Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. Most problems 5pts here.

Problem 93 If E_1, E_2, E_3 is an orthonormal frame field on $\mathbb{R}^3$ for which E_1, E_2 restrict to tangent fields to $M \subset \mathbb{R}^3$ and $E_3(p) \in T_pM^\perp$ then we say E_1, E_2, E_3 is an **adapted frame** on M.

- **(a.)** Show $S(v) = \omega_{13}(v)E_1 + \omega_{23}(v)E_2$
- **(b.)** Show $\omega_{13} \wedge \omega_{23} = K\theta^1 \wedge \theta^2$
- **(c.)** Show $\omega_{13} \wedge \theta^2 + \theta^1 \wedge \omega_{23} = 2H\theta^1 \wedge \theta^2$
- **(d.)** Show $det(S) = \omega_{13}(E_1)\omega_{23}(E_2) - \omega_{13}(E_2)\omega_{23}(E_1)$
- **(e.)** Show $d\omega_{12} = -K\theta^1 \wedge \theta^2$,

Problem 94 A **principal frame field** adapted to M is an orthonormal frame field E_1, E_2, E_3 adapted to M for which there exist **principal curvature functions** k_1, k_2 such that $S(E_1) = k_1E_1$ and $S(E_2) = k_2E_2$. Show $E_1[k_2] = (k_1 - k_2)\omega_{12}(E_2)$ and $E_2[k_1] = (k_1 - k_2)\omega_{12}(E_1)$.

Problem 95 Consider the catenoid M given by parametric equations

$$x = b\cosh(v/b)\cos(u), \qquad y = b\cosh(v/b)\sin(u), \qquad z = v.$$

Calculate the following:

- **(a.)** Find an adapted frame E_1, E_2, E_3 to M by normalizing $\partial_u X$ and $\partial_v X$ and setting $E_3 = E_1 \times E_2$.
- **(b.)** Find coframe $\theta^1, \theta^2, \theta^3$ of the adapted frame, check that $\theta^3 = 0$ on M.
- **(c.)** Find ω_{12} from solving Cartan's Structure Equations.
- **(d.)** Calculate the Gaussian curvature from $d\omega_{12} = -K\theta^1 \wedge \theta^2$.

Problem 96 Let $F : M \to N$ be a smooth surface map. For each patch $X : D \to M$ consider the map $\bar{X} = F \circ X : D \to N$. Then F is a **local isometry** if and only if for each patch X we have $E = \bar{E}$ and $F = \bar{F}$ and $G = \bar{G}$ where $E = \partial_u X \bullet \partial_u X$ and $F = \partial_u X \bullet \partial_v X$ and $G = \partial_v X \bullet \partial_v X$. Use this theorem to find a local isometry of the

- **(a.)** plane and cylinder which have patches $X(u, v) = (u, v, 0)$ and $Y(u, v) = (R\cos(u/R), R\sin(u/R), v)$ respective.
- **(b.)** the helicoid and catenoid which have patches $X(u, v) = (u\cos v, u\sin v, v)$ and $Y(u, v) = (g, h\cos v, h\sin v)$ where $g(u) = \sinh^{-1}(u)$ and $h(u) = \sqrt{1 + u^2}$ (it's a little algebra, but you can take the implicit formulation provided by the theorem and make it explicit)

Problem 97 Consider a surface $M \subset \mathbb{R}^3$ with adapted frame E_i and coframe θ^i. In this problem we seek to argue for the intrinsic character of the Gaussian curvature:

- **(a.)** Define $\gamma = (d\theta^1(E_1, E_2))\theta^1 + d\theta^2(E_1, E_2))\theta^2$ and show $\gamma = \omega_{12}$.

(b.) Let $F : M \to \overline{M}$ be an isometry and let $\overline{E}_j = F_*(E_j)$ for $j = 1, 2$. Show $\theta^j = F^*(\overline{\theta}^j)$ for $j = 1, 2$ and $\omega_{12} = F^*(\overline{\omega}_{12})$ where $\overline{\omega}_{12}$ is the connection form on $\overline{M}$.

(c.) Show $K = \overline{K} \circ F$

Problem 98 Let $g = \frac{4}{(1-x^2-y^2)^2}(dx^2 + dy^2)$ then you can show $E_i = \frac{1}{2}(1 - x^2 - y^2)\frac{\partial}{\partial x^i}$ is a g-orthonormal frame in the sense that $g(E_i, E_j) = \delta_{ij}$. You could also show the dual coframe of 1-forms θ^1, θ^2

$$\theta^1 = \frac{2dx}{1 - x^2 - y^2}, \qquad \& \qquad \theta^2 = \frac{2dy}{1 - x^2 - y^2}.$$

With all of this given, calculate the Gaussian curvature. The set $M = \{(x^1, x^2) \mid \|(x^1, x^2)\| < 1\}$ paired with the metric g is known as the hyperbolic disk. It is an example of how a subset of the plane can be given a non-Euclidean geometry.

Problem 99 Find a metric on the plane which gives a subset of the plane curvature $K = 1$

Problem 100 Find a metric on a subset of the sphere which makes it flat.

Problem 101 State the Gauss Bonnet Theorem. Also, in a surface of constant curvature what can we say about the interior angles of a triangle ?

Problem 102 Explain why the sphere is not isometric to the torus.

Problem 103 Show that the ellipsoid $\mathcal{E} = \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ is mapped to the unit-sphere Σ via the map $F : \Sigma \to \mathcal{E}$ defined by $F(u, v, w) = (au, bv, cw)$. Also, explain why F is a diffeomorphism. Find the total curvature of $\mathcal{E}$ via the Gauss Bonnet Theorem as well as the observation that diffeomorphisms are also homeomorphisms and as such preserve the Euler characteristic. Remember, we already know the total curvature of Σ from our work in class.

Extra Problems you don't have to do:

(I.) MDG, Exercise 4.42 on page 168 (formula for Christoffel symbols in terms of metric)

(II.) Let $E_1 = -\sin\theta U_1 + \cos\theta U_2$ and $E_2 = U_3$ and $E_3 = \cos\theta U_1 + \sin\theta U_2$. Find the attitude matrix A for the cylindrical frame E_1, E_2, E_3 and calculate the matrix of connection forms $\omega = (dA)A^T$

(III.) In cylindrical coordinates, the frame of the above problem is simply $E_1 = \frac{1}{r}\frac{\partial}{\partial\theta}$, $E_2 = \frac{\partial}{\partial z}$ and $E_3 = \frac{\partial}{\partial r}$. This frame adapts nicely to the cylinder M given by $r = R$ where R is a fixed positive constant. Notice $E_1 = \frac{1}{R}\frac{\partial}{\partial\theta}$ and $E_2 = \frac{\partial}{\partial z}$ have coframe $\theta^1 = Rd\theta$ and $\theta^2 = dz$. We found $\omega_{13} = -d\theta = -\omega_{31}$ whereas $\omega_{12} = \omega_{21} = \omega_{23} = \omega_{32} = 0$.

(a.) Recall $S(E_1) = -\omega_{31}(E_1)E_1 - \omega_{32}(E_1)E_2$ and $S(E_2) = -\omega_{31}(E_2)E_1 - \omega_{32}(E_2)E_2$. Show E_1, E_2 is a principle frame.

(b.) Calculate H and K

(c.) Give an example of an asymptotic curve on M

(d.) Give an example of a geodesic curve on M

(IV.) Let θ^1, θ^2 be a coframe of E_1, E_2 on a surface such that $\theta^1 \wedge \theta^2 \neq 0$. Notice $f\theta^1 \wedge \theta^2 = g\theta^1 \wedge \theta^2$ implies $f = g$. Furthermore, if α is a one-form then

$$\alpha = \alpha[E_1]\theta^1 + \alpha[E_2]\theta^2$$

Use the assumptions above to find α given that $\alpha \wedge \theta^1 = A\theta^1 \wedge \theta^2$ and $\alpha \wedge \theta^2 = B\theta^1 \wedge \theta^2$. (your formula for α will involve A and B)

(V.) The helicoid has patch $X(u,v) = (u\cos v, u\sin v, bv)$ where $b > 0$. Since this is an orthogonal patch, we can use frame $E_1 = \frac{1}{\sqrt{E}}X_u$ and $E_2 = \frac{1}{\sqrt{G}}X_v$ and coframe $\theta^1 = \sqrt{E}du$ and $\theta^2 = \sqrt{G}dv$. Derive the connection form ω_{12} and the Gaussian curvature K from the equations $d\theta^1 = \omega_{12} \wedge \theta^2$, $d\theta^2 = \omega_{21} \wedge \theta^1$ and $d\omega_{12} = -K\theta^1 \wedge \theta^2$.

(VI.) The paraboloid of revolution has patch $X(u,v) = (u\cos v, u\sin v, u^2/2)$ where $b > 0$. Since this is an orthogonal patch, we can use frame $E_1 = \frac{1}{\sqrt{E}}X_u$ and $E_2 = \frac{1}{\sqrt{G}}X_v$ and coframe $\theta^1 = \sqrt{E}du$ and $\theta^2 = \sqrt{G}dv$. Derive the connection form ω_{12} and the Gaussian curvature K from the equations $d\theta^1 = \omega_{12} \wedge \theta^2$, $d\theta^2 = \omega_{21} \wedge \theta^1$ and $d\omega_{12} = -K\theta^1 \wedge \theta^2$.

(VII.) The cone has patch $X(u,v) = (u\cos v, u\sin v, au)$ where $a > 0$. Since this is an orthogonal patch, we can use frame $E_1 = \frac{1}{\sqrt{E}}X_u$ and $E_2 = \frac{1}{\sqrt{G}}X_v$ and coframe $\theta^1 = \sqrt{E}du$ and $\theta^2 = \sqrt{G}dv$. Derive the connection form ω_{12} and the Gaussian curvature K from the equations $d\theta^1 = \omega_{12} \wedge \theta^2$, $d\theta^2 = \omega_{21} \wedge \theta^1$ and $d\omega_{12} = -K\theta^1 \wedge \theta^2$.

(VIII.) Let $X(u,v) = (u, v, f(u,v))$ where f is a smooth function of u, v on $\mathbb{R}^2$. Let $M = X(\mathbb{R}^2)$. Calculate the Gaussian curvature of M. What condition on f is needed for M to be flat ?

(IX.) Let

$$X(u,v) = (u\cos(v), u\sin(v), v).$$

Calculate E, F, G and L, M, N. Use the standard formulas to calculate the Gaussian and mean curvature.

(X.) Suppose $u > 0$ and let

$$X(u,v) = (u\cos(v), u\sin(v), u^2).$$

Calculate the associated frame field E_1, E_2 as well as the dual frame θ^1, θ^2. Find the connection form ω_{12} and calculate the Gaussian curvature K. For bonus, also find ω_{12}, ω_{23} and calculate the mean curvature H.