y] = g$ identifying that $L = (D-1)^2$ and $g = e^x + e^{2x} + x^2$. Next, we wish to determine the correct guess for the particular solution using the method of annihilators. That means to start with we need to find an operator A such that $A[g] = 0$. Observe that

$$A_1[e^x] = 0 \quad \Longrightarrow \quad A_1 = D - 1.$$
$$A_2[e^{2x}] = 0 \quad \Longrightarrow \quad A_2 = D - 2.$$
$$A_3[x^2] = 0 \quad \Longrightarrow \quad A_3 = D^3.$$

Next, we can form the total annihilator by multiplying these together. It is clear that $A = (D-1)(D-2)D^3$ will annihilate $g = e^x + e^{2x} + x^2$. Then we operate on our equation $L[y] = g$ by A on both sides,

$$(D-1)(D-2)D^3(D-1)^2[y] = 0$$

Now we can write the aux. equation, I'll gather together the $(D-1)$ factors,

$$(\lambda - 1)^3(\lambda - 2)\lambda^3 = 0$$

Which yields $\lambda_1 = \lambda_2 = \lambda_3 = 1, \ \lambda_4 = 2$ and $\lambda_5 = \lambda_6 = \lambda_7 = 0$ thus the general solution to $AL[y] = 0$ (which is also the general solution to $L[y] = g$ as is clear from our calculations) is simply,

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 e^{2x} + c_5 + c_6 x + c_7 x^2$$

We identify that the first two terms are y_h and we for our future convenience relabel the coefficients on the remaining terms which form y_p

$$\boxed{y_p = Ax^2 e^x + Be^{2x} + C + Dx + Ex^2}$$

Now we must determine the undetermined coefficients A, B, C, D, E.

Lets calculate the derivatives of y_p ,

$$\begin{aligned} {y'}_p &= 2Axe^x + Ax^2e^x + 2Be^{2x} + D + 2Ex \\ &= (2Ax + Ax^2)e^x + 2Be^{2x} + D + 2Ex \end{aligned}$$

Now differentiate again,

$$\begin{aligned} {y''}_p &= (2A + 2Ax)e^x + (2Ax + Ax^2)e^x + 4Be^{2x} + 2E \\ &= (2A + 4Ax + Ax^2)e^x + 4Be^{2x} + 2E \end{aligned}$$

Now plug y_p into the differential equation,

$${y''}_p - 2{y'}_p + y_p = e^x + e^{2x} + x^2$$

This yields,

$$\begin{aligned} &(2A + 4Ax + Ax^2)e^x + 4Be^{2x} + 2E \\ &\quad - 2\big[(2Ax + Ax^2)e^x + 2Be^{2x} + D + 2Ex\big] \\ &\qquad + Ax^2e^x + Be^{2x} + C + Dx + Ex^2 = e^x + e^{2x} + x^2 \end{aligned}$$

Now we must be careful with parenthesis and signs and such,

$$2Ae^x + Be^{2x} + 2E - 2D - 4Ex + C + Dx + Ex^2 = e^x + e^{2x} + x^2$$

From which it follows that

$$\begin{aligned} e^x &: \ 2A = 1 \\ e^{2x} &: \ B = 1 \\ 1 &: \ 2E - 2D + C = 0 \\ x &: \ -4E + D = 0 \\ x^2 &: \ E = 1 \end{aligned}$$

These equations are not too bad to solve, clearly $A = 1/2,\ B = 1,\ E = 1$ then we see that $D = 4E = 4$ and finally $C = 2D - 2E = 8 - 2 = 6$. Thus the general solution is

$$\boxed{y = c_1e^x + c_2xe^x + \frac{1}{2}x^2e^x + e^{2x} + 6 + 4x + x^2}$$