Review Test I

February 20, 2001

(1) Know the definition of the derivative of a function with domain an open subset of a normed linear space E and range a normed linear space F. Also be able to apply the definition in case E=F=gl(n).

(2) Know the statements of the inverse and implicit function theorems well enough to be able to correctly use them.Be able to identify regions on which equations can be inverted or solved as the case may be. Be able to actually solve systems of equations for simple cases and to relate the result to what the theorems tell you.

(3) Know the proofs of the following statements:
(a) The product of two manifolds is a manifold.
(b) Every derivation X_p can be written in the form

$$X_p = \sum_{i=1}^{m} a^i (\frac{\partial}{\partial x^i}|_p).$$

(c) There is a bijection from the set of all equivalence classes of curves on a manifold onto the set of all derivations on the manifold.
(d) The tangent bundle of a manifold has an atlas of adapted charts.

(4) Be able to do simple problems such as the following:
(a) Show that an open subset of a manifold has an atlas induced from the atlas of the manifold. Here recall that a subset $\mathcal{O}$ of a manifold is open iff each point $p \in \mathcal{O}$ is contained in a chart domain U such that $p \in U \subset \mathcal{O}$.
(b) $\frac{\partial}{\partial x^i}|_p$ is a derivation.
(c) The projections $M \times N \longrightarrow M$ and $M \times N \longrightarrow N$ are smooth mappings.
(d) Every chart (U, x) produces a diffeomorphism from U onto $x(U)$.
(e) The composition of smooth maps is smooth.

$$\begin{tikzcd} U \arrow[d, "x"'] \arrow[r, bend left, "x"] & x(U) \arrow[d, "Id"] \\ x(U) & x(U) \end{tikzcd}$$

$id \circ x \circ x^{-1}$
$x \circ x^{-1} \circ id$

MA: 555 — Manifold Theory — Test One Review

Def^n/ A Linear Space is a Vector Space. Formally it is a set with 2 operations $+$ and $\cdot$, vector addition and scalar multiplication. The vector addition is commutative, distributive and closed and the scalar multipication is distributive and associative w/ respect to $+$.

Def^n/ A normed linear space is a vector space (E) paired with function called the norm; $\|\ \| : E \longrightarrow \mathbb{R}$ such that if $x, y \in E$, $\alpha, \beta \in \mathbb{F}$

1.) $\|x+y\| \leq \|x\| + \|y\|$
2.) $\|cx\| = |c|\,\|x\|$
3.) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = \hat{0}$.

Typical Norms

1.) $\|x\|_2 = \sqrt{\sum_{i=1}^{n} (x^i)^2}$ for $x \in \mathbb{R}^n$

2.) $\|A\|_m = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} (A^i_j)^2}$ for $(A^i_j) \in \mathbb{R}^{m \times n}$ matrices. (Frobenius)

Def^n/ Let E and F be normed vector spaces with $U \subset E$ and $V \subset F$ open then a function $f : U \to V$ is differentiable at $x \in U$ iff $\exists$ a continous linear mapping $D_x f : E \to F$

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - D_x f\|_F}{\|h\|_E} = 0$$

We denote $D_x f = Df(x)$ or on $h \in U$ we may write

$$D_x f(h) = Df(x)(h) = f'(x)h.$$

Theorem

Let $U \subseteq \mathbb{R}^n$ be open, $V \subseteq \mathbb{R}^m$ be open and $f : U \to V$
If f is differentiable at $x \in U$ then $\frac{\partial f^i}{\partial u^i}$ exists at x and

$$Df(x)(h) = h J_f(x)^T$$

THEOREM : IMPLICIT FUNCTION THEOREM

Assume that $U \subseteq \mathbb{R}^{n+m}$ is open and that $F: U \longrightarrow \mathbb{R}^m$ is smooth

If $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$ and the following conditions hold,

1.) $F(a,b) = 0$

2.) $\det\left(\frac{\partial F^i}{\partial y_j}\right)(a,b) \neq 0 \quad \begin{matrix} i = 1,2,\dots,m \\ j = 1,2,\dots n \end{matrix}$

Then $\exists$ V_a open about a in $\mathbb{R}^n$ and W_b open about b in $\mathbb{R}^m$ and a smooth mapping $g: V_a \longrightarrow W_b$ such that

1.) $g(A) = b$

2.) for $(x,y) \in V_a \times W_b$ and $F(x,y) = 0 \iff y = g(x)$.

Def^n/ If M is a set then (U, x) is a chart on M iff $U \subseteq M$ and $x(U)$ is open subset of $\mathbb{R}^m$ then the map X

$$X: U \longrightarrow x(U)$$

Is a bijection. Further $X(U) = \{x(P) \mid P \in U\}$

Def^n/ Two charts (U, x) and (V, y) are compatible iff $U \cap V = \phi$ or $x(U \cap V)$ and $y(U \cap V)$ are open with $y \circ x^{-1}: X(U \cap V) \longrightarrow y(U \cap V)$ a diffeomorphism.

<u>Differentiation facts</u>

$D\varphi^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$id_{\mathbb{R}^n} = (D\varphi) \circ (D\varphi)^{-1} : \mathbb{R}^n \to \mathbb{R}^n$

$id_{\mathbb{R}^m} = (D\varphi^{-1}) \circ (D\varphi) : \mathbb{R}^m \to \mathbb{R}^n \to \mathbb{R}^m$

$$D(\varphi \circ \varphi^{-1}) = D\varphi \circ D\varphi^{-1} = id_{\mathbb{R}^n}$$

$$D(\varphi^{-1} \circ \varphi) = D\varphi^{-1} \circ D\varphi = id_{\mathbb{R}^m}$$

Also if $\begin{matrix} f: U \subseteq \mathbb{R}^k \longrightarrow V \subseteq \mathbb{R}^\ell \\ g: V \subseteq \mathbb{R}^\ell \longrightarrow \mathbb{R}^q \end{matrix}$

$$D(g \circ f) = Dg \circ Df$$

$$J_{g \circ f} = J_g J_f$$

Theorem

If $U \in \mathbb{R}^n$ is open and $V \subseteq \mathbb{R}^m$ is open and $f: U \to V$ then if Df exists $\forall x \in U$ and if Df is continous on U then $\frac{\partial f^j}{\partial u^i}$ all exist and are continous on U conversely.

Fact

If $A, B \in gl(n)$ then $\|AB\| \leq \|A\|\|B\|$

Def^n/ A normed linear space is complete iff every Cauchy sequence in E has a limit in E.

Fact

The series for $e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$ is convergent in $gl(n)$ $\forall A \in g$

Inverse Function Theorem

Let E and F be normed linear spaces (Banach) and let $U \subseteq E$ and $V \subseteq F$ be open. If $f: U \to V$ is a smooth function such that $Df(x_0)$ is invertible for some $x_0 \in U$ then $\exists$ open sets $U_{x_0} \subseteq U$ and $V_{y_0} \subseteq V$, $y_0 = f(x_0)$ such that $f|_{U_{x_0}} : U_{x_0} \to V_{y_0}$ is a diffeomorphism (that is $f|_{U_{x_0}}$ and $(f|_{U_0})^{-1}$ are smooth)

Useful Equivalence

Let $f: \mathbb{R}^m \longrightarrow \mathbb{R}^m$ with $Df(x_0)(h) = h J_f(x_0)^T$.

$$
\begin{aligned}
Df(x_0) \text{ is invertible} &\iff Df(x_0)h = 0 \Rightarrow h = 0 \\
&\iff h J_f(x_0)^T = 0 \Rightarrow h = 0 \\
&\iff \text{nullspace}(J_f(x_0)) = \{0\} \\
&\iff \text{rank}(J_f(x_0)) = m \quad (m = \text{rank}(J_f) + \nu(J_f)) \\
&\iff J_f(x_0) \text{ has an inverse} \\
&\iff \det(J_f(x_0)) \neq 0 .
\end{aligned}
$$

THEOREM

assume $\mathcal{A}$ an atlas on M and $\mathcal{A}^*$ be the maximal atlas containing $\mathcal{A}$ then every atlas which contains $\mathcal{A}$ is contained in $\mathcal{A}^*$

MANIFOLD

If M is a set then a maximal atlas on M is called a differentiable structure. A set with a differentiable structure is a Manifold.

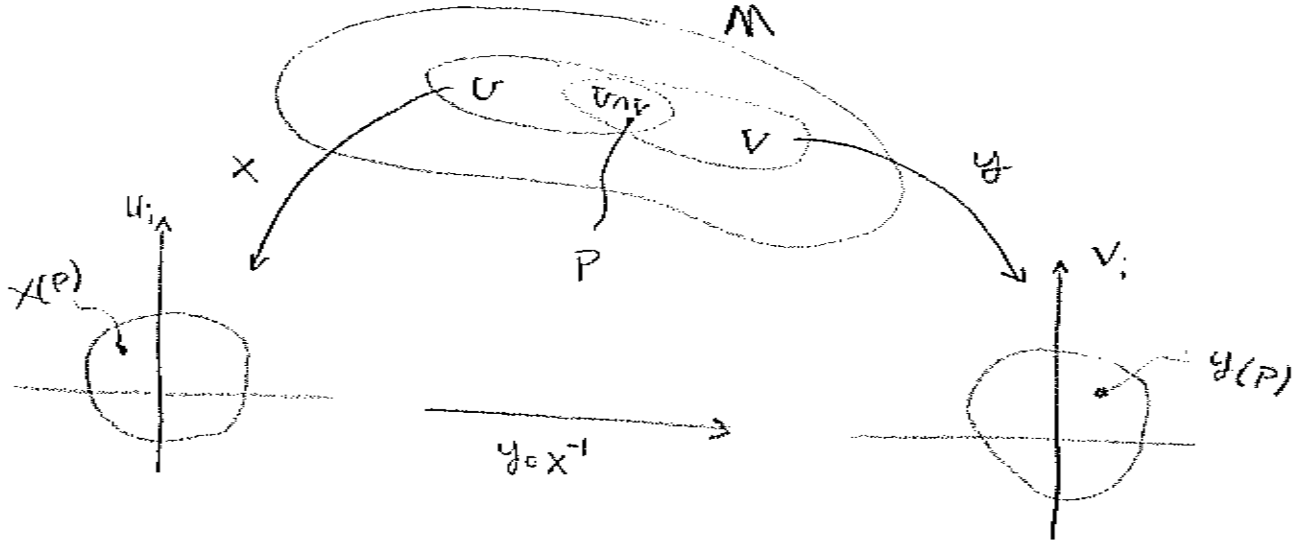
TOPOLOGY ON MANIFOLDS : Def^n of open set

If M is a manifold then $\Theta \subseteq M$ is open iff $\forall p \in \Theta$ $\exists$ chart (U, x) such that

(1.) $p \in U$
(2.) $\exists$ open set $W \subseteq X(U)$ such that $x(p) \in W$ and $p \in X^{-1}(W) \subseteq \Theta$

Better Definition

$\Theta \subseteq M$ is open iff $\forall p \in \Theta$ $\exists$ a chart (U, x) such that $p \in U \subseteq \Theta$.



$$y(P) = (y \circ x^{-1} \circ x)(P) = (y \circ x^{-1})(x(P))$$

$$(y \circ x^{-1})(\bar{u}) = \bar{v} \quad ; \quad (y \circ x^{-1})(u^1, u^2, \ldots, u^m) = (v^1, v^2, \ldots, v^m)$$

Defⁿ/

$$\frac{\partial y^i}{\partial x^j}(P) \equiv \frac{\partial}{\partial u^j}\left(y^i \circ x^{-1}\right)(x(P))$$

Differentiation with respect to a chart defined.

<u>Lemma</u> : If $L : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is linear injection then $m \leq n$

<u>Lemma</u> : If $L : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and $S : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are linear and $S \circ L = id_{\mathbb{R}^m}$ then L is injective

<u>Propostion</u> : Let M be a set with charts (U, x), (V, y) compatible. Then if $x(U) \subseteq \mathbb{R}^m$ and $y(V) \subseteq \mathbb{R}^n$ with $U \cap V \neq \phi$ then $n = m$.

(Def^n of smooth ?)

THEOREM

If M and N are Manifolds and $F: M \to N$ with atlases $\mathcal{A}_M$ & $\mathcal{A}_N$ of M and N respectively. Then F is smooth iff $\forall (U,x) \in \mathcal{A}_M$ and $(V,y) \in \mathcal{A}_N$ we have $y \circ F \circ x^{-1}$ is smooth.

THEOREM

If M, N are manifolds then $F: M \to N$ is smooth iff $\forall$ smooth real valued function y defined on an open subset V of N then $y \circ F$ is smooth. $(y: V \subseteq N \to \mathbb{R})$

THEOREM

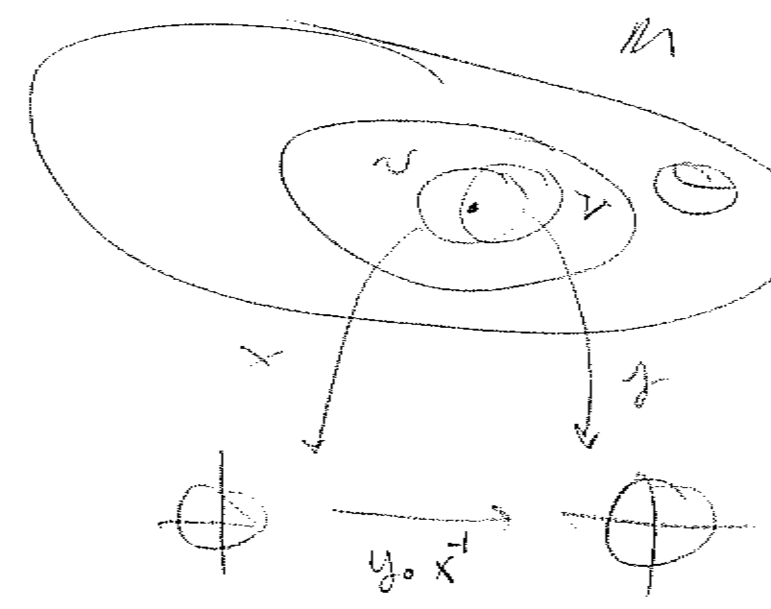
If M is a manifold then M is connected $\Longleftrightarrow$ it is pathwise connected

THEOREM

If $\exists$ a continuos curve from p to q then $\exists$ a smooth curve from p to q in a manifold.

Def^n/ If $\mathcal{M}$ is a manifold, we say that $f \in C^\infty_p \mathcal{M}$ iff f is a smooth function from $\mathcal{M}$ into $\mathbb{R}$ such that $p \in U$

A chart is a bijection that maps onto an open set



the set of all charts whose domain is in Θ

$$\mathcal{A}_\Theta = \{ (U,x) \mid U \subset \Theta, (U,x) \in \mathcal{A}_M \}$$

Product of Two Manifolds

Assume $\mathcal{A}$ is an atlas on $\mathcal{M}$ and $\mathcal{B}$ is an atlas on $\mathcal{N}$. Further if $(U, \mu) \in \mathcal{A}$ and $(V, \nu) \in \mathcal{B}$ then we may define $(U \times V, \mu \times \nu)$ as follows,

$$U \times V = \{ (a,b) \mid a \in U, b \in V \}$$

$$\mu \times \nu : U \times V \to \mu(U) \times \nu(V) \subseteq \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$$

$$(\mu \times \nu)(a,b) \equiv (\mu(a), \nu(b))$$

Composition across Cartesian Products

$$\begin{aligned}(f_1 \times g_1) \circ ((f_2 \times g_2)(u,v)) &\equiv (f_1 \times g_1)(f_2(u) \times g_2(v)) \\ &= f_1(f_2(u)) \times g_1(g_2(v)) \\ &= (f_1 \circ f_2)(u) \times (g_1 \circ g_2)(v) \\ &= (f_1 \circ f_2) \times (g_1 \circ g_2)(u,v)\end{aligned}$$

atlas on $M \times N$

Let $(U, x) \in \mathcal{A}_M$ and $(V, y) \in \mathcal{A}_N$ we propose that $(U \times V, x \times y)$ is a chart on $M \times N$. Note that

$$(x^{-1} \times y^{-1}) \circ (x \times y) = (x^{-1} \circ x) \times (y^{-1} \circ y) = id_U \times id_V$$

$$(x \times y) \circ (x^{-1} \times y^{-1}) = (x \circ x^{-1}) \times (y \circ y^{-1}) = id_{x(U)} \times id_{y(V)}$$

So $x \times y$ is a bijection. Thus $(U \times V, x \times y)$ is a chart on $M \times N$. Further $\mathcal{A}_M \times \mathcal{A}_N$ forms a atlas on $M \times N$

$$\mathcal{A}_M \times \mathcal{A}_N = \{ (U \times V, x \times y) \mid (U, x) \in \mathcal{A}_M \text{ and } (V, y) \in \mathcal{A}_N \}$$

We show $\mathcal{A}_M \times \mathcal{A}_N$ is an atlas, pick two charts

$$(U_1 \times V_1, x_1 \times y_1) \quad \text{and} \quad (U_2 \times V_2, x_2 \times y_2)$$

Then pick $(a,b) \in (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ where both x_1 x_2 y_1 y_2 are defined then

$$\begin{aligned}(x_2 \times y_2) \circ (x_1 \times y_1)^{-1} &= (x_2 \times y_2) \circ (x_1^{-1} \times y_1^{-1}) \\ &= (x_2 \circ x_1^{-1}) \times (y_2 \circ y_1^{-1})\end{aligned}$$

smooth since $x_1, x_2 \in \mathcal{A}_M$ smooth $(y_1, y_2 \in \mathcal{A}_N)$

$$(x_1 \times y_1)[(U_1 \cap U_2) \times (V_1 \cap V_2)] = x_1(U_1 \cap U_2) \times y_1(V_1 \cap V_2)$$

$$(x_2 \times y_2)[U_1 \cap U_2 \times V_1 \cap V_2] = x_2(U_1 \cap U_2) \times y_2(V_1 \cap V_2)$$

Let $f, g \in C_p^\infty M$ we define operations on $C_p^\infty M$ pointwise

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = c\, f(x)$$
$$(fg)(x) = f(x)\, g(x)$$

Where $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ for $f+g$, fg and $x \in \mathrm{dom} f$ for cf.

Def^{n}/ We say that X_p is a derivation at $p \in M$ iff $X_p : C_p^\infty M \longrightarrow \mathbb{R}$ such that

1.) $X_p(f+g) = X_p(f) + X_p(g)$
2.) $X_p(cf) = c\, X_p(f)$
3.) $X_p(fg) = f(p)\, X(g) + g(p)\, X(f)$

Notice derivations act on smooth functions to yield real numbered results.

Def^{n}/ Let (U, x) be a chart on M. We define

$$\left.\frac{\partial}{\partial x^i}\right|_p f \equiv \frac{\partial}{\partial u^i}(f \circ x^{-1})(x(p))$$

Where u^i are coordinates on $\mathbb{R}^n$ and $x(U) \subseteq \mathbb{R}^n$

Also we may write $\left.\frac{\partial}{\partial x^i}\right|_p \equiv \frac{\partial}{\partial x^i}(p) \equiv \partial_i$.

Properties of Function Composition

1.) $(f+g)\circ x^{-1} = f\circ x^{-1} + g\circ x^{-1}$

2.) $(fg)\circ x^{-1} = (f\circ x^{-1})(g\circ x^{-1})$

3.) $(cf)\circ x^{-1} = c(f\circ x^{-1})$

Pf/ 1.) $((f+g)\circ x^{-1})(p) = (f+g)(x^{-1}(p)) = f(x^{-1}(p)) + g(x^{-1}(p))$
$= ((f\circ x^{-1}) + (g\circ x^{-1}))(p) \quad \forall p$

2.) $((fg)\circ x^{-1})(p) = f(x^{-1}(p))\, g(x^{-1}(p)) = (f\circ x^{-1})(g\circ x^{-1})(p) \quad \forall p$

3.) $((cf)\circ x^{-1})(p) = (cf)\circ(x^{-1}(p)) = c f(x^{-1}(p)) = c(f\circ x^{-1})(p) \quad \forall p.$

Prove that $\left.\frac{\partial}{\partial x^i}\right|_p$ is a <u>derivation</u>. Let $f, g \in C^\infty_p M$

$$\left.\frac{\partial}{\partial x^i}\right|_p (f+g) \equiv \frac{\partial}{\partial u^i}\left((f+g)\circ x^{-1}\right)(x(p))$$
$$= \frac{\partial}{\partial u^i}\left(f\circ x^{-1} + g\circ x^{-1}\right)(x(p))$$
$$= \frac{\partial}{\partial u^i}\left(f\circ x^{-1}\right)(x(p)) + \frac{\partial}{\partial u^i}\left(g\circ x^{-1}\right)(x(p))$$
$$= \left.\frac{\partial}{\partial x^i}\right|_p f + \left.\frac{\partial}{\partial x^i}\right|_p g \qquad \therefore \text{ linear}$$

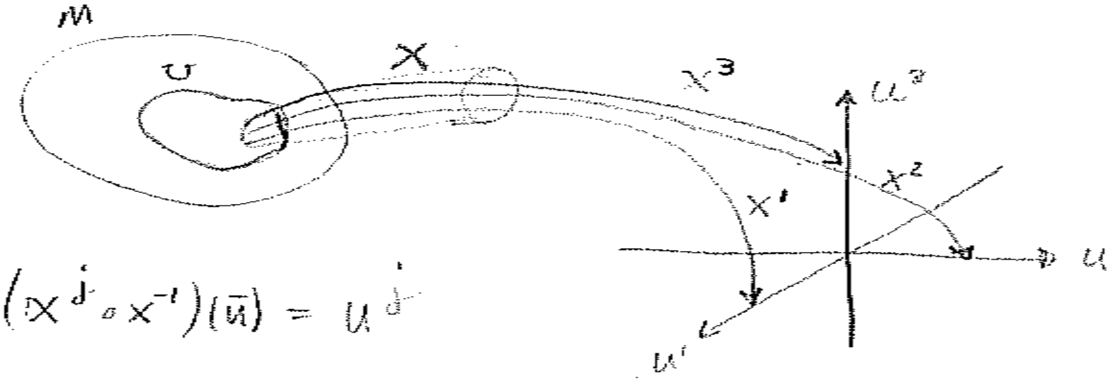
$$\left.\frac{\partial}{\partial x^i}\right|_p (cf) = \frac{\partial}{\partial u^i}\left((cf)\circ x^{-1}\right)(x(p)) = \frac{\partial}{\partial u^i}\left(c\,(f\circ x^{-1})\right)x(p)$$
$$= c\,\frac{\partial}{\partial u^i}\left(f\circ x^{-1}\right)(x(p)) = c\left.\frac{\partial}{\partial x^i}\right|_p f \qquad \therefore \text{ preserves scalar mult}$$

$$\left.\frac{\partial}{\partial x^i}\right|_p (fg) = \frac{\partial}{\partial u^i}\left((fg)\circ x^{-1}\right)(x(p)) = \frac{\partial}{\partial u^i}\left((f\circ x^{-1})(g\circ x^{-1})\right)(x(p))$$
$$= \frac{\partial}{\partial u^i}\left(f\circ x^{-1}\right)(x(p))\,(g\circ x^{-1})(x(p)) + \frac{\partial}{\partial u^i}\left(g\circ x^{-1}\right)(x(p))\,(f\circ x^{-1})(x(p))$$
$$= \left(\left.\frac{\partial}{\partial x^i}\right|_p f\right) g(p) + \left(\left.\frac{\partial}{\partial x^i}\right|_p g\right) f(p) \qquad \therefore \text{ obeys Leibniz rule}$$

Def^{n}/ Relation of chart X and coordinates u on $X(U)$.

$$X(q) = (X^1(q), X^2(q), \ldots, X^m(q))$$

$$X^j \circ X^{-1}(\bar{u}) = u^j \quad \text{where } \bar{u} \in \mathbb{R}^m$$



$$(X^j \circ X^{-1})(\bar{u}) = u^j$$

$$\frac{\partial}{\partial x^i}(p)(X^j) = \frac{\partial}{\partial u^i}\left(X^j \circ X^{-1}\right)(X(p)) = \frac{\partial}{\partial u^i}(u^j) = \delta_i^j$$

Proposition

$\boxed{\text{If } f(x) = c \text{ is constant } \forall x \in U \subseteq M \text{ and } X_p \in T_pM \text{ then } X_p(f) = 0.}$

Pf/ $X_p(f \cdot 1) = 1 \cdot X_p(f) + f(p) X_p(1) = X_p(f) + c X_p(1)$

$= X_p(f) + X_p(c)$

$= X_p(f) + X_p(f) = X_p(f) \therefore X_p(f) = 0$

Proposition:

$\boxed{\text{If } f, g \in C_p^\infty M \text{ and } \exists \text{ open } U \text{ with } p \in U \text{ and } f(x) = g(x) \; \forall x \in U \text{ then } X_p(f) = X_p(g)}$

Pf/ Let $h(x) = f(x) - g(x) = 0 \quad \forall x \in U$ then $h(x)$ is constant on U thus $X_p(h) = 0$

$X_p(h) = X_p(f - g) = 0$

$\therefore X_p(f) = X_p(g)$

THEOREM

For $f \in C_p^\infty M$ and $\mathbb{X}_p$ a derivation of $C_p^\infty M$ then $\exists\ (U,x) \in \mathcal{Q}_m$ with $p \in U$ so that

$$\mathbb{X}_p(f) = \sum_{i=1}^{m} \mathbb{X}_p(x^i) \frac{\partial}{\partial x^i}\Big|_p (f)$$

$$\mathbb{X}_p = \sum_{i=1}^{m} \mathbb{X}_p(x^i) \frac{\partial}{\partial x^i}\Big|_p$$

Changing Charts

Let $(U,x), (V,y) \in \mathcal{Q}_m$ with U and V containing p

If $\mathbb{X}_p$ is a derivation at p on T_pM then

$$\mathbb{X}_p = \sum_{i=1}^{m} \mathbb{X}_p(x^i) \frac{\partial}{\partial x^i}\Big|_p$$

$$\mathbb{X}_p = \sum_{i=1}^{m} \mathbb{X}_p(y^i) \frac{\partial}{\partial y^i}\Big|_p$$

Notice that $y^j \in C_p^\infty M$ thus $\mathbb{X}_p$ may act on it,

$$\begin{aligned}
\mathbb{X}_p(y^j) &= \sum_{i=1}^{m} \mathbb{X}_p(x^i) \frac{\partial}{\partial x^i}\Big|_p y^j \\
&= \sum_{i=1}^{m} \mathbb{X}_p(x^i) \frac{\partial y^j}{\partial x^i}(p) \\
&= \sum_{i=1}^{m} \mathbb{X}_p(x^i) \frac{\partial}{\partial u^i}(y^j \circ x^{-1})(x(p)) \\
&= \sum_{i}^{m} \mathbb{X}_p(x^i) \left(J_{y \circ x^{-1}}(x(p))\right)_i^j
\end{aligned}$$

Tangent Structures on M

$$G_p = \{ \gamma \mid \exists a > 0 \text{ where } \gamma : (-a,a) \to M \text{ is smooth}, \ \gamma(0) = p \}$$
$$D_p = \{ X_p \mid X_p \text{ is a derivation of } C_p^\infty M \}$$

Let $\Delta : G_p \longrightarrow D_p$ by $\Delta_x(\gamma) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(0) \left(\frac{\partial}{\partial x^i}\Big|_p \right)$

We show that Δ def^n is chart indep., $\Delta_x(\gamma) = \Delta_y(\gamma)$.

$$\begin{aligned}
\Delta_x(\gamma) &= \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(0) \frac{\partial}{\partial x^i}\Big|_p \\
&= \sum_i \sum_j \frac{d(y^j \circ \gamma)}{dt}(0) \frac{d(x^i \circ y^{-1})}{dv^j} \frac{\partial}{\partial x^i}\Big|_p \\
&= \sum_i \sum_j (y^j \circ \gamma)'(0) \frac{dx}{dy^j} \frac{\partial}{\partial x^i}\Big|_p \\
&= \sum_j (y^j \circ \gamma)'(0) \frac{\partial}{\partial y^j}\Big|_p \\
&= \Delta_y(\gamma)
\end{aligned}$$

Lemma: $\sum_{i,j} \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}\Big|_p = \sum_j \frac{\partial}{\partial y^j}\Big|_p \longrightarrow \sum_j \frac{\partial (f \circ y^{-1})}{\partial v^j}$

$$\begin{aligned}
\sum_{i,j} \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}\Big|_p f &= \sum_{i,j} \frac{\partial (x^i \circ y^{-1})}{\partial v^j} \frac{\partial (f \circ x^{-1})}{\partial u^i} \\
&= \sum_j \frac{\partial (f \circ y^{-1})}{\partial v^j}
\end{aligned}$$

$$(f \circ x^{-1})(x(p)) = (f \circ x^{-1})(u^1, u^2, \ldots, u^n)$$

So clearly Δ is chart independent

$$\frac{d}{dt}(x^i \circ \gamma)(0) = (x^i \circ \gamma)'(0) = X_{\gamma(0)}(x^i)$$

$$\Delta_x(\gamma) = \sum_{i=1}^{n} (x^i \circ \gamma)'(0) \frac{\partial}{\partial x^i}\Big|_{\gamma(0)}$$

Let $\Xi_p \in \mathcal{D}_p$ then we see γ maps to it.
Show that $\Delta : \mathcal{C}_p \to \mathcal{D}_p$ is onto. define $\gamma \in \mathcal{C}_p$ by

$$\gamma(t) \equiv x^{-1}(x(p) + t\vec{\Xi}_x)$$

Then using $x \circ x^{-1} =$ identity

$$x \circ \gamma(t) = x(p) + t(\Xi_x^1, \Xi_x^2, \ldots, \Xi_x^m)$$

$$\frac{d}{dt}(x \circ \gamma(t)) = (\Xi_x^1, \Xi_x^2, \ldots, \Xi_x^m)$$

$$\therefore \quad \Delta(\gamma) = \sum_{i=1}^{m} \frac{d(x^i \circ \gamma)}{dt}(0) \frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^{m} \Xi_x^i \frac{\partial}{\partial x^i}\Big|_p = \Xi_p$$

thus Δ is onto.

Now if γ_1 and $\gamma_2 \in \mathcal{C}_p$ with $\Delta(\gamma_1) = \Delta(\gamma_2)$ then

$$\sum_i (x^i \circ \gamma_1)'(0)\, \partial_i = \sum_i (x^i \circ \gamma_2)'(0)\, \partial_i$$

Act on x^j to find

$$\sum_i (x^i \circ \gamma_1)'(0)\, \partial_i x^j = \sum_i (x^i \circ \gamma_2)'(0)\, \partial_i x^j$$

$$(x^j \circ \gamma_1)'(0) = (x^j \circ \gamma_2)'(0) \qquad \forall j$$

Also from def$^{\underline{n}}$ of $\mathcal{C}_p$ we have that $\gamma_1(0) = \gamma_2(0) = P$.
So we say that $\gamma_1 \sim \gamma_2$ aka $[\gamma_1] = [\gamma_2]$ thus
We define $\tilde{\Delta} : \mathcal{C}_p/\sim \;\longrightarrow \mathcal{D}_p$ by $\tilde{\Delta}([\gamma]) = \Delta(\gamma)$
which is cleary 1-1 and onto. Thus $\mathcal{C}_p/\sim$ and
$\mathcal{D}_p$ are formally equivalent.

Equivalent Elements of T_pM.

$$\gamma'(0) = X_p = \sum_i \frac{d(x^i\circ\gamma)}{dt}(0)\left(\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}\right)$$

Let $F: M \longrightarrow N$ then we define the differential of F at p by $d_pF : T_pM \longrightarrow T_pN$ also denoted $F_*(p)$ by,

$$d_pF(X_p) = Y_p \iff D_{x(p)}(y\circ F\circ x^{-1})(X_x^1,\dots,X_x^n) = (Y_y^1,\dots,Y_y^m)$$

Equivalently

$$(Y_y^1, Y_y^2,\dots, Y_y^m) = (X_x^1, X_x^2,\dots, X_x^n)\, J_{y\circ F\circ x^{-1}}(x(p))^T$$

$$d_pF\left(\sum X_x^i \frac{\partial}{\partial x^i}\Big|_p\right) = \sum_j \sum_i X_x^i \left[J_{y\circ F\circ x^{-1}}(x(p))\right]^i_j \frac{\partial}{\partial y^j}\Big|_{F(p)}$$

$$dF\left(\sum X^i \frac{\partial}{\partial x^i}\right) = \sum_{i,j} X^j \frac{\partial(y^i\circ F)}{\partial x^j}\frac{\partial}{\partial y^i}$$

$$d_pF(\gamma'(0)) = \sum_{i,j}\frac{d(x^j\circ\gamma)}{dt}(0)\,\frac{\partial(y^i\circ F)}{\partial x^j}\frac{\partial}{\partial y^i}$$

$$= \sum_i \frac{d}{dt}(y^i\circ F\circ\gamma)(0)\frac{\partial}{\partial y^i}\Big|_{F(p)}$$

Which is Δ equivalent to

$$d_pF(\gamma'(0)) = (F\circ\gamma)'(0)$$

$$d_pF([\gamma]) = [F\circ\gamma]$$

What does $\gamma'(0)f$ mean? Let $f \in C^\infty_p M$.

$$\gamma'(t)f = \sum_i \frac{d(x^i \circ \gamma)}{dt}(t) \frac{\partial f}{\partial x^i}(\gamma(t)) = \frac{d(f \circ \gamma)}{dt}(t)$$

Since

$$\frac{d}{dt}(f\circ\gamma)(t) = \frac{d}{dt}\left((f\circ x^{-1})\circ(x\circ\gamma)\right)(t)$$

$$= \sum_i \frac{\partial(f\circ x^{-1})}{\partial u^i} \frac{d}{dt}\left(x^i\circ\gamma\right)(t) \quad \text{chain rule!}$$

$$= \sum_i \frac{d(x^i\circ\gamma)}{dt}(t) \frac{\partial f}{\partial x^i}(\gamma(t))$$

$$\boxed{\gamma'(t)f = (f\circ\gamma)'(t)}$$

Let $f : M \to \mathbb{R}^n$ then $d_pf : T_pM \to T_p\mathbb{R}^n \approx \mathbb{R}^n$

$$\frac{\partial}{\partial u^i} \longrightarrow e_i$$

$$d_pf\left(\sum_i X^i_x \left(\frac{\partial}{\partial x^i}\Big|_p\right)\right) = \sum_i \sum_j X^j_x \frac{\partial u^i \circ f}{\partial x^j} \frac{\partial}{\partial u^i}$$

$$= \sum_i \sum_j X^j_x \frac{\partial(u^i\circ f)}{\partial x^j} e_i$$

$$= \sum_j X^j_x \frac{\partial f^i}{\partial x^j} e_i$$

$$= \sum_j \left(X^j_x \frac{\partial f^1}{\partial x^j}, X^j_x \frac{\partial f^2}{\partial x^j}, \dots\right)$$

Take the case where $n=1$ to find

$$\boxed{d_pf(X_p) = X_p(f)}$$

$$\gamma'(t) \longleftrightarrow \sum_i (x^i\circ\gamma)'(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

Atlas on Tangent Bundle

$$\mathcal{a}_{TM} = \{ (TU, T_x) \mid (U, x) \in \mathcal{a}_M \}$$

We begin by verifying that T_x is a bijection onto open sets, that is that T_x is a chart.

$\text{Image}(T_x) = x(U) \times \mathbb{R}^n$ which is open as $x(U)$ is open

Now $T_x(P, V) = (x(P), d_P x(V))$ is 1-1 let $T_x(P,V) = T_x(\tilde{P}, \tilde{V})$

$$T_x(P,V) = (x(P), (V^1, V^2, \dots, V^n)) = (x(\tilde{P}), (\tilde{V}^1, \tilde{V}^2, \dots, \tilde{V}^n)) = T_x(\tilde{P}, \tilde{V})$$

$\therefore$ $x(P) = x(\tilde{P}) \Rightarrow P = \tilde{P}$ as x is 1-1.

$$d_P x(V) = d_P x(\tilde{V})$$

$$V = \sum_i V^i \frac{\partial}{\partial x^i}\Big|_P$$

$$\tilde{V} = \sum_i \tilde{V}^i \frac{\partial}{\partial x^i}\Big|_P$$

$$\sum_i V^i d_P x\left(\frac{\partial}{\partial x^i}\Big|_P\right) = \sum_i \tilde{V}^i d_P x\left(\frac{\partial}{\partial x^i}\Big|_P\right)$$

$$(V^1, V^2, \dots, V^n) = (\tilde{V}^1, \tilde{V}^2, \dots, \tilde{V}^n) \quad \therefore V = \tilde{V}$$

$$\therefore (P, V) = (\tilde{P}, \tilde{V})$$

So T_x is a chart.

Tangent Bundle

$$TM = \{(p,v) \mid p \in M \text{ and } v \in T_pM\}$$

Let $(U,x) \in \mathcal{A}_M$ we define a chart on TM where $TU = \{(p,v) \mid p \in U,\ v \in T_pU = T_pM\}$ and

$$T_x : TU \longrightarrow x(U) \times \mathbb{R}^m \subseteq \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$$
$$(T_x)(p,v) = (x(p), (d_px^1(v), d_px^2(v), \dots, d_px^m(v))) \in \mathbb{R}^{2m}$$

Other ways to express. Since $v \in T_pM \Rightarrow v = \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p$
also d_px^j is linear map thus

$$d_px^j\Big(\sum_i v^i \frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_i v^i d_px^j\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p (x^j)$$

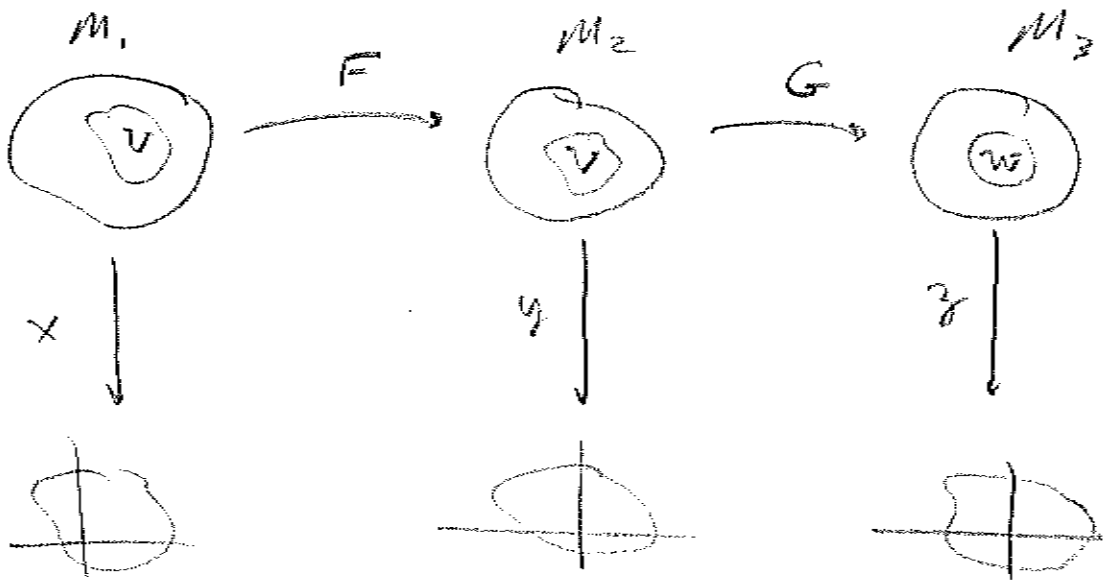
$$d_px^j(v) = \sum_i v^i \delta_i^j = v^j$$

Thus $(d_px^1(v), \dots, d_px^m(v)) = (v^1, v^2, \dots, v^m)$ thus

$$\boxed{(T_x)(p,v) = (x(p), (v^1, v^2, \dots, v^m))}$$

Also if we have $f : M \to \mathbb{R}^n \Rightarrow d_pf(v) = \sum_{i,j} \frac{\partial f^j}{\partial x^i}(p) v^i e$

$$\boxed{(T_x)(p,v) = (x(p), d_px(v))}$$



Given F and G are smooth.

$y \circ F \circ x^{-1}$ is smooth

$z \circ G \circ y^{-1}$ is smooth

We want to show $F \circ G : M_1 \longrightarrow M_3$ smooth

$z \circ G \circ F \circ x^{-1}$ smooth

$\underbrace{(z \circ G \circ y^{-1})}_{\substack{\text{smooth} \\ \text{on } \mathbb{R}^n}} \circ \underbrace{(y \circ F \circ x^{-1})}_{\substack{\text{smooth} \\ \text{on } \mathbb{R}^m}} \implies$ smooth

$\therefore$ $F \circ G$ smooth

Py #6

$$\pi : M \times N \to M, \quad \pi(m,n) = m$$

For $(m,n) \in (M \times N)$ take $(U \times V, x \times y) \in \mathcal{Q}_{M \times N}$ about (m,n). Note $(U, x) \in \mathcal{Q}_M$ about m.

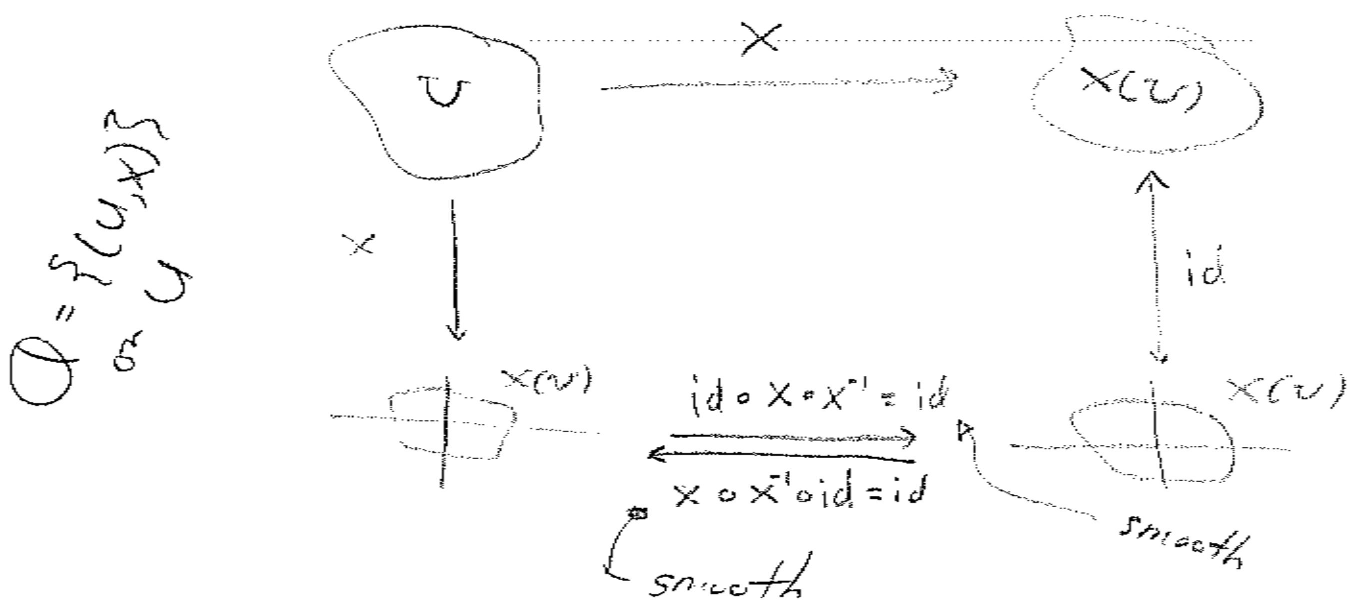
$\left(x \circ \pi \circ (x \times y)^{-1}\right)(a,b)$ where $(a,b) \in x(U) \times y(V)$

$\Rightarrow \exists\ u \in U$ and $v \in V$ such that $(x \times y)(u,v) = (a,b)$

$$\begin{aligned}
\left(x \circ \pi \circ (x \times y)^{-1}\right)(a,b) &= x \circ \pi \circ (x \times y)^{-1} \circ (x \times y)(u,v) \\
&= x \circ \pi \circ (u,v) \\
&= x(u) \\
&= a \quad \text{this is smooth.} \\
&= (a^1, a^2, \ldots, a^m)
\end{aligned}$$

$(a^1, \ldots, a^m, b^1, \ldots, b^n) \longrightarrow (a^1, a^2, \ldots, a^m)$ by clearly

$\therefore\ \pi$ is smooth



$\therefore\ x$ is a bijection which is smooth with smooth inverse.

Let $\mathcal{A}_{TM} = \{ (TU, T_x) \mid (U, x) \in \mathcal{A}_M \}$ then T_x is a chart since it is a bijection onto an open set. Now we check compatibility.

Let $(U, x), (V, y) \in \mathcal{A}_M$ with $U \cap V \neq \phi$. Then $(TU, T_x), (TV, T_y) \in \mathcal{A}_{TM}$ such that if $q \in U$, $v \in T_q U$

$$(T_x)(q, v) \equiv (x(q), (v_x^1, v_x^2, \ldots, v_x^m))$$

$$(T_x)^{-1}(u, \vec{v}) = (T_x)^{-1}(u, v^i e_i)$$
$$\equiv (x^{-1}(u), \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p)$$

? — Def$^{\underline{n}}$ — ?

$$(T_x)(T_x)^{-1}(u, \vec{v}) = T_x (x^{-1}(u), \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p)$$
$$= (x(x^{-1}(u)), d_p x (\sum_i v^i \frac{\partial}{\partial x^i}\Big|_p))$$
$$= (u, v^i e_i)$$
$$= (u, \vec{v})$$

$$(T_x)^{-1}(T_x)(q, v) = (T_x)^{-1}(x(q), (v_x^1, v_x^2, \ldots, v_x^m))$$
$$= (x^{-1}(x(q)), \sum_i v_x^i \frac{\partial}{\partial x^i})$$
$$= (q, v)$$

$\therefore$ T_x is bijection

Compatibility? $u \in U \cap V$ and $\vec{v} \in T_u U = T_u V = T_u M$.

$$(T_y)(T_x)^{-1}(u, \vec{v}) = T_y (x^{-1}(u), \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p)$$
$$= (y \circ x^{-1}(u), dy (\sum_i v^i \frac{\partial}{\partial x^i}\Big|_p))$$
$$= ((y \circ x^{-1})(u), \sum_i v^i \, dy (\frac{\partial}{\partial x^i}\Big|_p))$$
$$= ((y \circ x^{-1})(u), \sum_i v^i \frac{\partial y}{\partial x^i}\Big|_p)$$

$$\sum_i v^i \frac{\partial y}{\partial x^i}\Big|_p = \sum_i v^i \frac{\partial}{\partial u^i}(y \circ x^{-1})(x(p)) \leftarrow \Sigma \text{ of smooth fnctn}$$

$$\therefore (T_y)(T_x)^{-1}(u, \vec{v}) = (\text{smooth}, \text{smooth}) = \text{smooth}$$

$\therefore$ T_y and T_x are compatible.