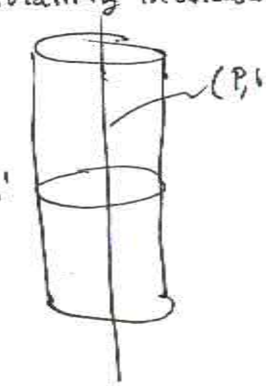
The Tangent Bundle

$TM = \{ (p,v) \mid v \in T_pM,\ p \in M \}$ V is not free, this prevents the bundles' triviallity in all case

$T_pM = \{ (p,v) \mid v \in T_pM \}$

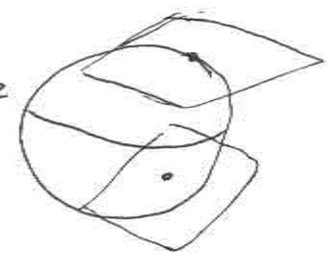
$TS^1 = S^1 \times \mathbb{R}$ $\longrightarrow$ S^1



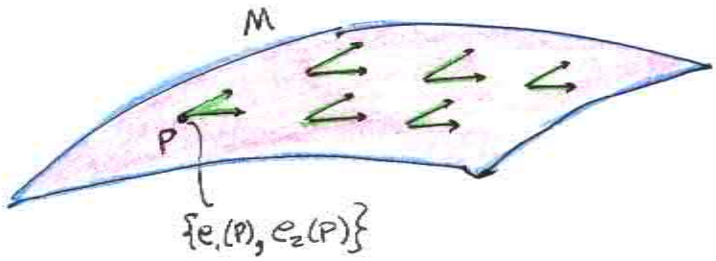
$TS^2 \neq S^2 \times \mathbb{R}^2$ $\leftarrow$ nontrivial vector field

$T_pS^2 \cong \mathbb{R}^2$

S^2



We don't think of tangent planes intersecting



If we had two global linearly independent vector fields on M of dim(M) = 2 which form a basis at T_pM ... Then every vector could be written in terms of global basis $\{e_1(P), e_2(P)\}$

$V \in T_pM \Rightarrow V = v^1 e_1(P) + v^2 e_2(P)$

$(P,V) \longrightarrow (P,(v^1,v^2)) \in M \times \mathbb{R}^2$

Tangent Bundle is trivial if this occurs.

S^1

$V \in T_pS^1 \Rightarrow V = v^1 e_1(P)$

$(P,V) \longrightarrow (P,v^1) \in S^1 \times \mathbb{R}$

Millner proved S^7 had infinitely many unique differentiable structures

$$TM = \{ (p,v) \mid p \in M,\ v \in T_pM \}$$

Let (U, x) be a chart on M. I want to define a chart on TM by the following formula;

$$T_x : TU \longrightarrow x(U) \times \mathbb{R}^m \subseteq \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$$

$$\boxed{(T_x)(p,v) = \left(x(p), (d_px^1(v), d_px^2(v), \ldots, d_px^m(v)) \right) \in \mathbb{R}^{2m}}$$

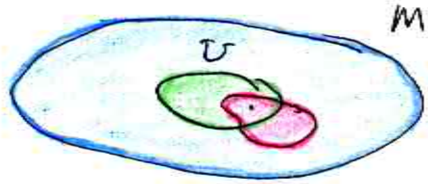
Where we have made the identification

$$TU = \{ (p,v) \mid p \in U,\ v \in T_pU = T_pM \}$$

Since $X_p = \sum X_x^i \frac{\partial}{\partial x^i}(p)$ is the same on U or $M \Rightarrow T_pM \cong T_pU$.

M

U



Now then $v \in T_pM \Rightarrow \sum_i v_x^i \frac{\partial}{\partial x^i} = v$ and dx^j a linear map thus

$$d_px^j(v) = \sum_i v_x^i \, dx^j\left(\frac{\partial}{\partial x^i}\right) = v_x^j$$

$$\left(d_px^1(v), d_px^2(v), \ldots, d_px^m(v) \right) = \left(v_x^1, v_x^2, \ldots, v_x^m \right)$$

Thus $(T_x)(p,v)$ is expressable in less fancy terms by

$$\boxed{(T_x)(p,v) = \left(x(p), (v_x^1, v_x^2, \ldots, v_x^m) \right)}$$

TEST

* Prove 3 out of 4

4d)

Now if $f: M \to \mathbb{R}^n$ then $d_p f(v) = \sum_{i,\gamma} \frac{\partial f^i}{\partial x^\gamma}(p)\, V_x^\gamma e_i$

Thus we could also be written as

$$\boxed{(T_x)(p,v) = (x(p), d_p x(v))}$$

Theorem

If $\mathcal{A}$ is an atlas on M then

$$T\mathcal{A} = \{(TU, T_x) \mid (U,x) \in \mathcal{A}\}$$

is an atlas for TM.

Pf/ Let (U,x) and (V,y) be charts in $\mathcal{A}$ such that $U \cap V \neq \phi$. Then (TU, T_x), (TV, T_y) are charts in $T\mathcal{A}$ and $T_x : TU \to x(U) \times \mathbb{R}^m$ Is defined by $(T_x)(q,v) = (x(q), (V_x^1, V_x^2, \dots, V_x^m))$

where $\mathbf{v} = \sum_i V_x^i \frac{\partial}{\partial x^i}\Big|_p$ so then

$$\begin{aligned}(T_x)^{-1}(u,\vec{v}) &= (T_x^{-1})(u, \sum v^i e_i) \\ &= \left(x^{-1}(u), \sum_i v^i \left(\frac{\partial}{\partial x^i}\Big|_p\right)\right)\end{aligned}$$

don't need to prove injective that's clear.

$$\begin{aligned}(T_y)(T_x^{-1})(u,\vec{v}) &= (T_y)\left(x^{-1}(u), \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p\right) \\ &= \left(y(x^{-1}(u)), \left(dy^1\left(\sum_i v^i \frac{\partial}{\partial x^i}\Big|_p\right), dy^2\left(\sum_i v^i \frac{\partial}{\partial x^i}\right), \dots, dy^m\left(\sum_i v^i \frac{\partial}{\partial x^i}\right)\right)\right) \\ &= \left((y \circ x^{-1})(u), \left(\sum_i v^i dy^1\left(\frac{\partial}{\partial x^i}\right), \dots, \sum_i v^i dy^m\left(\frac{\partial}{\partial x^i}\right)\right)\right)\end{aligned}$$

Now we can use, $f: U \to \mathbb{R}$ then $d_p f(X_p) = X_p f$ thus $X_p \to \frac{\partial}{\partial x^i}, dy^m$:

$$(T_y)(T_x^{-1})(u,\vec{v}) = \left((y \circ x^{-1})(u), \left(\sum_i v^i \frac{\partial y^1}{\partial x^i}, \dots, \sum_i v^i \frac{\partial y^m}{\partial x^i}\right)\right)$$

$$(T_y)\circ(T_x^{-1})\circ(u,\vec{v}) = \left((y\circ x^{-1})(u),\ \left(\sum_i v^i \frac{\partial y^1}{\partial x^i}, \ldots, \sum_i v^i \frac{\partial y^m}{\partial x^i}\right.\right.$$

$$\frac{\partial y^j}{\partial x^i} = \frac{\partial(y^j\circ x^{-1})}{\partial u^i}\circ x$$

Thus as

we have that $T_y \circ T_x^{-1}$ is smooth.

what is this and why $(dx)^{-1} = dx^{-1}$

$$\left[(T_y)\circ(T_x^{-1})\right](u,\vec{v}) = T_y\left(x^{-1}(u),\ (dx)^{-1}(\vec{v})\right)$$
$$= \left(y\circ x^{-1}(u),\ dy\left((dx)^{-1}(\vec{v})\right)\right)$$
$$= \left(y\circ x^{-1}(u),\ dy\left(dx^{-1}(\vec{v})\right)\right)$$
$$= \left((y\circ x^{-1})(u),\ d(y\circ x^{-1})(\vec{V})\right)$$

4(d) If we have chart (U,x) is $U \xrightarrow{x} x(U)$ are they diffeomorphic? are U and $x(U)$ manifolds?

well yes

$$\begin{array}{ccc} U & & x(U) \\ x\downarrow & & \downarrow \text{identity} \\ x(U) & \longrightarrow & x(U) \end{array}$$

$x^{-1}\circ x\circ \text{identity} = \text{identity} \leftarrow$ smooth.

Now then $dx = ?$

$P \in U \rightarrow x(U)$

$V \in T_P U \Rightarrow V = \sum V_x^i \left(\frac{\partial}{\partial x^i}\Big|_P\right)$

$(dx)(V) = W$ $\quad u^i$ varidbles on open subset of $\mathbb{R}^n$.

$W = \sum w^i \frac{\partial}{\partial u^i} = (w^1, w^2, \ldots, w^m)$

$dx(V) = (w^1, w^2, w^3)$

$$\begin{tikzcd} p \in U \arrow[r] \arrow[d, "x"'] & x(U) \arrow[d, "id."] \\ x(U) \arrow[r] & x(U) \end{tikzcd}$$

$$dx(v) = \sum v_x^i (J_{id})^j_i \frac{\partial}{\partial u^j} = \sum v_x^i \frac{\partial}{\partial u^i} = (v_x^1, v_x^2, \ldots, v_x^m)$$

dx is the projection operator on TM it projects tangent vectors down to $\mathbb{R}^m$. Just like x projects elements of the manifold down to $\mathbb{R}^m$.

$$\mathcal{A}_o = \{ (U, x) \in \mathcal{A}_m \mid U \subseteq O \}$$

$(U, x), (V, y) \in \mathcal{A}_m \Rightarrow$ (U, x) and (V, y) are compatible

$$\left.\frac{\partial}{\partial x^i}\right|_p (fg) = \frac{\partial}{\partial u^i}\left((fg)\circ x^{-1}\right)(x(p))$$

$$= \frac{\partial}{\partial u^i}\left[(f\circ x^{-1})(g\circ x^{-1}) \right] x(p)$$

Show X_p an abstract derivation can be written as $X_p = \sum_i X_x^i \left.\frac{\partial}{\partial x^i}\right|_p$

$$f(\ ) = f(p) + \sum h_i(x(\ ))\left[x^i(\ ) - x^i(p)\right]$$

$$X_p(f) = \sum X_p(x^i) \left.\frac{\partial f}{\partial x^i}\right|_p$$

$$X_p = \sum_i X_p(x^i) \left.\frac{\partial}{\partial x^i}\right|_p$$