

OUTLINE: MANIFOLD THEORY

The purpose of this course is to develop the basic properties of manifold theory. Manifold theory will be presented, not as a separate mathematical discipline (although this is in fact true) but rather as the appropriate arena in which to formulate any nonlinear mathematical theory.

1. Preliminary definitions and results from multivariable calculus. Definition of the derivative of a mapping f as a linear mapping with the property that it is the best linear approximation to f . Basic properties of the derivative will be stated (but not proven) including its relation to the Jacobian matrix. Examples from matrix theory will be provided to show the utility of thinking of the derivative as a linear mapping. The inverse and implicit function theorems will be stated and problems given to show how they are used. Two days.
2. Charts, atlases, differentiable structures, and manifolds will be defined along with relevant examples. The concept of a Lie group will be defined and some standard examples presented. These examples will be emphasized throughout the course to illustrate the various constructions utilized in the development of manifold theory. A day and a half.
3. Techniques for constructing new manifolds from known manifolds will be presented. These include: finite products of manifolds, regular submanifolds, manifolds obtained as level “surfaces” of maps from \mathbf{R}^n to \mathbf{R}^m . Manifold structures of standard groups of matrices such as $O(n)$, $SO(n)$, the Lorentz group, and the symplectic group. Three days.
4. Three equivalent definitions of tangent vectors: equivalence classes of curves, tensorial definition, derivations. Tangent spaces and derivatives of mappings from one manifold to another. Immersions, imbeddings, local diffeomorphisms and (global) diffeomorphisms. Three and a half days.
5. Manifold structure of tangent bundles and tangent mappings. Vector fields on a manifold and their relation to differential equations. The flow of a vector field. The Lie brackets of two vector fields. Left invariant vector fields on a Lie group. Lie algebra of a Lie group and its structure constants. Three and a half days.

6. Vector space duals, tensor algebras, and exterior algebras. Two days.
7. Cotangent spaces, Cotangent bundles, other tensor bundles of a manifold. Tensor fields, metrics, symplectic structures. Differential forms, exterior derivatives of forms, pullbacks of forms, interior products. The exterior derivative of a form applied to vector fields and application to Lie algebra cohomology. Mayer-Cartan form of a Lie group, and the structure equation. Five or six days.
8. Lie derivatives of vector fields, tensors, and differential forms. The magic formula of Cartan: $L_X = i_X d + d i_X$. Two days.
9. Distributions, involutive distributions, Frobenius' Theorem, foliations. Application to Lie groups Theorem 3.19 of Warner page 94. Five days.
10. Stoke's Theorem. Remaining time.

MA 555 — MANIFOLD THEORY

Cook

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$



$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} - f'(x) \right] = 0$$

$$\lim_{h \rightarrow 0} \left\{ \frac{f(x+h) - f(x) - f'(x)h}{h} \right\} = 0$$

Note: $\lim_{x \rightarrow a} g(x) = 0 \iff \lim_{x \rightarrow a} |g(x)| = 0$ works only for zero

Thus

$$\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|} = 0$$

If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ then f is differentiable at x iff

$\exists v \in \mathbb{R}^n$ such that

$$\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - v \cdot h|}{\|h\|} = 0$$

Def^b/ A linear space is a set E with 2 operations $+$, \cdot with

- 1.) $x, y \in E \Rightarrow x+y \in E$
- 2.) $x+(y+z) = (x+y)+z$
- 3.) $\exists \hat{0} \in E \ni x+\hat{0} = \hat{0}+x = x, \forall x \in E$
- 4.) $\forall x \in E, \exists -x \in E \ni x+(-x) = \hat{0} = (-x)+x$
- 5.) $x+y = y+x, x, y \in E$
- 6.) $\forall c \in \mathbb{R}, x \in E, c \cdot x \in E$
- 7.) $c \cdot (x+y) = c \cdot x + c \cdot y$
 $(c+d) \cdot x = c \cdot x + d \cdot x$
- 8.) $(cd) \cdot x = c \cdot (dx)$
- 9.) $1 \cdot x = x \quad \forall x \in E$

Def^b/ A linear space E is a normed space if \exists a function
 $x \rightarrow \|x\|$ from $E \rightarrow \mathbb{R}$ such that

- 1.) $\|x+y\| \leq \|x\| + \|y\|$
- 2.) $\|cx\| = |c| \|x\|$
- 3.) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = \hat{0}$.

Notation .. $E = \mathbb{R}^n, x \in \mathbb{R}^n, x = (x^1, x^2, \dots, x^n)$

$$x+y = (x^1+y^1, \dots, x^n+y^n)$$

$$cx = (cx^1, cx^2, \dots, cx^n)$$

Several Norms are of interest,

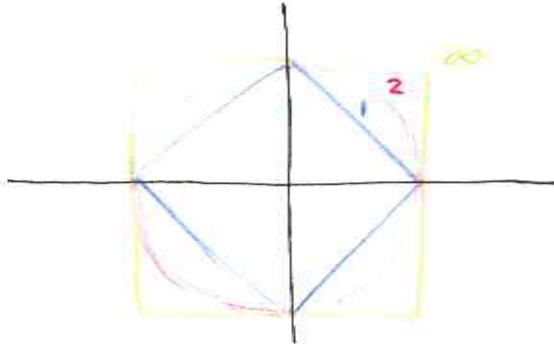
$$\|x\|_2 = \sqrt{\sum_{i=1}^n (x^i)^2} \quad \leftarrow \text{can induce from dot product } x \cdot y = \sum x^i y^i \\ \text{then } \|x\|_2 = \sqrt{x \cdot x}$$

$$\|x\|_1 = \sum_{i=1}^n |x^i|$$

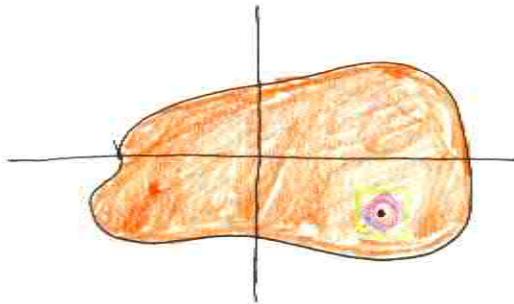
$$\|x\|_\infty = \max_{1 \leq i \leq n} |x^i|$$

$$B_r(x) = \{y \mid \|y-x\| < r\}$$

These norms are equivalent, let $r=1$ for example and $n=2$



For a set to be open it means that for every point in the set there is a ball around it, it turns out that all the above norms generate the same classes of open sets



Def* If E and F are normed linear spaces and $U \subseteq E$, $V \subseteq F$ are open then $f: U \rightarrow V$ is differentiable at $x \in U$ iff \exists a continuous linear mapping $L_x: E \rightarrow F$ \exists

$$\lim_{\|h\|_E} \frac{\|f(x+h) - f(x) - L_x h\|_F}{\|h\|_E} = 0$$

Can prove for fixed x that L_x is unique. Denote it by $Df(x) = D_x f$

$$D_x f(h) = Df(x)(h)$$

$$Df(x)(h) = f'(x)h$$

THEOREM

Assume $U \subseteq \mathbb{R}^n$ is open, $V \subseteq \mathbb{R}^m$ is open and $f: U \rightarrow V$. If f is differentiable at $x \in U$ then $\frac{\partial f^i}{\partial u_j}$ exists at x and

$$Df(x)(h) = h J_f(x)^t$$

$$Df(x)(h)^t = J_f(x) h^t$$

THEOREM

If $U \subseteq \mathbb{R}^n$ is open, $V \subseteq \mathbb{R}^m$ is open and $f: U \rightarrow V$. Then if Df exists at each point of U and if Df is continuous on U then $\frac{\partial f^i}{\partial u_j}$ all exist and are continuous on U and conversely.

$$Df: U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \quad \leftarrow \text{no theorem.}$$

$$J_f: U \rightarrow \mathbb{R}^{m \times n} \quad \leftarrow \text{theorem.}$$

Let $E = GL(n, \mathbb{R}) = \{A \mid A \text{ is a } (n \times n) \text{ real matrix}\}$

$$A = A_{ij}^i \quad \begin{array}{l} \text{row} \\ \text{column} \end{array}$$

$$\left. \begin{array}{l} A + B = (A_{ij}^i + B_{ij}^i) \\ (A + B)_{ij}^i = A_{ij}^i + B_{ij}^i \\ c A = (c A_{ij}^i) \\ (c A)_{ij}^i = c A_{ij}^i \end{array} \right\} \text{the matrix operations}$$

$$A = (A_1^1, \dots, A_n^1, A_1^2, \dots, A_n^2, \dots, A_1^n, \dots, A_n^n) \in \mathbb{R}^{n^2}$$

$$\|A\| = \sqrt{\sum_{i,j} (A_{ij}^i)^2}$$

(Frobenius Norm)

basically know this is
a norm due to
the isomorphism of $\mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$

$$\begin{aligned}\|A\|^2 &= \sum_i \sum_j A_{ij}^i A_{ij}^i & (AB)_{ij}^i &= \sum_{k=1}^n A_{ik}^i B_{kj}^k \\ &= \sum_i \left(\sum_j A_{ij}^i (A^t)_{ij}^i \right) \\ &= \sum_i (AA^t)_{ii}^i \\ &= \boxed{\text{Tr}(AA^t) = \|A\|^2}\end{aligned}$$

Cauchy Schwartz : $|x \cdot y| \leq \|x\| \|y\|$

Exercise 1 If $A, B \in gl(n)$ then $\|AB\| \leq \|A\| \|B\|$

$gl(n), +, \cdot, \circ, \cdot, \parallel \parallel \Rightarrow$ Banach Algebra

Taylor Series $f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \dots$

$$f : gl(n) \longrightarrow gl(n)$$

$$f(A) = A^2$$

$$f(A+H) = (A+H)^2 = (A+H)(A+H) = A^2 + AH + HA + H^2$$

$$f(A+H) = f(A) + AH + HA + H^2$$

$$Df(x)(h) = h J_f(x)^t$$

$$Df(x)(h) = f'(x)h$$

↑
linear in h

Fulip conjecture is that the following is indeed the derivative

$$Df(A)(H) \equiv AH + HA$$

$$\begin{aligned} Df(A)(H_1 + H_2) &= A(H_1 + H_2) + (H_1 + H_2)A \\ &= AH_1 + AH_2 + H_1A + H_2A \\ &= Df(A)(H_1) + Df(A)(H_2) \end{aligned}$$

So it is really the derivative

$$0 \leq \frac{\|f(A+H) - f(A) - Df(A)(H)\|}{\|H\|} = \frac{\|H^2\|}{\|H\|} \leq \frac{\|H\|\|H\|}{\|H\|} = \|H\|$$

Thus as $\|H\| \rightarrow 0$ as $H \rightarrow 0$ by the squeezing lemma we can send the middle to zero. We find something we believe to be the derivative by some linear approximation then check a limit too see if it is indeed the derivative.

Exercise 2: Let $f, g : gl(n) \rightarrow gl(n)$ and let ~~$f(A) = A^3$~~ and let $f(A) = A^3$ find linear part and show the derivative exist.

THE MATRIX EXPONENTIAL

1/10/2001

$$A \in gl(n)$$

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$$

$$S_n = \sum_{p=0}^n \frac{1}{p!} A^p$$

$m > n$ then we have

$$S_m - S_n = \sum_{p=0}^m \frac{1}{p!} A^p - \sum_{p=0}^n \frac{1}{p!} A^p = \sum_{\substack{p=n+1 \\ p \text{ odd}}}^m \frac{1}{p!} A^p$$

$$\|S_m - S_n\| = \left\| \sum_{p=n+1}^m \frac{1}{p!} A^p \right\|$$

$$\leq \sum_{p=n+1}^m \frac{1}{p!} \|A^p\|$$

$$\leq \sum_{p=n+1}^m \frac{1}{p!} \|A\|^p$$

$$\text{Define } D_n = \sum_{p=0}^n \frac{1}{p!} \|A\|^p$$

so then we examine the limit of D_n

$$\lim_{n \rightarrow \infty} D_n = \sum_{p=0}^{\infty} \frac{1}{p!} \|A\|^p = e^{\|A\|}$$

$$\|S_m - S_n\| \leq \|D_m - D_n\| < \epsilon \quad \forall \epsilon > 0 \quad \exists n, m \geq N \Rightarrow$$

$$|D_m - D_n| < \epsilon \quad \therefore \{S_n\}_{n=0}^{\infty} \text{ is a Cauchy Seq. in } gl(n)$$

Def^e/ A normed linear space E is complete iff every Cauchy sequence in E has a limit in E .

\mathbb{R}^n is complete in $\|\cdot\|_2$ as proven in standard analysis.

$gl(n) \approx \mathbb{R}^{n^2} \Rightarrow gl(n)$ is complete.

Proposition

The series for e^A is convergent in $gl(n)$ $\forall A \in gl(n)$

FACT: $e^{A+B} = e^A e^B$ if $AB = BA$

INVERSE FUNCTION THEOREM : Assume E, F are complete normed linear spaces (Banach spc.) and that $U \subseteq E$ and $V \subseteq F$ are open. If $f: U \rightarrow V$ is a smooth function such that $Df(x_0)$ is invertible for some $x_0 \in U$ then \exists open sets $U_{x_0} \subseteq U$, $V_{y_0} \subseteq V$, $y_0 = f(x_0)$ such that $f|_{U_{x_0}}: U_{x_0} \rightarrow V_{y_0}$ is a diffeomorphism (that is $f|_{U_{x_0}}$ and $(f|_{U_{x_0}})^{-1}$ are smooth)

$Df(x_0)$ is invertible $\Rightarrow Df(x_0)$ is an isomorphism of $E \rightarrow F$
So we might as well call $E = F = \mathbb{R}^n$ in $\dim(E, F) < \infty$ case.

Example

$$f(x, y) = (x^2 + y^2, x^2 - y^2)$$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$Df(x_0)$ invertible $\Leftrightarrow J_f(x_0)$ is invertible

$$J_f(x_0, y_0) = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix} \Rightarrow \det J_f(x_0, y_0) = -4xy - 4xy = -8xy$$

then $J_f(x_0, y_0)$ is invertible iff $\det J_f(x_0, y_0) \neq 0$

$$u = x^2 + y^2 \geq 0$$

$$v = x^2 - y^2$$

$$2x^2 = u+v \geq 0$$

$$x^2 = \frac{u+v}{2}$$

$$2y^2 = u-v \geq 0$$

$$x = \pm \sqrt{\frac{u+v}{2}}$$

$$y = \pm \sqrt{\frac{u-v}{2}}$$