

Definition: A smooth manifold of dimension  $m$ ,  $M$  is manifold giving the set  
 $M$  is a set, w/ a collection of open sets  $U_i \subseteq \mathbb{R}^n$   
 and a collection of mappings  $\phi_i : U_i \subseteq \mathbb{R}^m \rightarrow V_i \subseteq M$   
 and these satisfy:

smooth mapping on  $\mathbb{R}^m$   
 $\Rightarrow$  continuous partials of all orders

- 1) each  $\phi_i : U_i \rightarrow V_i$  is injective.
- 2) if  $V_i \cap V_j \neq \emptyset$  then:  $\exists$  a <sup>( $C^\infty$ )</sup> smooth  $\theta_{ij}$ .

$$\theta_{ij} : \phi_j^{-1}(V_i \cap V_j) \rightarrow \phi_i^{-1}(V_i \cap V_j)$$

$$\text{s.t. } \phi_j = \phi_i \circ \theta_{ij}$$

$$3) M = \bigcup_{i \in I} \phi_i(U_i) = \bigcup_{i \in I} V_i$$

We call  $\phi_i$  local parametrizations or patches of  $M$   
 and  $U_i$  is parameter space. The range  $V_i$  together  
 w/  $\phi_i^{-1}$  is called coordinate chart on  $M$

The component functions of  $(V, \phi^{-1})$  are usually  
 denoted  $\phi^{-1} = (x^1, x^2, \dots, x^m)$  where  $x^j : V \rightarrow \mathbb{R}$   
 for  $j = 1, 2, \dots, m$

(Another way)

- 1)  $x_i : V_i \rightarrow U_i$  injective
- 2) if  $V_i \cap V_j \neq \emptyset$  then  $\exists \theta_{ij}$  s.t.  
 $\theta_{ij} : x_j(V_i \cap V_j) \rightarrow x_i(V_i \cap V_j)$  s.t.  
 $\theta_{ij} = \underbrace{x_i \circ x_j^{-1}}$   
 transition functions of manifold.

Defn:  $M$  together w/  $\{(V_i, \chi_i) \mid i \in \Lambda\} = \mathcal{A}$   
 is a manifold w/ atlas  $\mathcal{A}$ .

Theorem:  $\exists!$  maximal atlas  $\mathcal{A}_{\max}$  which contains  $\mathcal{A} \subseteq \mathcal{A}$ ,  
 $\mathcal{A}_{\max} = \{(\tilde{V}, \tilde{\chi}) \mid (\tilde{V}, \tilde{\chi}) \text{ compatible w/ all } (V_i, \chi_i) \in \mathcal{A}\}$

Example:

$$M = \mathbb{R}^n$$

$$\chi(p) = p \quad \forall p \in \mathbb{R}^n$$

$$U = \mathbb{R}^n, \quad V = \mathbb{R}^n$$

$M = \mathbb{R}^n$  is a manifold w/ global coordinate chart

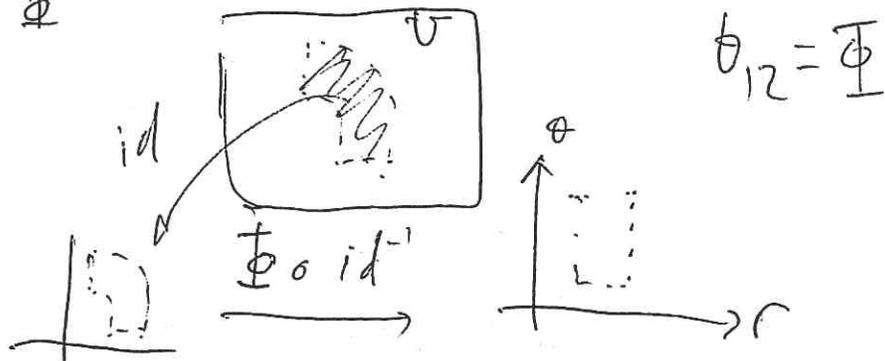
$\text{id}_{\mathbb{R}^n}$

$$\mathcal{A} = \{(\mathbb{R}^n, \text{id})\}$$

$(U, \text{id}|_U)$  for  $U$  open connected in  $\mathbb{R}^n$

Ex:  $(U, (r, \theta))$  for  $U = \{p \in \mathbb{R}^2 \mid 1 < r < 2, 0 < \theta < \frac{\pi}{2}\}$

where  $\underbrace{(r, \theta)}_{\Phi}(\varphi) = \left( \|\varphi\|, \tan^{-1}\left(\frac{\varphi_2}{\varphi_1}\right) \right)$



\*  $(M, \mathcal{A}_{\max})$  is called a differentiable structure.

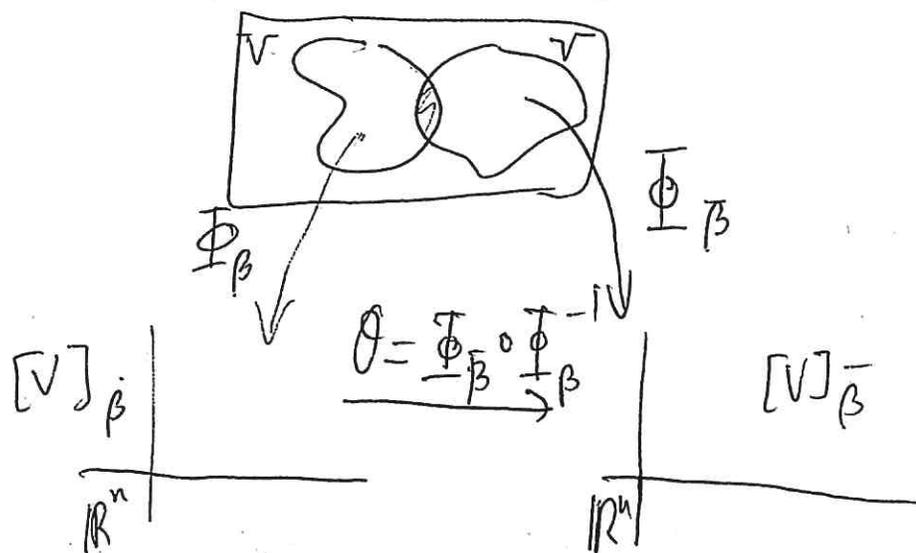
Ex:  $V$  is a vector space w/  $\beta = \{f_1, f_2, \dots, f_n\}$   
 $\bar{\beta} = \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n\}$

$$\Phi_{\beta}(v) = [v]_{\beta}$$

$$\Phi_{\bar{\beta}}(v) = [v]_{\bar{\beta}}$$

paired w/  $\{(V, \Phi_{\beta}), (V, \Phi_{\bar{\beta}})\}$

Conditions 1 & 3 are clearly met



$$\theta(x) = Ax \quad \forall x \in \mathbb{R}^n$$

$$\theta'(x) = A$$

Ex:  $M = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

1)  $V_+ = \{(x, y) \in M \mid y > 0\} = \text{dom}(\chi_+)$

$$\chi_+(x, y) = x$$

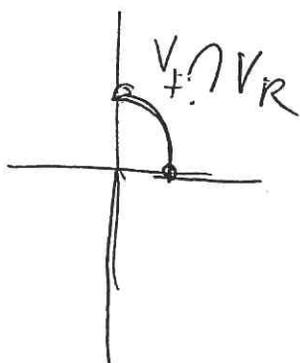
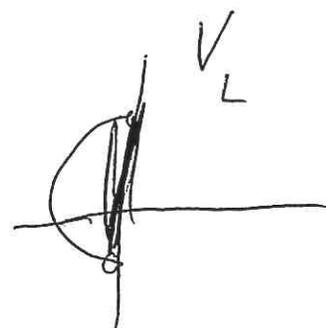
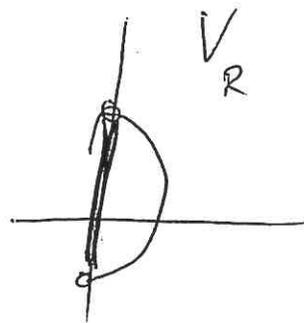
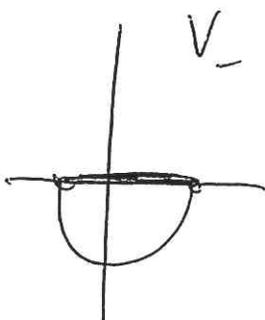
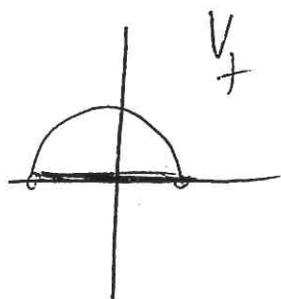
2)  $V_- = \{(x, y) \in M \mid y < 0\} \Rightarrow \chi_-(x, y) = x$

$$3) V_R = \{(x,y) \in M \mid x > 0\} \quad \chi_R(x,y) = y$$

$$4) V_L = \{(x,y) \in M \mid x < 0\} \quad \chi_L(x,y) = y$$

$$A = \{(V_+, \chi_+), (V_-, \chi_-), (V_L, \chi_L), (V_R, \chi_R)\}$$

forms an atlas



$$\chi_R \circ \chi_+^{-1}(t) = \sqrt{1-t^2} \text{ smooth as } 0 < t < 1$$

Construction: If  $M$  &  $N$  are manifolds w/ charts  $(\bar{U}_i, \chi_i)$  for  $M$  and  $(\bar{V}_j, \bar{\chi}_j)$  for  $N$ .

Then  $M \times N$  is a manifold w/ charts

$$\chi_i \times \bar{\chi}_j : \bar{U}_i \times \bar{V}_j \rightarrow V_i \times \bar{V}_j \quad \forall i,j$$

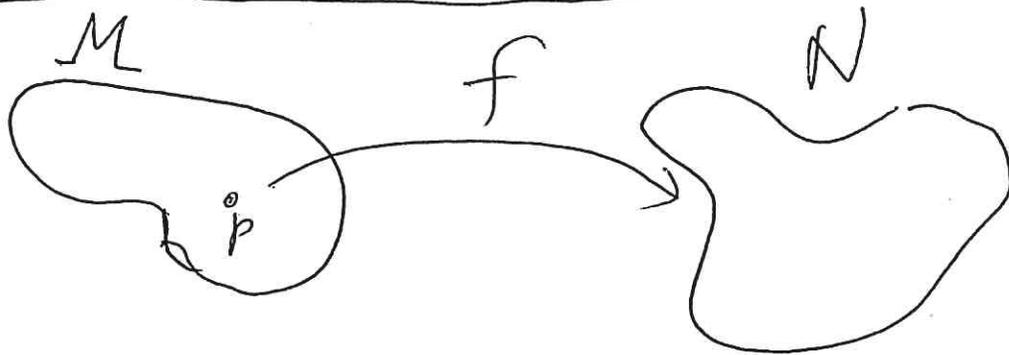
Ex. Could also be seen as using  $M$  from previous Ex and  $N = \mathbb{R}$  paired w/ id chart.

$$M \times N = \{ (x, y, z) \mid x^2 + y^2 = 1, z \in \mathbb{R} \}$$

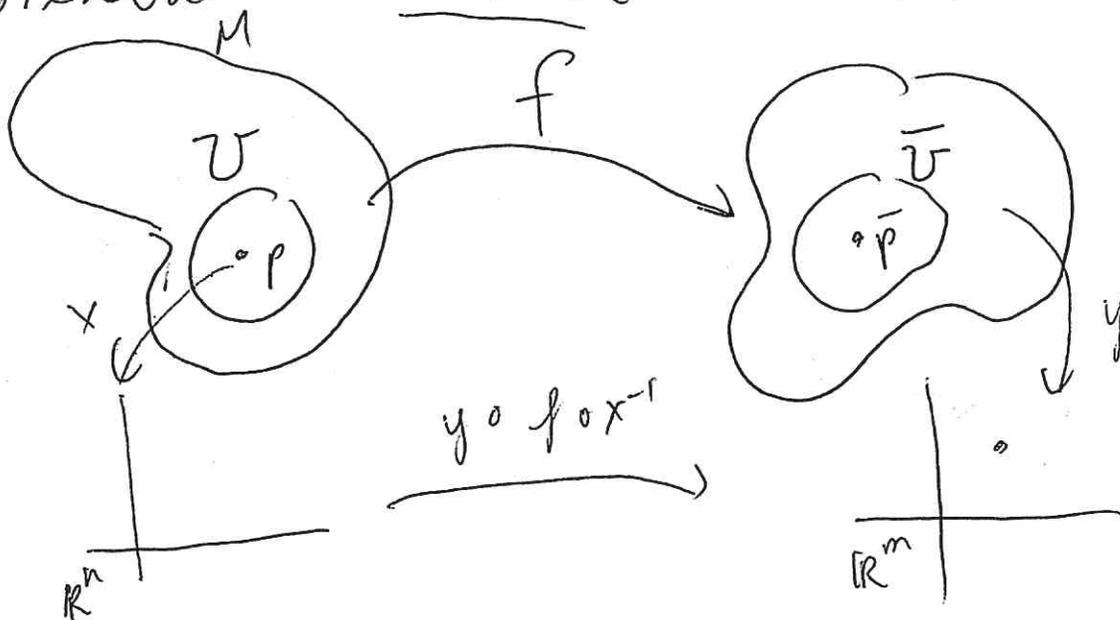
Charts on  $M \times N$

$$\mathcal{X}_L \times \mathbb{Z}, \quad \mathcal{X}_{L,R} \times \mathbb{Z}$$

### Smooth maps between Manifolds



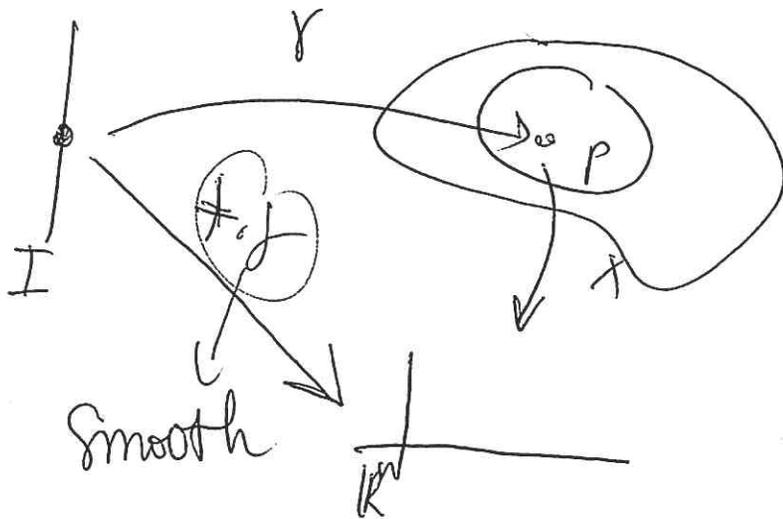
Def:  $f$  is smooth at  $p$  iff  $\forall$  charts containing  $p$  and all charts containing  $f(p)$  we have the local coordinate representative is smooth (as a mapping from  $\mathbb{R}^m \rightarrow \mathbb{R}^n$ )



# Tangent Space & Manifold (define is to define $T_p M$ )

## • Equivalence classes of Curves

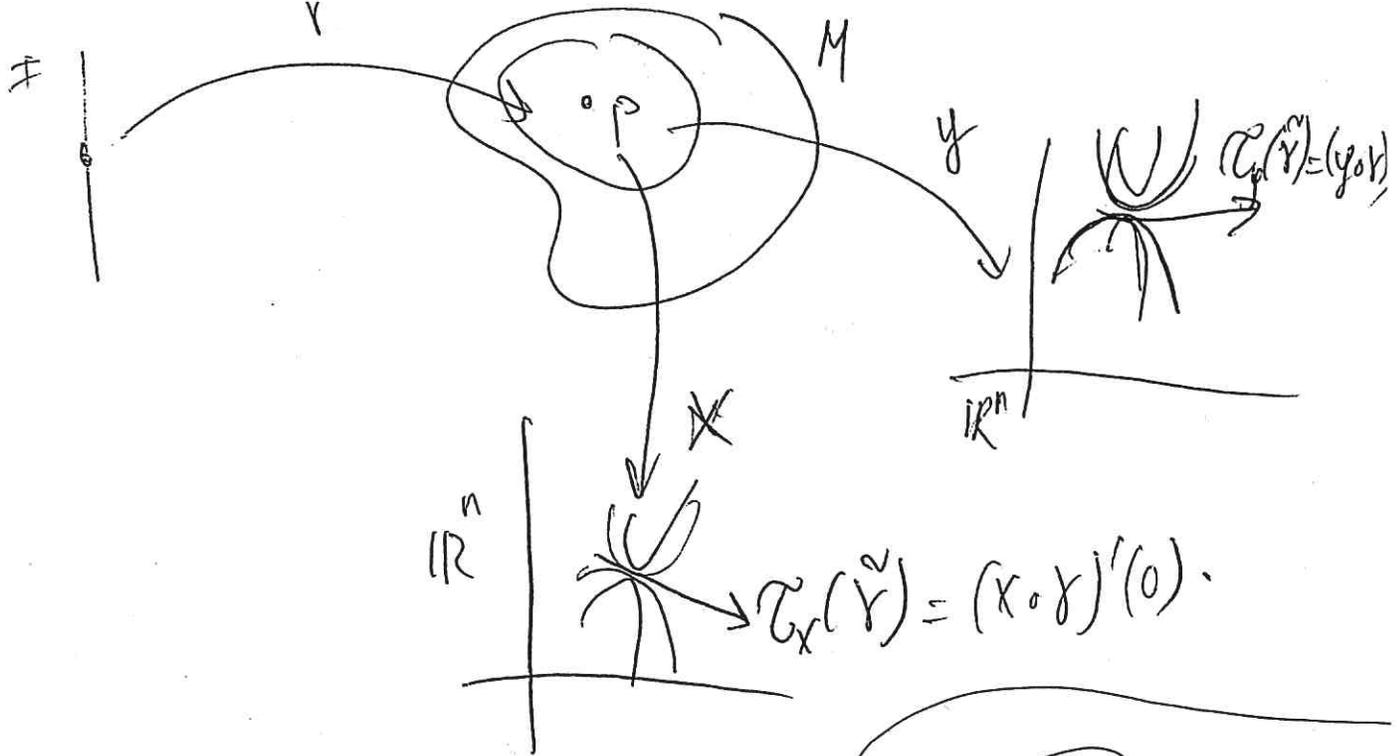
Let  $\gamma: I \rightarrow M$  w/  $\gamma(0) = p$ .  
and  $\gamma$  smooth



Definition: Given a coordinate system  $\chi$  containing  $p \in M$   
we say curves  $\gamma_1, \gamma_2: I \rightarrow M$  are equivalent.  
iff  $(\chi \circ \gamma_1)'(0) = (\chi \circ \gamma_2)'(0)$

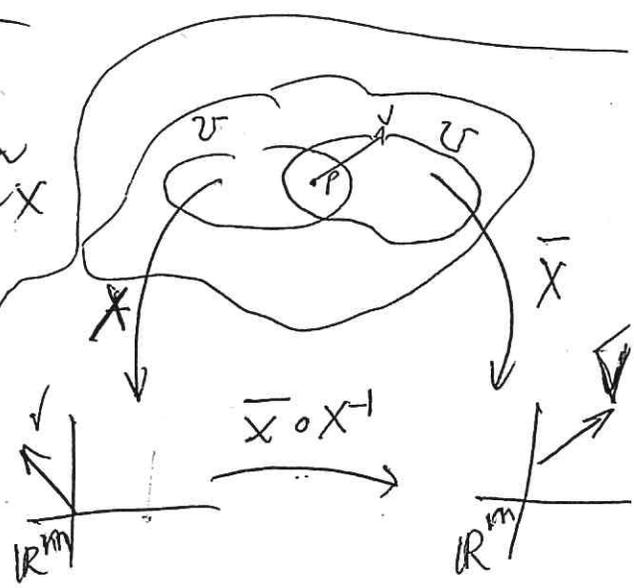
$$[\gamma] = \tilde{\gamma} = \{ \beta: I \rightarrow M \mid \beta \sim_p \gamma \} \xrightarrow{\tilde{\chi}} \underbrace{(\chi \circ \gamma)'(0)}_{\text{in } \mathbb{R}^n}$$

$$T_p^{-1} = \{ \beta: I \rightarrow M \mid \beta \sim_p \gamma \text{ w/ } \gamma(t) = p + tv \}$$



$$T_y = (y \circ X^{-1})^{-1} \circ T_x$$

at  $X(p)$



2nd view point: for tangent space

$$\text{vect}(T_p M) = \{ (p, v) \mid v \in \mathbb{R}^n \}$$

Relative to coordinate system  $X$  at  $p$ .

$$\left. \begin{aligned} (p, v_1 + v_2) &= (p, v_1) + (p, v_2) \\ c(p, v_1) &= c(p, cv_1) \end{aligned} \right\} \text{ give } \text{vect}(T_p M) \text{ a vector space structure}$$

If we change the  $\bar{x}$  coordinate then likewise

$$(p, \bar{v} + \bar{w}) = (p, \bar{v}) + (p, \bar{w}) \text{ etc...}$$

$$\bar{v} = P v \text{ where } P = (\bar{X} \circ X^{-1})'(x(p))$$

Really,  $\text{vect}(T_p M) = \{ (p, v, x) \mid p \in M, v \in \mathbb{R}^n, x \in \mathcal{A}_m \}$

where  $(p, v, x) \sim (q, w, y)$

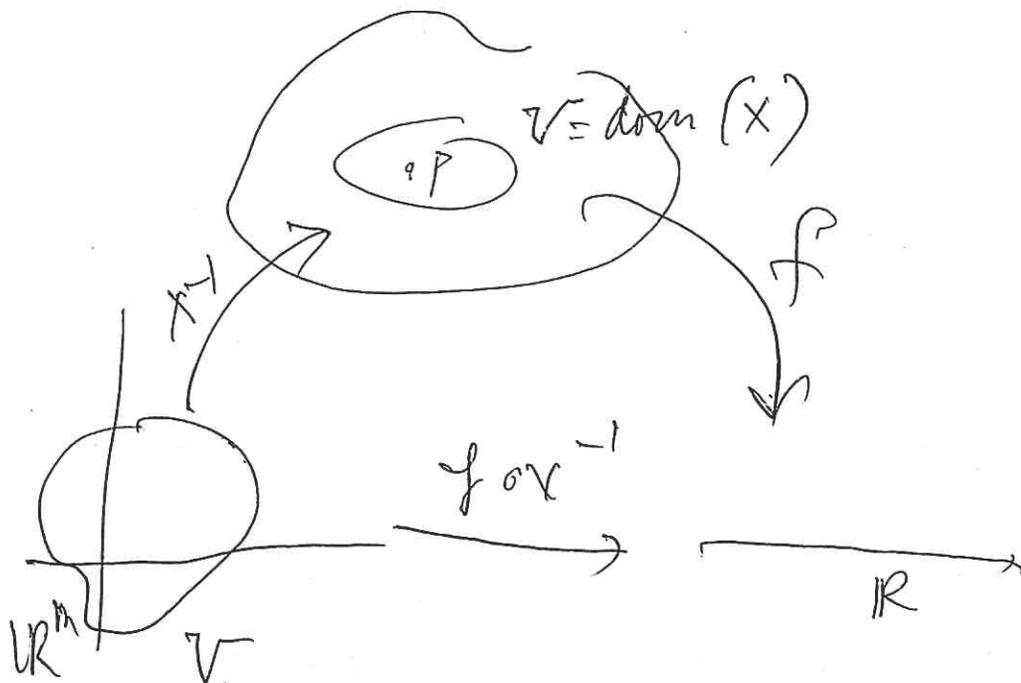
$p = q$  and  $w = (y \circ x^{-1})'(x(p)) v$

3rd viewpoint

Derivation on smooth functions at  $\varphi$ .

$C^\infty(p) = \{ f: M \rightarrow \mathbb{R} \mid f \text{ is smooth on open set containing } p \}$

$\Rightarrow \exists x \in \mathcal{A}_m$  s.t.  $f \circ x^{-1}$  is smooth.



Def<sup>n</sup>: Suppose  $\Sigma_p : C^\infty(p) \rightarrow \mathbb{R}$  is linear transformation which satisfies the Leibniz Rule then  $\Sigma_p$  is a derivation on  $C^\infty(p)$  and we write

$$\Sigma_p \in D_p M = \text{Der}_p(\mathbb{R} \langle C^\infty(p) \rangle)$$

In particular;

$$\Sigma_p(f + cg) = \Sigma_p(f) + c\Sigma_p(g)$$

$$\Sigma_p(fg) = \Sigma_p(f)g(p) + f(p)\Sigma_p(g)$$

~~z~~

Ex: Let  $M = \mathbb{R}$  and Consider  $\Sigma_{t_0} = \frac{d}{dt} \Big|_{t_0}$  is a derivation.

~~Example~~

Ex: Let  $M = \mathbb{R}^2$  and  $P = (x_0, y_0)$ .  $\Sigma_P = \frac{\partial}{\partial x} \Big|_P$   $\bar{\Sigma}_P = \frac{\partial}{\partial y} \Big|_P$

Clear  $\Sigma_P, \bar{\Sigma}_P \in D_P \mathbb{R}^2$

Definition:  $M$  a smooth  $m$ -manifold

Let  $x^j$  coordinate function of  $X: U \rightarrow V$  is a  $j^{\text{th}}$  component function of  $X$ ,

$$X(p) = (x^1(p), x^2(p), \dots, x^m(p)) \in V \subseteq \mathbb{R}^m$$

Let  $x^j$  are the manifold coordinates.

Denote coordinates of  $V \subseteq \mathbb{R}^m$  by  $u^1, u^2, \dots, u^m$   
(standard Cartesian coordinates)

Clearly  $u^j: \mathbb{R}^m \rightarrow \mathbb{R}$  the f-la is  $u^j(x(p)) = e_j \cdot x(p)$

We define: for  $f \in C^0(p)$  and  $x$  a coord. chart containing  $P$ ,  $\frac{\delta f}{\delta x^j}(p) = \left. \frac{\partial}{\partial u^j} [(f \circ x^{-1})(u)] \right|_{u=x(p)}$

Ex:  $f = x^i \rightarrow \mathbb{R}$ . Let calculate  $\frac{\delta x^i}{\delta x^j}(p)$

$$\left. \frac{\partial}{\partial u^j} [(x^i \circ x^{-1})(u)] \right|_{u=x(p)}$$

$$(x^i \circ x^{-1})(u) = x^i(p) = u^i \quad u = x(p)$$

$$\left. \frac{\partial}{\partial u^j} [u^i] \right|_{u=x(p)} = \delta_j^i$$

Concept:  $\frac{\delta}{\delta u^j} [u^i] = \delta_j^i \Rightarrow \frac{\delta x^i}{\delta x^j} = \delta_j^i$

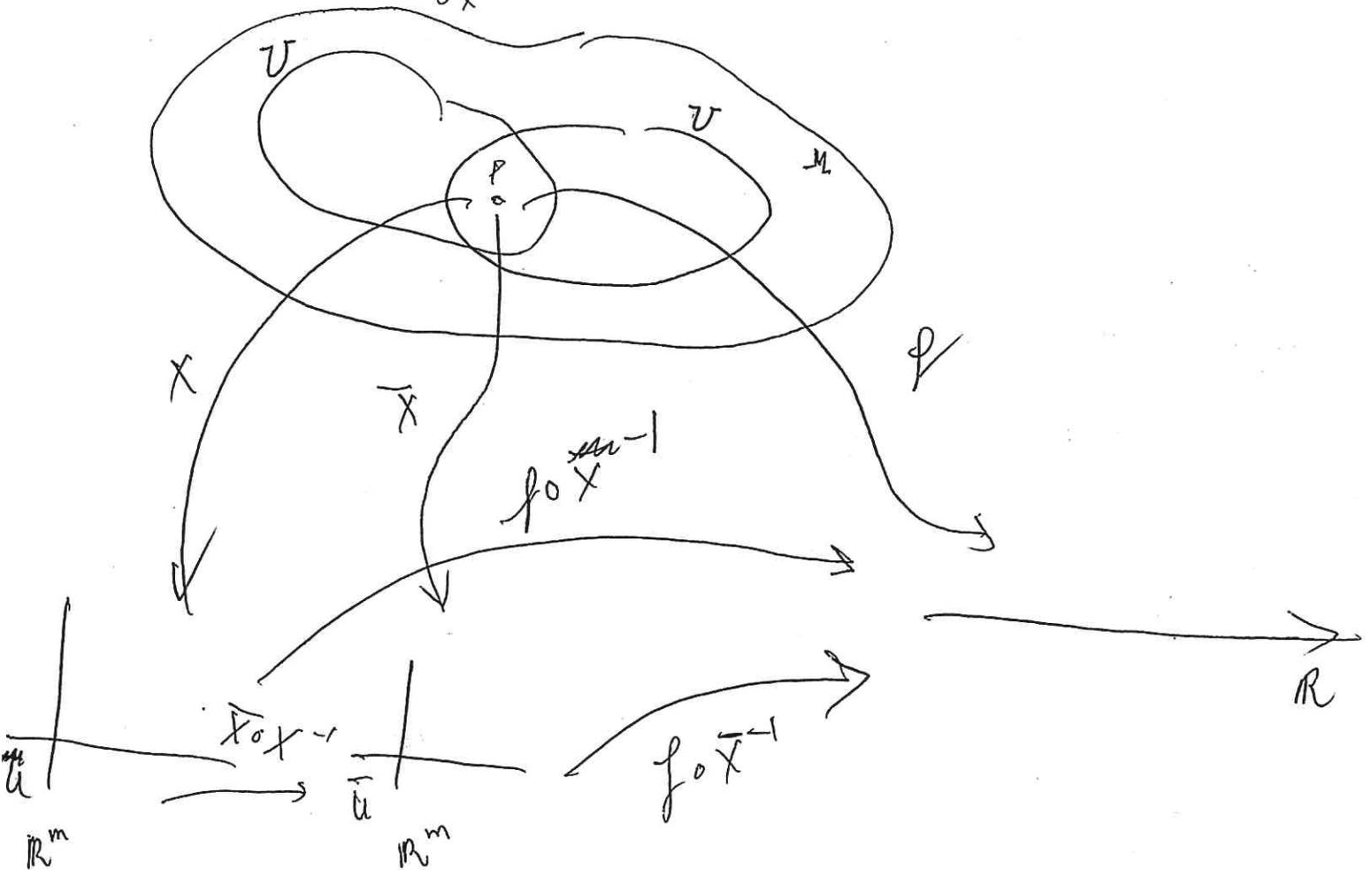
Ex: Show  $\frac{\delta}{\delta x} (cf \pm g) = c \frac{\delta f}{\delta x} \pm \frac{\delta g}{\delta x}$

Product Rule?

$$\begin{aligned}
 \frac{\delta}{\delta x^j} \Big|_p (fg) &= \frac{\delta}{\delta u^i} \left( (fg) \circ x^{-1} \right) \Big|_{x(p)} \\
 &= \frac{\delta}{\delta u^i} \left[ (f \circ x^{-1})(g \circ x^{-1}) \right] \Big|_{x(p)} \\
 &= \left( \frac{\delta (f \circ x^{-1})}{\delta u^i} (g \circ x^{-1}) + (f \circ x^{-1}) \frac{d(g \circ x^{-1})}{\delta u^i} \right) \Big|_{x(p)} \\
 &= \frac{\delta f}{\delta x} \Big|_p g(p) + f(p) \frac{dg}{\delta x} \Big|_p
 \end{aligned}$$

Chain Rule?

$$\frac{\delta f}{\delta x^i} \text{ w/ } \frac{\delta f}{\delta x^j}$$



$$\frac{\partial f}{\partial x^i} = \sum_{j=1}^n \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial f}{\partial \bar{x}^j}$$

$$f \circ \bar{x}^{-1} = (f \circ x^{-1}) \circ (x \circ \bar{x}^{-1})$$

$$(f \circ \bar{x}^{-1})' = (f \circ x^{-1})' (x \circ \bar{x}^{-1})' \longrightarrow \frac{\partial f}{\partial \bar{x}^j} = \sum_{i=1}^n \frac{\partial x^i}{\partial \bar{x}^j} \frac{\partial f}{\partial x^i}$$

Question:

① If  $\text{Der}_p(M) = \{ \text{all derivations of } C^\infty(P) \}$   
then why is  $\text{Der}_p(M) \cong \mathbb{R}^m$ ?

② Isomorphism to curve  $(T_p M)$ ?  
 $\text{vect}(T_p M)$ ?

Proposition:  $\text{Der}(C^\infty_p(M)) = \text{span} \left\{ \frac{\partial}{\partial x^j} \Big|_p \right\}_{j=1}^n$

Moreover,  $X_p \in \text{Der}(C^\infty_p(M)) \Rightarrow X_p = \sum_{j=1}^n X(x^j) \frac{\partial}{\partial x^j} \Big|_p$

Lemma: If  $p \in M$  is smooth and  $f: M \rightarrow \mathbb{R}$ , smooth

$X: U \rightarrow V$  a chart w/  $p \in U$  and  $x(p) = 0$  then

$\exists$  smooth function  $g_j: M \rightarrow \mathbb{R}$  whose values at  $p$  satisfy

$$g_j(p) = \frac{\partial f}{\partial x^j}(p)$$

In addition,  $\forall q$ , near to  $p$ , we have

$$f(q) = f(p) + \sum_{k=1}^n x^k(q) g_k(q)$$

(Burns & Gidean, p. 92)

Proof:

Back to proposition.

$$\mathbb{X}_p(f) = \mathbb{X}_p\left(f(p) + \sum_{k=1}^m x^j(q) g_j(q)\right) \quad q \text{ is variable.}$$

$$= \sum_{k=1}^m \left[ \mathbb{X}_p(x^j) g_j(p) + x^j(p) \mathbb{X}_p(g_j) \right]$$

$$= \sum_{k=1}^m \mathbb{X}_p(x^j) \frac{\partial f}{\partial x^j} \Big|_p \quad \checkmark$$

Def<sup>n</sup>  $\mathbb{X}, \mathbb{Y} \in \text{Der}(T_p M) \quad c \in \mathbb{R}$

$$(\mathbb{X} + c\mathbb{Y})(f) = \mathbb{X}(f) + c\mathbb{Y}(f) \quad \forall f \in C^\infty(p)$$

$$c_1 \frac{\partial}{\partial x_1} \Big|_p + c_2 \frac{\partial}{\partial x_2} \Big|_p + \dots + c_m \frac{\partial}{\partial x_m} \Big|_p = 0$$

Feeding this  $x^j$  and use  $\frac{\partial x^j}{\partial x^i} \Big|_p = \delta_i^j \Rightarrow \mathbb{X}^j = 0$

$\Rightarrow$  (L.I)

Compare:  $V_p = \sum V_p(x^i) \frac{\partial}{\partial x^i} \Big|_p$   $\leftarrow$  how related

$V_p = \sum V_p(\bar{x}^i) \frac{\partial}{\partial \bar{x}^i} \Big|_p$

$V_p = \sum_{i=1}^m V_p(\bar{x}^i) \sum_{j=1}^m \frac{\partial x^j}{\partial \bar{x}^i} \Big|_p \frac{\partial}{\partial x^j} \Big|_p$

$= \sum_{j=1}^m \left( \underbrace{\sum_{i=1}^m \frac{\partial x^j}{\partial \bar{x}^i} V_p(\bar{x}^i)}_{V_p(\bar{x}^j)} \right) \frac{\partial}{\partial x^j} \Big|_p$

Isomorphism:

$\Phi: \text{Der}(T_p M) \leftrightarrow \text{Curve}(T_p M)$

$\Phi(\gamma) = V \rightarrow V_i = \frac{d}{dt} (x^i \circ \gamma)(t) \Big|_{t=0}$

$\gamma \rightarrow x \circ \gamma \rightarrow v^i = \frac{d}{dx} (x \circ \gamma)(t) \Big|_{t=0}$



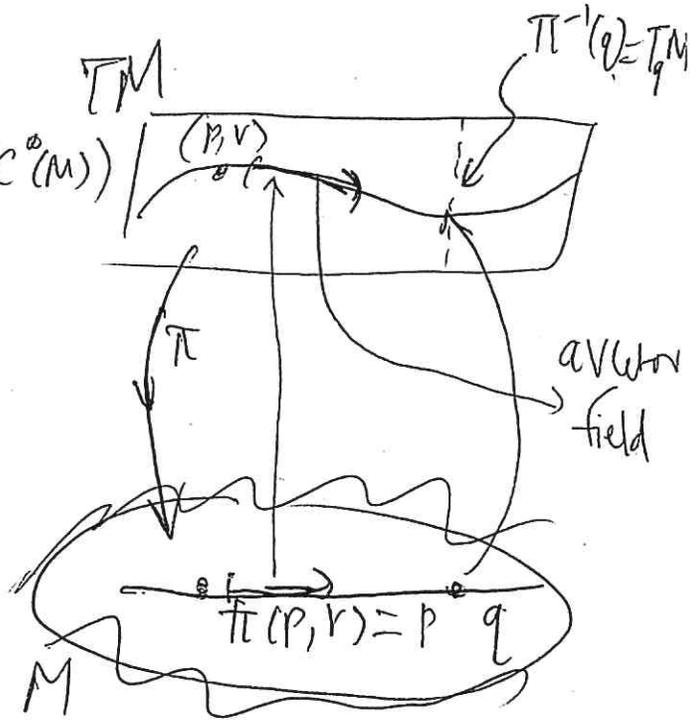
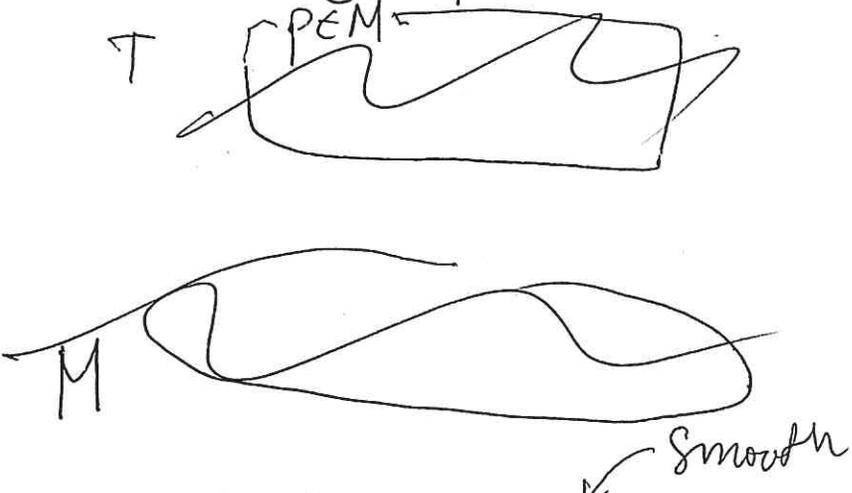
$V_p = \sum_{j=1}^m v^j \frac{\partial}{\partial x^j} \Big|_p$

$\Phi(\gamma) = \sum_{j=1}^m \frac{d}{dt} (x^j \circ \gamma) \Big|_{t=0} \frac{\partial}{\partial x^j} \Big|_{x(0)}$

$\Phi^{-1}(V_p) = x^{-1}(x(p) + t \sum v_p(x^i) e_i)$

# Vector Field on a manifold?

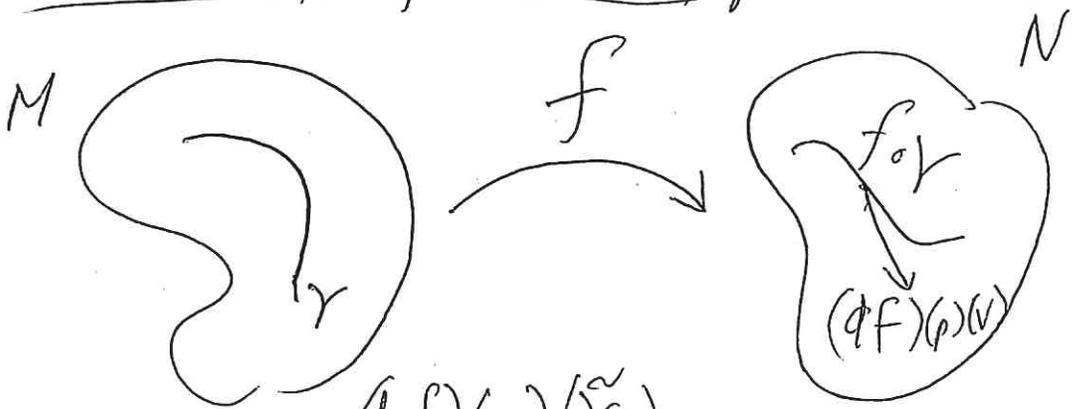
$$TM = \bigcup_{p \in M} T_p M, \text{ here } T_p M = \text{Der}(C^\infty(M))$$



- a vector field an assignment of a vector  $X_p$  for each  $p \in M$
- a section of  $\pi, s(U)$

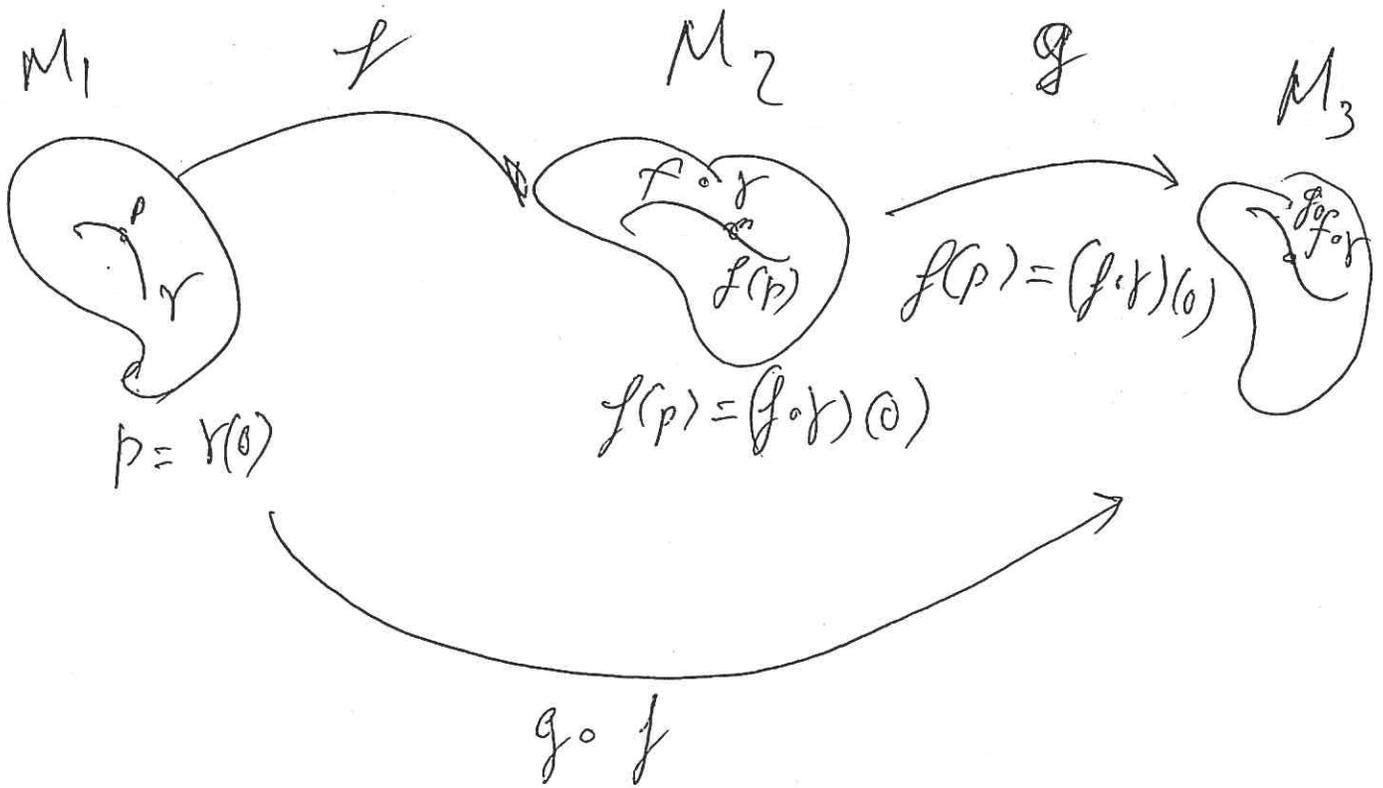
Theorem: If  $X$  is a vector field on  $M$  such that  $X_p \neq 0$  then  $\exists$  coordinate chart  $X$  for which  $p \in \text{dom}(X)$  and  $X_p = \frac{\partial}{\partial x^i} \Big|_p$ .

Derivatives for  $f: M \rightarrow N$ ,  $f$  is smooth



$$(df)_p(\tilde{\gamma}) = \tilde{f} \circ \tilde{\gamma}$$

$$d_p f(\tilde{\gamma}) = \tilde{f} \circ \tilde{\gamma}$$



$$d_p(g \circ f)(\tilde{v}) = \widetilde{g \circ f \circ \gamma}$$

$$d_{f(p)} g(d_p f(\tilde{v})) = d_{f(p)} g(\widetilde{f \circ \gamma})$$

$$= \widetilde{g \circ f \circ \gamma}$$

$$\therefore d_p(g \circ f) = d_{f(p)} g \circ d_p f$$