Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems.

Problem 1 Spheres are a nice example. Let's build some background.

- **(a.)** Find an atlas for the circle of radius R in $\mathbb{R}^2$.
- **(b.)** Find an atlas for the the sphere of radius R in $\mathbb{R}^3$ and show your charts are compatible.
- **(c.)** Find an atlas for the n-sphere of radius R in $\mathbb{R}^{n+1}$
- **(d.)** Find a different atlas that covers most of the sphere from **(b.)** and check compatibility.

Problem 2 MDG exercise 1.40

Problem 3 MDG exercise 1.42

Problem 4 MDG exercise 1.45

Problem 5 MDG exercise 1.46

Problem 6 MDG exercise 1.47

Problem 7 MDG exercise 1.48

Problem 8 MDG exercise 1.54

Problem 9 MDG exercise 1.56

Problem 10 MDG exercise 1.60

Problem 11 MDG exercise 1.61

Problem 12 MDG Problem 4 from around page 51.

Problem 13 MDG Problem 7 from around page 51.

Problem 14 MDG Problem 16 from around page 51.

Problem 15 Let $f, g \in C^\infty(M)$. Prove the following for (U, x) a chart in the atlas of M.

(a.)

$$\frac{\partial}{\partial x^i}[f+g] = \frac{\partial f}{\partial x^i} + \frac{\partial g}{\partial x^i}$$

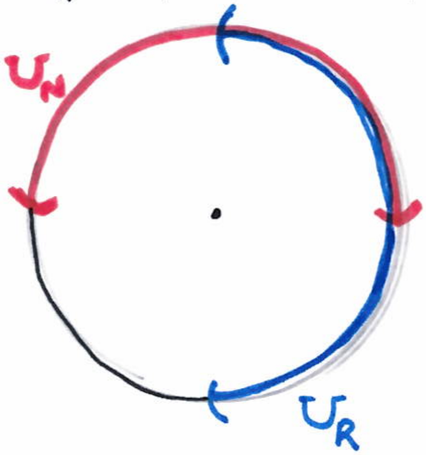
(b.)

$$\frac{\partial}{\partial x^i}[fg] = \frac{\partial f}{\partial x^i}g + f\frac{\partial g}{\partial x^i}.$$

MANIFOLDS : MISSION 1 SOLUTION

$\boxed{P1}$ atlas building,

(a.) $S^1_R = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2 \}$



$x_N(x,y) = x \qquad U_N = \{ (x,y) \in S^1_R \mid y > 0 \}$

$x_S(x,y) = x \qquad U_S = \{ (x,y) \in S^1_R \mid y < 0 \}$

$x_L(x,y) = y \qquad U_L = \{ (x,y) \in S^1_R \mid x < 0 \}$

$x_R(x,y) = y \qquad U_R = \{ (x,y) \in S^1_R \mid x > 0 \}$

Notice $x_N^{-1}(x) = (x, \sqrt{R^2 - x^2})$ for $x \in (-R, R)$

$x_S^{-1}(x) = (x, -\sqrt{R^2 - x^2})$ for $x \in (-R, R)$

$x_L^{-1}(y) = (-\sqrt{R^2 - y^2}, y)$ for $y \in (-R, R)$

$x_R^{-1}(y) = (\sqrt{R^2 - y^2}, y)$ for $y \in (-R, R)$

To see compatibility of these charts, notice for appropriate x, y

$$(x_R \circ x_N^{-1})(x) = x_R(x, \sqrt{R^2 - x^2}) = \sqrt{R^2 - x^2}$$

$$(x_N \circ x_R^{-1})(y) = x_N(\sqrt{R^2 - y^2}, y) = \sqrt{R^2 - y^2}$$

where $x_N^{-1}(x) \in U_R \cap U_N$ and $x_R^{-1}(y) \in U_R \cap U_N$ hence $0 < x, y < R$.
Long story short, all overlaps give smooth transition functions.

Since

$$U_N \cup U_S \cup U_R \cup U_S = S^1_R$$

we find $\{ (U_N, x_N), (U_S, x_S), (U_R, x_R), (U_L, x_L) \}$
is an atlas for S^1_R.

P1

(6.) Consider $S_R^2 = \{(x,y,z) \mid x^2+y^2+z^2 = R^2\}$

define $U_1^\pm = \{(x,y,z) \in S_R^2 \mid \quad 0 < \pm x < R\}$

so U_1^+ has $0<x<R$ whereas U_1^- has $-R<x<0$.

Likewise U_2^+ has $0<y<R$ and U_2^- has $-R<y<0$

and U_3^+ has $0<z<R$ and U_3^- has $-R<z<0$

$$\varphi_1^\pm(x,y,z) = (y,z) \ \& \ (\varphi_1^\pm)^{-1}(y,z) = (\pm\sqrt{R^2-y^2-z^2}, y, z)$$
$$\varphi_2^\pm(x,y,z) = (x,z) \ \& \ (\varphi_2^\pm)^{-1}(x,z) = (x, \pm\sqrt{R^2-x^2-z^2}, z)$$
$$\varphi_3^\pm(x,y,z) = (x,y) \ \& \ (\varphi_3^\pm)^{-1}(x,y) = (x,y,\pm\sqrt{R^2-x^2-y^2})$$

Clearly $U_1^+ \cup U_1^- \cup U_2^+ \cup U_2^- \cup U_3^+ \cup U_3^- = S_R^2$. Moreover, the fact that $\varphi_1^\pm, \varphi_2^\pm, \varphi_3^\pm$ are bijections is evident from the given inverse maps shown above. I'll check compatibility, consider,

$$\left(\varphi_1^+ \circ (\varphi_2^+)^{-1}\right)(x,z) = \varphi_1^+\left(x, \sqrt{R^2-x^2-z^2}, z\right)$$
$$= \left(\sqrt{R^2-x^2-z^2}, z\right)$$

$\mathrm{dom}\left((\varphi_2^+)^{-1}\right) = \varphi_2^+(U_2^+) = \{(x,z) \mid x^2+z^2 < R^2\}$, we don't find $x^2+z^2=R^2$ in range of φ_2^+ since U_2^+ disallows $y=0$ or $y=R$. Then for $(\varphi_2^+)^{-1}(x,z) \in \mathrm{dom}(\varphi_1^+)$ we need $0<x<R$ thus the (x,z) considered in calculation above are from an open half-disk ; (x,z) for which $x^2+z^2<R^2$ and $0<x<R$.

$$\left(\varphi_3^- \circ (\varphi_1^+)^{-1}\right)(y,z) = \varphi_3^-\left(\sqrt{R^2-y^2-z^2}, y, z\right) = \left(\sqrt{R^2-y^2-z^2}, y\right)$$

(for $y^2+z^2<R^2$ and $-R<z<0$) etc.

P1

(c.) $U_j^+ = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = R, \ x^j > 0 \}$

$U_j^- = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = R, \ x^j < 0 \}$

$S_R^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = R \}$

Observe $\left(\bigcup_{j=1}^{n+1} U_j^+ \right) \cup \left(\bigcup_{j=1}^{n+1} U_j^- \right) = S_R^n$

Define $\varphi_j^\pm(x) = (x^1, \dots, \widehat{x^j}, \dots, x^n) \quad \forall x \in U_j^\pm$

where $\widehat{x^j}$ means to delete x^j from the list, hence

$\varphi_j^\pm(U_j^\pm) \subseteq \mathbb{R}^n$ indeed $\varphi_j^\pm(U_j^\pm) = \{ y \in \mathbb{R}^n \mid \|y\| < R \}$

Notice, the inverse of the coord. charts are easy to write down,

$$(\varphi_j^\pm)^{-1}(y) = (y^1, \dots, y^{j-1}, \pm\sqrt{1 - \|y\|^2}, y^{j+1}, \dots, y^n) \in S_R^n$$

<u>All</u>* these charts overlap. Consider, for $i \neq j$, wlog $\ni$ $i < j$

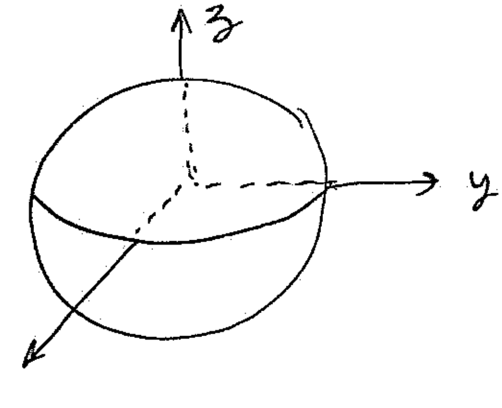
$$\boxed{\begin{aligned}\left(\varphi_i^+ \circ (\varphi_j^-)^{-1}\right)(y) &= \varphi_i^+\left(y^1, \dots, y^{j-1}, -\sqrt{1 - \|y\|^2}, y^{j+1}, \dots, y^n\right) \\ &= (y^1, \dots, y^{i-1}, y^{i+1}, \dots, y^{j-1}, \sqrt{1 - \|y\|^2}, y^{j+1}, \dots, y^n)\end{aligned}}$$

The transition function above is <u>smooth</u> since $\|y\|^2 < 1$ by construction of the chart domains.

* $\mathrm{dom}(\varphi_i^+) \cap \mathrm{dom}(\varphi_i^-) = \phi$ so these are compatible by default. The rest are compatible in the smooth sense by a calculation similar to that given above.

P1

(d) $x = R\cos\theta\sin\phi$
$y = R\sin\theta\sin\phi$
$z = R\cos\phi$



ϕ degenerate along the z-axis.

θ degenerate on a half-plane where the necessary 2π- angle jump must happen.

$$U = \{ (x,y,z) \in S_R^2 \mid x^2+y^2 \neq 0 \text{ and } \underbrace{y=0 \Rightarrow x>0}_{\text{deleting the half-plane where } \theta = \pm\pi} \}$$

For $(x,y,z) \in U$ we define,

$$\chi(x,y,z) = (\phi,\theta)$$

[thank you Math 331.]

$$= \left(\cos^{-1}\left(\frac{z}{R}\right), \operatorname{Arg}(x+iy)\right)$$
$$= \left(\cos^{-1}\left(\frac{z}{R}\right), \tan^{-1}\left(\frac{y}{x}\right)\right) \quad (\text{if } x>0)$$
$$= \left(\cos^{-1}\left(\frac{z}{R}\right), \cot^{-1}\left(\frac{x}{y}\right)\right) \quad (\text{if } y>0)$$

Consider compatibility with (U_1^+, φ_1^+)

$$(\varphi_1^+ \circ \chi^{-1})(\phi,\theta) = \varphi_1^+(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$$
$$= (R\sin\theta\sin\phi, R\cos\phi).$$

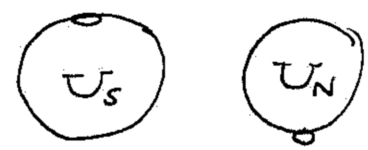
$$(\chi \circ (\varphi_1^+)^{-1})(y,z) = \chi\left(\sqrt{R^2-y^2-z^2}, y, z\right)$$
$$= \left(\cos^{-1}\left(\frac{z}{R}\right), \operatorname{Arg}\left(\sqrt{R^2-y^2-z^2}+iy\right)\right)$$

Smooth! (in context of domain considered, $y^2+z^2<R^2$ ~~etc.~~)

(<u>Remark</u>: sorry, I should have forced $x>0$ for part d.)

Example 1.39, pg. 17 of Jeffrey Lee's MANIFOLDS & DIFFERENTIAL GEOMETRY

n-sphere $S^n \subset \mathbb{R}^{n+1}$



$U_S \cup U_N = S^n$

$\mathcal{A} = \{(U_S, \psi_S), (U_N, \psi_N)\}$

$U_S = \{x = (x_1, .., x_{n+1}) \in S^n \mid x_{n+1} \neq 1\}$

$U_N = \{x = (x_1, .., x_{n+1}) \in S^n \mid x_{n+1} \neq -1\}$

Stereographic projection gives charts,

$$\psi_S(x) = \frac{1}{1 - x_{n+1}}(x_1, .., x_n) \in \mathbb{R}^n$$

$$\psi_N(x) = \frac{1}{1 + x_{n+1}}(x_1, ..., x_n) \in \mathbb{R}^n$$

Calculate inverse for $\psi_S : U_S \longrightarrow \mathbb{R}^n$ for $x_{n+1} \neq 1$,

$$(y_1, y_2, .., y_n) = \frac{1}{1 - x_{n+1}}(x_1, .., x_n)$$

$$y_j = \frac{x_j}{1 - x_{n+1}} \Rightarrow x_j = y_j(1 - x_{n+1}) \quad \text{where } x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = 1$$

after some algebra, using $\|y\|^2 = y_1^2 + y_2^2 + \cdots + y_n^2$, see

$$\psi_S^{-1}(y) = \frac{1}{1 + \|y\|^2}(2y_1, 2y_2, ..., 2y_n, \|y\|^2 - 1)$$

$$\psi_N^{-1}(y) = \frac{1}{1 + \|y\|^2}(2y_1, 2y_2, ..., 2y_n, 1 - \|y\|^2)$$

$$y_j = \frac{x_j}{1 - x_{n+1}} \quad \therefore \quad \boxed{x_j = y_j(1 - x_{n+1})}\ *$$

$$x_1^2 + \cdots + x_n^2 = 1 - x_{n+1}^2 = (1 - x_{n+1})(1 + x_{n+1})$$

By $*$, $y_1^2(1-x_{n+1})^2 + \cdots + y_n^2(1-x_{n+1})^2 = \underbrace{(y_1^2 + \cdots + y_n^2)}_{\|y\|^2}(1-x_{n+1})^2 = (1-x_{n+1})(1+x_{n+1})$

Thus $(1-x_{n+1})^2\|y\|^2 = (1-x_{n+1})(1+x_{n+1})$

Since $x_{n+1} \neq 1$ we find $\underbrace{(1-x_{n+1})\|y\|^2 = 1 + x_{n+1}}$

Solve for x_{n+1} $(1 + \|y\|^2) = \|y\|^2 - 1$

$$\boxed{x_{n+1} = \frac{\|y\|^2 - 1}{\|y\|^2 + 1}}\ **$$

From $**$, $1 - x_{n+1} = 1 - \dfrac{\|y\|^2 - 1}{\|y\|^2 + 1} = \dfrac{\|y\|^2 + 1 - (\|y\|^2 - 1)}{\|y\|^2 + 1} = \dfrac{2}{\|y\|^2 + 1}$

From $*$ we find $\boxed{x_j = \dfrac{2y_j}{\|y\|^2 + 1}}$

Therefore,

$$\boxed{\psi_S^{-1}(y) = \frac{(2y_1, 2y_2, \ldots, 2y_n, \|y\|^2 - 1)}{\|y\|^2 + 1}}$$

Exercise 1.40

$$U_N \cap U_S = \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} \neq \pm 1, \ x_1^2 + \cdots + x_{n+1}^2 = 1 \}$$

$P_N = (0, \ldots, 0, 1)$ NORTH POLE

$P_S = (0, \ldots, 0, -1)$ SOUTH POLE

$$\psi_N(P_N) = \frac{1}{1+1}(0, 0, \ldots, 0) = 0$$

$$\psi_S(P_S) = \frac{1}{1-(-1)}(0, 0, \ldots, 0) = 0$$

Notice if $y \in \mathbb{R}^n$ then

$$\begin{aligned}
\psi_S(\psi_S^{-1}(y)) &= \psi_S\left(\frac{(2y_1, \ldots, 2y_n, \|y\|^2 - 1)}{1 + \|y\|^2} \right) \\
&= \frac{1}{1 - \frac{\|y\|^2 - 1}{1 + \|y\|^2}} \frac{(2y_1, 2y_2, \ldots, 2y_n)}{1 + \|y\|^2} \\
&= \frac{1}{1 + \|y\|^2 - \|y\|^2 + 1}(2y_1, \ldots, 2y_n) \\
&= (y_1, \ldots, y_n) \\
&= y \qquad \Rightarrow \underline{\psi_S(U_S) = \mathbb{R}^n}.
\end{aligned}$$

Likewise, $\underline{\psi_N(U_N) = \mathbb{R}^n}$. Hence, as $U_N \cap U_S = S^n - \{P_N, P_S\}$

$$\psi_N(U_N \cap U_S) = \psi_S(U_N \cap U_S) = \mathbb{R}^n - \{0\}$$

<u>Exercise 1.40</u>

$$\psi_S \circ \psi_N^{-1}(y) = \psi_S\left(\frac{(2y_1, \ldots, 2y_n, 1 - \|y\|^2)}{1 + \|y\|^2}\right)$$

$$= \frac{1}{1 - \frac{1 - \|y\|^2}{1 + \|y\|^2}} \left[\frac{(2y_1, \ldots, 2y_n)}{1 + \|y\|^2}\right]$$

$$= \frac{1}{1 + \|y\|^2 - 1 + \|y\|^2} (2y_1, \ldots, 2y_n)$$

$$= \frac{1}{2\|y\|^2} (2y_1, \ldots, 2y_n)$$

$$= \frac{y}{\|y\|^2}.$$

Likewise, for slightly different reasons, $\psi_N \circ \psi_S^{-1}(y) = \dfrac{y}{\|y\|^2}$

Hence $\psi_S \circ \psi_N^{-1}$ and $\psi_N \circ \psi_S^{-1}$ are smooth transition functions on $\mathbb{R}^n - \{0\}$ $\therefore$ A is atlas for S^n

<u>Remark</u>: A has just <u>two</u> charts, but it was fairly troublesome to calculate the transition functions. In contrast, the projection atlas $\varphi_i^{\pm}(x) = (x_1, \ldots, \hat{x}_i, \ldots, x_n)$ has much easier algebra but a proliferation of charts.

Example 1.41 (Projective Spaces)

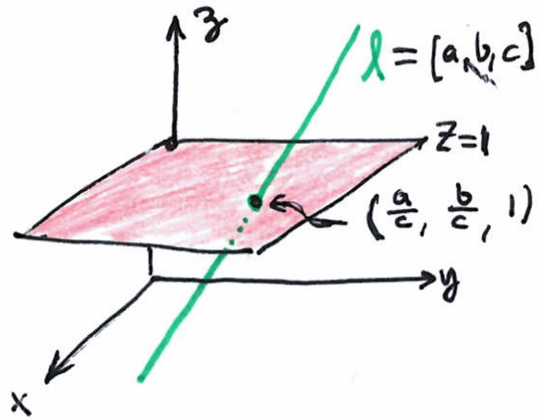
$\mathbb{R}P^2$ = set of all lines through origin in $\mathbb{R}^3 = \{ \ell \subseteq \mathbb{R}^3 \,/\, \ell = [x] = \{ tx \mid t \in \mathbb{R},\ x \neq 0 \}_{x \in \mathbb{R}^3} \}$

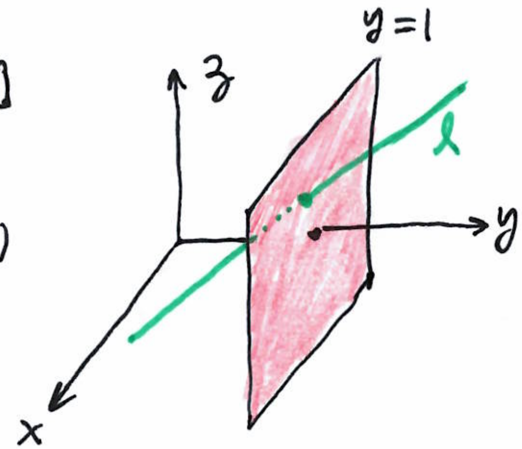
REAL PROJECTIVE PLANE

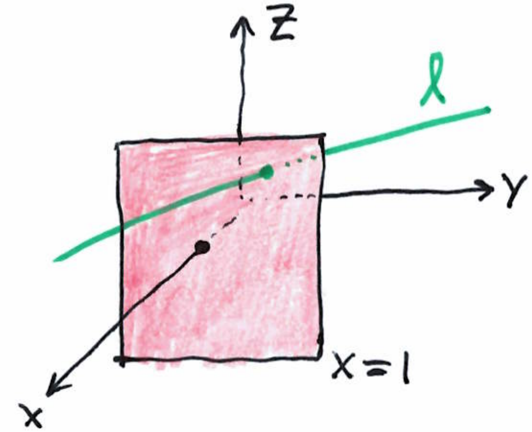
$U_z = \{ \ell \in \mathbb{R}P^2 \,/\, \ell \cap \{ (x,y,1) \mid x,y \in \mathbb{R} \} \neq \phi \}$

$U_x = \{ \ell \in \mathbb{R}P^2 \,/\, \ell \cap \{ (1,y,z) \} \neq \phi \}$

$U_y = \{ \ell \in \mathbb{R}P^2 \,/\, \ell \cap \{ (x,1,z) \} \neq \phi \}$







$\varphi_z(\ell) = (x(\ell), y(\ell))$

ℓ intersects $z=1$ at $(x(\ell), y(\ell), 1)$

$\varphi_z(\ell) = (u^1, u^2)$ where $(u^1, u^2, 1) \in \ell$

$\varphi_y(\ell) = (x(\ell), z(\ell))$

ℓ intersects $y=1$ at $(x(\ell), z(\ell))$

$\varphi_y(\ell) = (u^1, u^3)$ where $(u^1, 1, u^3) \in \ell$

$\varphi_x(\ell) = (y(\ell), z(\ell))$

ℓ intersects $x=1$ $(y(\ell), z(\ell))$

$\varphi_x(\ell) = (u^2, u^3)$ $(1, u^2, u^3) \in \ell$

What does $\varphi_z(\ell) = (x(\ell), y(\ell))$ mean formula-wise?

$\ell = \{ t(a,b,c) \mid t \in \mathbb{R},\ c \neq 0 \} = [(a,b,c)]$ ← notation for span $\{(a,b,c)\}$

for $(x,y,z) \in \ell$ with $z=1$ we need $ct = 1 \Rightarrow t = 1/c \Rightarrow x = \frac{a}{c}$ & $y = \frac{b}{c}$

$$\varphi_z([(a,b,c)]) = \left(\frac{a}{c}, \frac{b}{c}\right)$$

Similarly for $(x,y,z) \in \ell$ with $x = 1$ need $at = 1 \Rightarrow t = 1/a \Rightarrow y = \frac{b}{a}$ & $z = \frac{c}{a}$

$$\varphi_x([a,b,c]) = \left(\frac{b}{a}, \frac{c}{a}\right)$$

Likewise, $(x,y,z) \in \ell$ with $y=1$ need $bt = 1 \Rightarrow t = 1/b \Rightarrow x = \frac{a}{b}$ & $z = \frac{c}{b}$

$$\varphi_y([a,b,c]) = \left(\frac{a}{b}, \frac{c}{b}\right)$$

$\boxed{\text{Th}^{\text{m}}/\ [\lambda(a,b,c)] = [(a,b,c)]}$ $(\lambda \in \mathbb{R})$

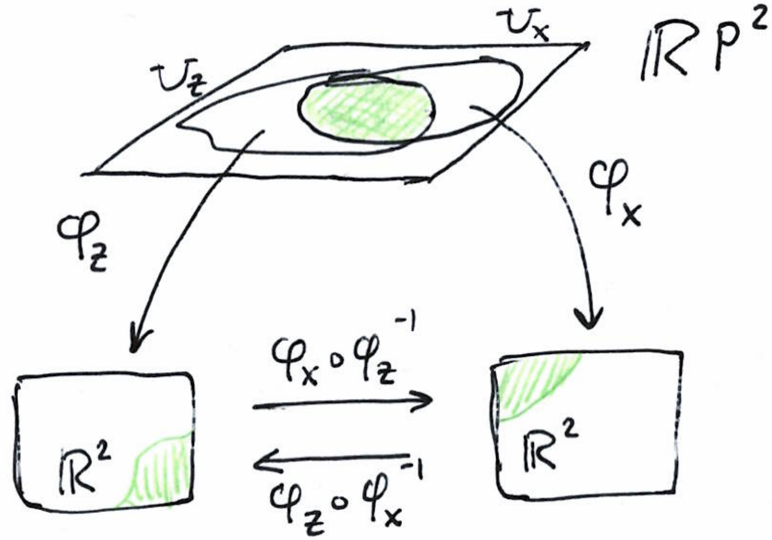
Claim,

$$\varphi_z^{-1}(u^1, u^2) = [(u^1, u^2, 1)]$$
$$\varphi_y^{-1}(u^1, u^3) = [(u^1, 1, u^3)]$$
$$\varphi_x^{-1}(u^2, u^3) = [(1, u^2, u^3)]$$

Try it out, $\varphi_z(\varphi_z^{-1}(u^1,u^2)) = \varphi_z[(u^1,u^2,1)] = \left(\frac{u_1}{1}, \frac{u_2}{1}\right) = (u^1, u^2)$.

$$\varphi_z^{-1}(\varphi_z(\underbrace{[(a,b,c)]}_{\text{assumed in } V_z \text{ so } c \neq 0})) = \varphi_z^{-1}\left(\frac{a}{c}, \frac{b}{c}\right) = \left[\left(\frac{a}{c}, \frac{b}{c}, 1\right)\right] = [(a,b,c)]$$

Transition functions for $\mathbb{R}P^2$?



$$U_x \cap U_z = \{[(a,b,c)] \mid a \neq 0, c \neq 0\}$$

$$(\varphi_x \circ \varphi_z^{-1})(u,v) = \varphi_x([(u,v,1)]) = \varphi_x([(1, \tfrac{v}{u}, \tfrac{1}{u})]) = (\tfrac{v}{u}, \tfrac{1}{u})$$

$$(\varphi_z \circ \varphi_x^{-1})(u,v) = \varphi_z([(1,u,v)]) = \varphi_z([(\tfrac{1}{v}, \tfrac{u}{v}, 1)]) = (\tfrac{1}{v}, \tfrac{u}{v})$$

For $(u,v) \in \varphi_x^{-1}(U_x \cap U_z)$ have $v \neq 0$ & $u \neq 0$

$(u,v) \in \varphi_z^{-1}(U_x \cap U_z)$ have $u \neq 0$ & $v \neq 0$

I'm moving on to $\mathbb{R}P^n$

$\mathbb{R}P^n$: REAL PROTECTIVE SPACE (all lines through 0 in $\mathbb{R}^{n+1}$)

$$\pi : \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}P^n$$

$$\pi(x) = [x] = \text{span}\{x\} = \{tx \mid t \in \mathbb{R}\} \quad \text{where } x \neq 0$$

Notice $[\lambda x] = [x]$ for $\lambda \neq 0$ so the choice of x is nonunique.

(Defn/ $U \subseteq \mathbb{R}P^n$ is <u>open</u> iff $\pi^{-1}(U)$ is open) later you'll learn about quotient topology in Topology. We're topologizing $\mathbb{R}P^n$ in such a way that π is continuous.

CHARTS ON $\mathbb{R}P^n$

(U_i, φ_i) for $i = 1, 2, \ldots, n+1$

$$\boxed{\begin{aligned} U_i &= \{ \ell \in \mathbb{R}P^n \mid \ell \text{ is not contained in } x^i = 0 \text{ hyperplane} \} \\ &= \{ [(x^1, \ldots, x^{i-1}, 1, x^{i+1}, \ldots, x^{n+1})] \mid x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1} \in \mathbb{R} \} \end{aligned}}$$

[Note $\ell = [(a^1, a^2, \ldots, a^{n+1})] = \{ (ta^1, ta^2, \ldots, ta^{n+1}) \mid t \in \mathbb{R} \}$

If we set $x^i = 0 = ta^i \Rightarrow \underbrace{a^i \neq 0}_{\text{I choose to rescale to 1 for convenience.}}$ so $x^i \not\equiv 0$ for points in ℓ.]

$$\boxed{\varphi_i([x]) = \left(\frac{x^1}{x^i}, \frac{x^2}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i} \right)}$$

$$\varphi_i^{-1}(u) = [(u^1, u^2, \ldots, u^{i-1}, 1, u^{i+1}, \ldots, u^n)]$$

TRANSITION FUNCTION FOR $\mathbb{R}P^n$ (oops, I think this is Ex. 1.42)

$$(\varphi_i \circ \varphi_j^{-1})(u) = \varphi_i\left([(u^1, \dots, u^{j-1}, 1, u^{j+1}, \dots, u^n)]\right)$$
$$= \left(\frac{u^1}{u^i}, \frac{u^2}{u^i}, \dots, \frac{u^{j-1}}{u^i}, \frac{1}{u^i}, \frac{u^{j+1}}{u^i}, \dots, \frac{u^n}{u^i}\right)$$

$$(\varphi_j \circ \varphi_i^{-1})(u) = \left(\frac{u^1}{u^j}, \frac{u^2}{u^j}, \dots, \frac{u^{i-1}}{u^j}, \frac{1}{u^j}, \frac{u^{i+1}}{u^j}, \dots, \frac{u^n}{u^{j+1}}\right)$$

For $u \in \varphi_i^{-1}(U_i \cap U_j)$ or $u \in \varphi_j^{-1}(U_i \cap U_j)$ we have $u^i, u^j \neq 0$.

Remark: $\varphi_i^{-1}(U_1 \cap U_2 \cap \dots \cap U_{n+1})$ has points with all nonzero entries.

Comment: since $[(x)] \cap S^n$ in the pair of antipodal points $\pm\hat{x}$ we can use S^n to model $\mathbb{R}P^n$ if we identify $\hat{x} \sim -\hat{x}$

In the language of p. 18 in Jeffrey Lee's text $S^n/\sim \,\cong \mathbb{R}P^n$

Homogeneous Coordinate Notation

$(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1}$ then $[x^1, \dots, x^{n+1}] = \ell \in \mathbb{R}P^n$ s.t. $(x^1, \dots, x^{n+1}) \in \ell$

As we've noted $[\lambda x^1, \dots, \lambda x^{n+1}] = [x^1, \dots, x^{n+1}]$

$$\varphi_i([x_1, \dots, x_{n+1}]) = (x_1/x_i, \dots, \hat{1}, \dots, x_{n+1}/x_i)$$
$$= (x_1/x_i, \dots, x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_{n+1}/x_i)$$

Complex Projective Space

$\mathbb{C}P^n \leftrightarrow$ all $\underbrace{\text{1-dim'l complex subspaces}}_{\text{a.k.a. 2-real dim'l}}$ of $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$

$$U_i = \{\ell \in \mathbb{C}P^n \mid \ell \text{ not contained in complex hyperplane } z^i = 0\}$$

$$\ell = [(z^1, \dots, z^{n+1})] = \mathrm{span}_{\mathbb{C}}(z) = \{(\lambda z^1, \dots, \lambda z^{n+1}) \mid \lambda \in \mathbb{C}\}$$

$$\varphi_i([z]) = \left(\frac{z^1}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^{n+1}}{z^i}\right)$$

$$\text{"}\ \varphi_i^{-1}(\underbrace{u^1, u^2, \dots, u^n}_{\mathbb{C}^n}) = [u^1, u^2, \dots, u^{i-1}, 1, u^{i+1}, \dots, u^n]\text{"} \in U_i \subset \mathbb{C}P^n$$

Exercise 1.46 (typo here) $U_1 \cap U_2$ supposed to be $\varphi_1(U_1 \cap U_2) = \varphi_2(U_1 \cap U_2) = \mathbb{C} - \{0\}$.

$$\begin{aligned}\mathbb{C}P^1 &= \{ [z] \mid z \in \mathbb{C}^2 \} \\ &= \{ [z^1, z^2] \mid (z^1, z^2) \in \mathbb{C}^2 \} \\ &= U_1 \cup U_2 \quad \text{where} \quad U_1 = \{ [(z_1, z_2)] \mid z_1 \neq 0 \} \\ & \qquad\qquad\qquad\qquad\;\; U_2 = \{ [(z_1, z_2)] \mid z_2 \neq 0 \}\end{aligned}$$

$$\varphi_1([(z^1, z^2)]) = \frac{z^2}{z^1} \qquad \varphi_1^{-1}(z^2) = [(1, z^2)] = [1, z^2]$$

$$\varphi_2([(z^1, z^2)]) = \frac{z^1}{z^2} \qquad \varphi_2^{-1}(z^1) = [(z^1, 1)] = [z^1, 1]$$

$$\varphi_1(U_1 \cap U_2) = \varphi_2(U_1 \cap U_2) = \mathbb{C} - \{0\}$$

$$(\varphi_2 \circ \varphi_1^{-1})(z) = \varphi_2([1, z]) = \frac{1}{z}$$

$$(\varphi_1 \circ \varphi_2^{-1})(w) = \varphi_1([w, 1]) = \frac{1}{w}$$

- Likewise, define conjugate chart $\overline{\varphi}_1 : U_1 \longrightarrow \mathbb{C}$ by $\overline{\varphi}_1(l) = \overline{\varphi_1(l)}$
  $\overline{\varphi}_2 : U_2 \longrightarrow \mathbb{C}$ by $\overline{\varphi}_2(l) = \overline{\varphi_2(l)}$

$$\overline{\varphi}_1([z^1, z^2]) = \overline{\frac{z^2}{z^1}} = \frac{\overline{z}^2}{\overline{z}^1} \qquad (\varphi_2 \circ \overline{\varphi}_1^{-1})(z) = \varphi_2([1, \overline{z}]) = \frac{1}{\overline{z}}$$

$$\overline{\varphi}_1^{-1}(z) = [1, \overline{z}]$$

continuing, typo $\varphi_1 \circ \varphi_2^{-1}(z)$ supposed to be $\overline{\varphi}_1 \circ \varphi_2^{-1}(z) = \dfrac{1}{\overline{z}}$

Consider,

$$(\overline{\varphi}_1 \circ \varphi_2^{-1})(z) = \overline{\varphi}_1([z,1]) = \overline{\left(\frac{1}{z}\right)} = \frac{1}{\overline{z}} = \frac{z}{z\overline{z}} = \frac{x+iy}{x^2+y^2}$$

Thus $\mathcal{A} = \{(U_1, \overline{\varphi}_1), (U_2, \varphi_2)\}$ gives smooth atlas for $\mathbb{C}P^1$

Exercise 1.40 $\quad (\psi_S \circ \psi_N^{-1})(y) = \dfrac{y}{\|y\|^2} \quad \forall y \in \mathbb{R}^n - \{0\}$

For $S^2 \subset \mathbb{R}^3$ have $y = (y^1, y^2)$ and $\dfrac{(y^1, y^2)}{(y^1)^2 + (y^2)^2}$ same formula as

 So, S^2 diffeomorphic to $\mathbb{C}P^1$?

$X = (X_1, X_2, X_3) \in S^2$ with $X_3 \neq -1$ define $f_1(x) = \dfrac{X_1 + iX_2}{1+X_3}$

with $X_3 \neq 1$ define $f_2(x) = \dfrac{X_1 - iX_2}{1-X_3}$

$$f(x) = \begin{cases} [f_1(x), 1], & X_3 \neq -1 \\ [1, f_2(x)], & X_3 \neq 1 \end{cases}$$

Exercise 1.47
Show $f: S^2 \to \mathbb{C}P^1$ diffeomorphism

Continuing, $f(x) = \begin{cases} [f_1(x), 1] & \text{if } x_3 \neq -1 \\ [1, f_2(x)] & \text{if } x_3 \neq 1 \end{cases}$

Why is $f(x)$ well-defined, what if $x_3 \neq 1$ and $x_3 \neq -1$

Following Jeff. Lee on p. 20,

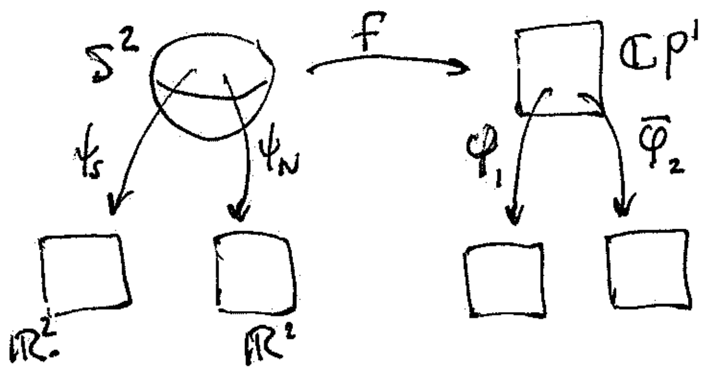
$$1 - x_3^2 = x_1^2 + x_2^2$$

If $x = (x_1, x_2, x_3) \in S^2$ with $-1 < x_3 < 1$ $(x_3 \neq 1$ & $x_3 \neq -1)$

then as $f_1(x) = \dfrac{x_1 + i x_2}{1 + x_3}$ and $f_2(x) = \dfrac{x_1 - i x_2}{1 - x_3}$

$$f_1(x) f_2(x) = \frac{(x_1 + i x_2)(x_1 - i x_2)}{(1 + x_3)(1 - x_3)} = \frac{x_1^2 + x_2^2}{1 - x_3^2} = \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} = 1.$$

Thus $[f_1(x), 1] = [f_1(x) f_2(x), 1 \cdot f_2(x)] = [1, f_2(x)]$ as desired.



To show f diffeomorphism need to show f bijection and local coord. rep. of f are smooth and f^{-1} as well,

$\varphi_1 \circ f \circ \psi_N^{-1}$ $\quad \bar{\varphi}_2 \circ f \circ \psi_N^{-1}$

$\varphi_1 \circ f \circ \psi_S^{-1}$ $\quad \bar{\varphi}_2 \circ f \circ \psi_S^{-1}$

I'd think through all of these...

P2 MDG exercise 1.40 (from Jeffrey Lee's text)

P3 ex. 1.42

P4 ex. 1.45 (actually not an exercise, but I worked through details in previous pages)

P5 ex. 1.46

P6 MDG exercise 1.47, Show $f: S^2 \to \mathbb{C}P^1$ discussed in previous pages is diffeomorphism.

$$f(x) = \begin{cases} [f_1(x), 1] & \text{if } x_3 \neq -1 \\ [1, f_2(x)] & \text{if } x_3 \neq 1 \end{cases}$$

$$f_1(x) = \frac{x_1 + i x_2}{1 + x_3} \quad \text{and} \quad f_2(x) = \frac{x_1 - i x_2}{1 - x_3}$$

As I commented on 13, we need to check f is bijection with smooth local coord. representatives.

$$\left(\varphi_1 \circ f \circ \psi_N^{-1}\right)(x, y) = \varphi_1\Big(f\Big(\underbrace{\tfrac{1}{1+x^2+y^2}(2x, 2y, 1-x^2-y^2)}_{z}\Big)\Big)$$

$$= \varphi_1\left(\left[1, f_2(z)\right]\right)$$

$$= \varphi_1\left(\left[1, \frac{1}{1 - \frac{1-x^2-y^2}{1+x^2+y^2}}\left(\frac{2x - 2iy}{1+x^2+y^2}\right)\right]\right)$$

$$= \varphi_1\left(\left[1, \frac{1}{1+x^2+y^2-(1-x^2-y^2)}(2x - 2iy)\right]\right)$$

$$= \varphi_1\left(\left[1, \frac{x-iy}{x^2+y^2}\right]\right)$$

$$= \varphi_1\left(\left[x^2+y^2, x-iy\right]\right)$$

$$= \frac{x-iy}{x^2+y^2}$$

$\boxed{P6}$ $f: S^2 \to \mathbb{C}P^1$ given by $f(x) = \begin{cases} [f_1(x), 1] & \text{if } x_3 \neq -1 \\ [1, f_2(x)] & \text{if } x_3 \neq 1 \end{cases}$

where $f_1(x) = \dfrac{x_1 + ix_2}{1+x_3}$ and $f_2(x) = \dfrac{x_1 - ix_2}{1-x_3}$

Let's study injectivity. Suppose $\underline{f(x) = f(y)}$ $*$

① $\underline{\text{If } x_3 \neq -1 \text{ and } y_3 \neq -1}$ then $*$ yields

$$[f_1(x), 1] = [f_1(y), 1]$$

$$(f_1(x), 1) \sim (f_1(y), 1) \iff (f_1(x), 1) = \lambda(f_1(y), 1)$$

Then $1 = \lambda$ and we find $f_1(x) = f_1(y)$. Hence,

$$\underbrace{\frac{x_1 + ix_2}{1+x_3} = \frac{y_1 + iy_2}{1+y_3}}_{\text{BOB}} \implies \underbrace{\frac{x_1^2 + x_2^2}{(1+x_3)^2} = \frac{y_1^2 + y_2^2}{(1+y_3)^2}}_{**}$$

However, $x, y \in S^2$ thus $x_1^2 + x_2^2 + x_3^2 = 1$ and $y_1^2 + y_2^2 + y_3^2 = 1$ hence $x_1^2 + x_2^2 = 1 - x_3^2 = (1+x_3)(1-x_3)$ and $y_1^2 + y_2^2 = (1+y_3)(1-y_3)$. Returning to $**$ we find

$$\frac{(1+x_3)(1-x_3)}{(1+x_3)^2} = \frac{(1+y_3)(1-y_3)}{(1+y_3)^2}$$

$$\implies (1+y_3)(1-x_3) = (1+x_3)(1-y_3)$$

$$\implies 1 + y_3 - x_3 - y_3x_3 = 1 - y_3 + x_3 - y_3x_3$$

$$\implies 2y_3 = 2x_3 \quad \therefore \underline{x_3 = y_3}.$$

Apply $x_3 = y_3$ to BOB and find $x_1 + ix_2 = y_1 + iy_2$ from which $x_1 = y_1$ and $x_2 = y_2$ follow. Thus $x = y$. (Case ① done)

P6 continued

② If $x_3 \neq -1$ and $y_3 \neq 1$ then $*$ yields

$$[f_1(x), 1] = [1, f_2(y)]$$

Then $\exists \lambda \in \mathbb{C}$ s.t. $(f_1(x), 1) = \lambda(1, f_2(y))$

$$f_1(x) = \lambda \quad \text{and} \quad 1 = \lambda f_2(y) \Rightarrow \underline{f_1(x) f_2(y) = 1} \; \star$$

From $\star$ we find,

$$\underline{\left(\frac{x_1 + i x_2}{1+x_3}\right)\left(\frac{y_1 - i y_2}{1 - y_3}\right) = 1} \text{ JANE}$$

Take the modulus and square the eq$^{\underline{n}}$ above,

$$\left(\frac{x_1^2 + x_2^2}{(1+x_3)^2}\right)\left(\frac{y_1^2 + y_2^2}{(1-y_3)^2}\right) = 1$$

Yet $x, y \in S^2$ thus $x_1^2 + x_2^2 = (1+x_3)(1-x_3)$ and $y_1^2 + y_2^2 = (1+y_3)(1-y_3)$

Consequently,

$$\left[\frac{(1+x_3)(1-x_3)}{(1+x_3)^2}\right]\left[\frac{(1+y_3)(1-y_3)}{(1-y_3)^2}\right] = 1$$

$$\Rightarrow \frac{1-x_3}{1+x_3} = \frac{1-y_3}{1+y_3} \Rightarrow (1-x_3)(1+y_3) = (1-y_3)(1+x_3)$$

$$\Rightarrow 1 - x_3 + y_3 - x_3 y_3 = 1 - y_3 + x_3 - y_3 x_3$$

From JANE,

$$\Rightarrow 2y_3 = 2x_3$$

$$\underbrace{(x_1 + i x_2)}_{z}\underbrace{(y_1 - i y_2)}_{\overline{w}} = (1+x_3)(1-x_3) \qquad \Rightarrow \underline{x_3 = y_3}.$$

$$= 1 - x_3^2$$

$$= x_1^2 + x_2^2 = y_1^2 + y_2^2 \text{ by same algebra.}$$

We have $|z| = |w|$ and $z\overline{w} = |z||w|$ hence $\overline{w} = \frac{|z|^2}{z}$

More to point, $z = |z|e^{i\theta}$ and $w = |w|e^{i\beta}$ have

$$z\overline{w} = |z||w|e^{i\theta}e^{-i\beta} = |z||w| \Rightarrow e^{i\theta}e^{-i\beta} = 1 \Rightarrow e^{i\beta} = e^{i\theta}$$

consequently $z = w \Rightarrow \underline{x_1 = y_1}$ & $\underline{x_2 = y_2}$ $\therefore$ $\underline{x = y}$ // (Case ② done)

P6 continued

(3) If $X_3 \neq 1$ and $y_3 \neq 1$

$$[1, f_2(x)] = [1, f_2(y)] \Rightarrow f_2(x) = f_2(y)$$

Then $\dfrac{x_1 - i x_2}{1 - x_3} = \dfrac{y_1 - i y_2}{1 - y_3} \Rightarrow \dfrac{x_1^2 + x_2^2}{(1-x_3)^2} = \dfrac{y_1^2 + y_2^2}{(1-y_3)^2}$

and similar algebra to (1) and (2) again shows $\underline{x = y}$.

Thus in every possible case (by symmetry of x & y) we find f is injective.

//

SURJECTIVE?

$$\mathbb{C}P^1 = \underbrace{\{[z_1, z_2] \mid z_1 \neq 0\}}_{U_1} \cup \underbrace{\{[z_1, z_2] \mid z_2 \neq 0\}}_{U_2}$$

Consider, $[z_1, z_2] \in U_1$ so $z_1 \neq 0$

We have $[z_1, z_2] = [1, \frac{z_2}{z_1}] = [1, a+ib]$

$[1, f_2(x)] = [1, a+ib]$ how to choose X?

$$\frac{x_1 - i x_2}{1 - x_3} = a + ib \quad \to \frac{x_1}{1-x_3} = a, \quad \to \frac{x_2}{x_3 - 1} = b$$

But, $x_1^2 + x_2^2 = 1 - x_3^2$ so we can consider that as well...

Well, $\dfrac{b}{a} = \dfrac{-x_2}{x_1} \hookrightarrow x_2 = \dfrac{-b}{a} x_1$

$$\frac{x_1}{(1-x_3)(1+x_3)} = \frac{a}{1+x_3} \Rightarrow \frac{x_1}{x_1^2+x_2^2} = \frac{a}{1+x_3} \ \& \ \frac{-x_2}{x_1^2+x_2^2} = b$$

P6 continued

$$\frac{x_1}{1-x_3} = a \hookrightarrow x_1 = a(1-x_3)$$

$$\frac{-x_2}{1-x_3} = b \hookrightarrow x_2 = b(x_3-1)$$

Now $x_1^2 + x_2^2 + x_3^2 = 1$

$$a^2(1-x_3)^2 + b^2(x_3-1)^2 + x_3^2 = 1$$

$$(a^2+b^2)(1-x_3)^2 = 1-x_3^2 = (1-x_3)(1+x_3)$$

$$(a^2+b^2)(1-x_3) = 1+x_3$$

$$a^2+b^2-1 = x_3(1+a^2+b^2)$$

$$\therefore \underline{x_3 = \frac{a^2+b^2-1}{1+a^2+b^2}} \neq 1.$$

Then $x_1 = a(1-x_3) = a\left[1 - \frac{a^2+b^2-1}{1+a^2+b^2}\right] = a\left[\frac{a^2+b^2+1-(a^2+b^2-1)}{1+a^2+b^2}\right]$

Thus $\underline{x_1 = \frac{2a}{1+a^2+b^2}}$. Likewise, $\underline{x_2 = \frac{-2b}{1+a^2+b^2}}$.

Hence for $[1, a+ib] \in U_1$ we argue by construction,

$$f\left(\frac{(2a, -2b, a^2+b^2-1)}{1+a^2+b^2}\right) = [1, a+ib]$$

I imagine $\exists$ a similar calculation to show f maps to all of U_2 as well.

P6 continuing

$$(\overline{\varphi}_2 \circ f \circ \psi_S^{-1})(x,y) = (\overline{\varphi}_2 \circ f)\Big(\underbrace{\frac{1}{1+x^2+y^2}(2x,\, 2y,\, x^2+y^2-1)}_{z}\Big)$$

$x^2+y^2-1 \neq -1$

I think.

$$= \overline{\varphi}_2\big([f_1(z), 1]\big)$$

$$= \overline{\varphi}_2\left(\left[\frac{1}{1+\left(\frac{x^2+y^2-1}{1+x^2+y^2}\right)}\left(\frac{2x+2iy}{1+x^2+y^2}\right),\ 1\right]\right)$$

$$= \overline{\varphi}_2\left(\left[\frac{1}{1+x^2+y^2+x^2+y^2-1}(2x+2iy),\ 1\right]\right)$$

$$= \overline{\varphi}_2\left(\left[\frac{x+iy}{x^2+y^2},\ 1\right]\right)$$

$$= \overline{\varphi}_2\big([x+iy,\ x^2+y^2]\big)$$

$$= \overline{\varphi_2\big([x+iy,\ x^2+y^2]\big)}$$

$$= \overline{\frac{x+iy}{x^2+y^2}}$$

$$= \frac{x-iy}{x^2+y^2}$$

I'll leave $\varphi_1 \circ f \circ \psi_S^{-1}$ and $\overline{\varphi}_2 \circ f \circ \psi_N^{-1}$ to the reader. Let's turn to the question of bijectivity.

P7 Show that $\mathbb{R}P^1$ is diffeomorphic to S^1

Let $f : S^1 \longrightarrow \mathbb{R}P^1$ be defined by

$$f(x) = \begin{cases} [f_1(x), 1] & \text{if } x_2 \neq -1 \\ [1, f_2(x)] & \text{if } x_2 \neq 1 \end{cases} \quad \& \quad \begin{aligned} f_1(x) &= \frac{x_1}{1+x_2} \\ f_2(x) &= \frac{x_1}{1-x_2} \end{aligned}$$

Notice $U_1 = \{ [1, x] \mid x \in \mathbb{R} \}$ and $U_2 = \{ [x, 1] \mid x \in \mathbb{R} \}$ provide sets whose union is $\mathbb{R}P^1 = U_1 \cup U_2$. Recall,

$$[a, b] = \{ t(a,b) \mid t \in \mathbb{R} \} \in \mathbb{R}P^1$$

Also, $x = (x_1, x_2) \in S^1$ means $x_1^2 + x_2^2 = 1$. Let us state a result we will require repeatedly in this sol$^{\underline{n}}$;

Lemma: If $x \in S^1$ then $x = (x_1, x_2)$ has

$$\frac{x_1^2}{(1+x_2)^2} = \frac{1-x_2}{1+x_2} \quad \text{and} \quad \frac{x_1^2}{(1-x_2)^2} = \frac{1+x_2}{1-x_2}$$

Proof: since $x_1^2 + x_2^2 = 1 \Rightarrow x_1^2 = 1 - x_2^2 = (1-x_2)(1+x_2)$ we find both results follow immediately by algebra. //

Is f defined above single-valued?

Clearly f is defined for each $x \in S^1$ and $f(x) \in \mathbb{R}P^1$.

- Consider the overlap of the cases $x_2 \neq 1$ and $x_2 \neq -1$

$\boxed{x_1 \neq 0}$

$$\left[\frac{x_1}{1+x_2}, 1 \right] = \left[\frac{1+x_2}{x_1} \frac{x_1}{1+x_2}, \frac{1+x_2}{x_1} \right]$$

Lemma

$$= \left[1, \frac{1+x_2}{x_1} \left(\frac{x_1^2 (1-x_2)}{(1-x_2)^2 (1+x_2)} \right) \right]$$

$$= \left[1, \frac{x_1}{1-x_2} \right]$$

Thus $[f_1(x), 1] = [1, f_2(x)]$ in the case $x_1 \neq 0$. If $x_1 = 0$ then $x_1^2 + x_2^2 = x_2^2 = 1 \Rightarrow x_2 = \pm 1$ thus we're done. //

P7 continued

Suppose $f(x) = f(y)$, we seek to show f injective.

① If $x_2 \neq -1$ and $y_2 \neq -1$ then

$$[f_1(x), 1] = [f_1(y), 1]$$

$$\Rightarrow (f_1(x), 1) = \lambda (f_1(y), 1)$$

$$\Rightarrow f_1(x) = f_1(y)$$

$$\Rightarrow \underline{\frac{x_1}{1+x_2} = \frac{y_1}{1+y_2}} \; *$$

$$\Rightarrow \frac{x_1^2}{(1+x_2)^2} = \frac{y_1^2}{(1+y_2)^2}$$

$$\Rightarrow \frac{1-x_2}{1+x_2} = \frac{1-y_2}{1+y_2} \quad \text{by Lemma}$$

$$\Rightarrow (1-x_2)(1+y_2) = (1-y_2)(1+x_2)$$

$$\Rightarrow 1 - x_2 + y_2 - x_2 y_2 = 1 - y_2 + x_2 - y_2 x_2$$

$$\Rightarrow 2y_2 = 2x_2 \Rightarrow \underline{x_2 = y_2}.$$

Then returning to $*$ we find $x_1 = y_1$ and $x_2 = y_2$ $\therefore$ $x = y$.

② If $x_2 \neq 1$ and $y_2 \neq 1$ a similar argument to ① shows $x = y$

③ If $x_2 \neq 1$ and $y_2 \neq -1$ then

$$[1, f_2(x)] = [f_1(y), 1] \Rightarrow (1, f_2(x)) = \lambda (f_1(y), 1)$$

Thus $1 = \lambda f_1(y)$ and $f_2(x) = \lambda \Rightarrow 1 = f_2(x) f_1(y)$ thus

$$\left(\frac{x_1}{1-x_2}\right)\left(\frac{y_1}{1+y_2}\right) = 1 \Rightarrow \frac{x_1^2}{(1-x_2)^2}\,\frac{y_1^2}{(1+y_2)^2} = 1$$

$$\Rightarrow \left(\frac{1+x_2}{1-x_2}\right)\left(\frac{1-y_2}{1+y_2}\right) = 1 \Rightarrow (1+x_2)(1-y_2) = (1-x_2)(1+y_2)$$

$$\Rightarrow x_2 = y_2 \Rightarrow x_1 = y_1$$

Hence $x = y$ and we've shown f is injective. //

P7 continued

To show f is surjective consider $[1, u] \in \mathbb{R}P^1$. Notice $x_1 = \frac{2u}{1+u^2}$ and $x_2 = \frac{u^2-1}{1+u^2}$ has $x_1^2 + x_2^2 = \frac{4u^2}{(1+u^2)^2} + \frac{u^4 - 2u^2 + 1}{(1+u^2)^2} = \frac{(1+u^2)^2}{(1+u^2)^2} = 1$ thus

$\left(\frac{2u}{1+u^2}, \frac{u^2-1}{1+u^2}\right) \in S^1$. Moreover,

$$f\left(\frac{2u}{1+u^2}, \frac{u^2-1}{1+u^2}\right) = \left[1, \frac{\frac{2u}{1+u^2}}{1 - \frac{u^2-1}{1+u^2}}\right]$$

$$= \left[1, \frac{2u}{1+u^2-(u^2-1)}\right]$$

$$= [1, u].$$

Similarly, if $[u, 1] \in \mathbb{R}P^1$

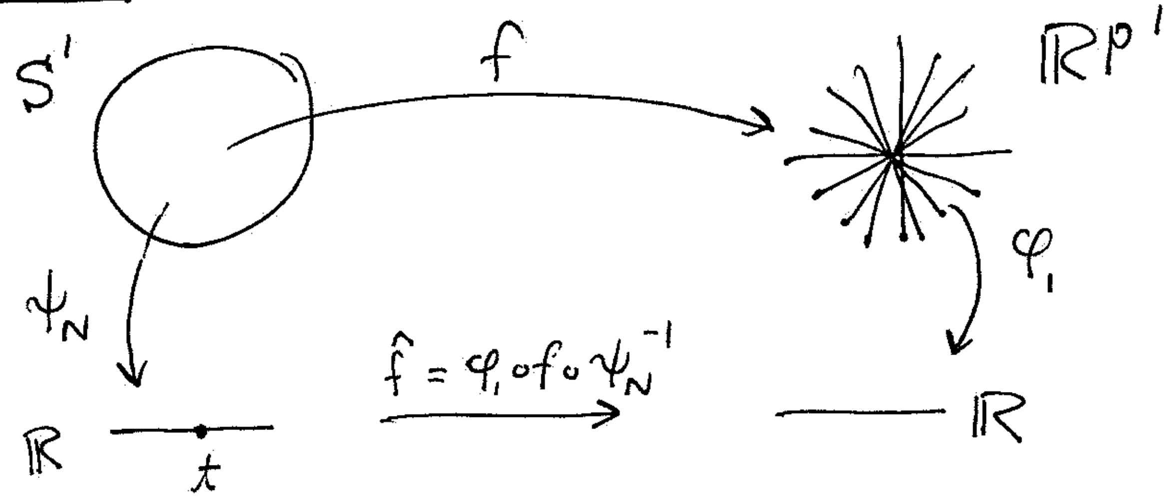
$$f\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right) = \left[\frac{\frac{2u}{1+u^2}}{1 + \frac{1-u^2}{1+u^2}}, 1\right]$$

$$= \left[\frac{2u}{1+u^2+1-u^2}, 1\right]$$

$$= [u, 1].$$

Hence f is a bijection from S^1 to $\mathbb{R}P^1$. It remains to show f is smooth.

P7 continued



$$\psi_N^{-1}(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \qquad \left(\frac{1-t^2}{1+t^2} \neq -1\right)$$

$$\varphi_1([1,u]) = u$$

$$\hat{f}(t) = \varphi_1\left(f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\right)$$

$$= \varphi_1\left(\left[f_1(\psi_N^{-1}(t)), 1\right]\right)$$

$$= \varphi_1\left(\left[\frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}, 1\right]\right)$$

$$= \varphi_1\left(\left[\frac{2t}{2}, 1\right]\right)$$

$$= \varphi_1\left(\left[1, 1/t\right]\right)$$

$$= \frac{1}{t}$$

$\mathrm{dom}\,\varphi_1 = U_1 = \{[1,x] \mid x \in \mathbb{R}\}$ and $\psi_N(f^{-1}(U_1) \cap U_N) \ni t$ implies $t \neq 0$ thus $t \longmapsto \frac{1}{t}$ smooth.

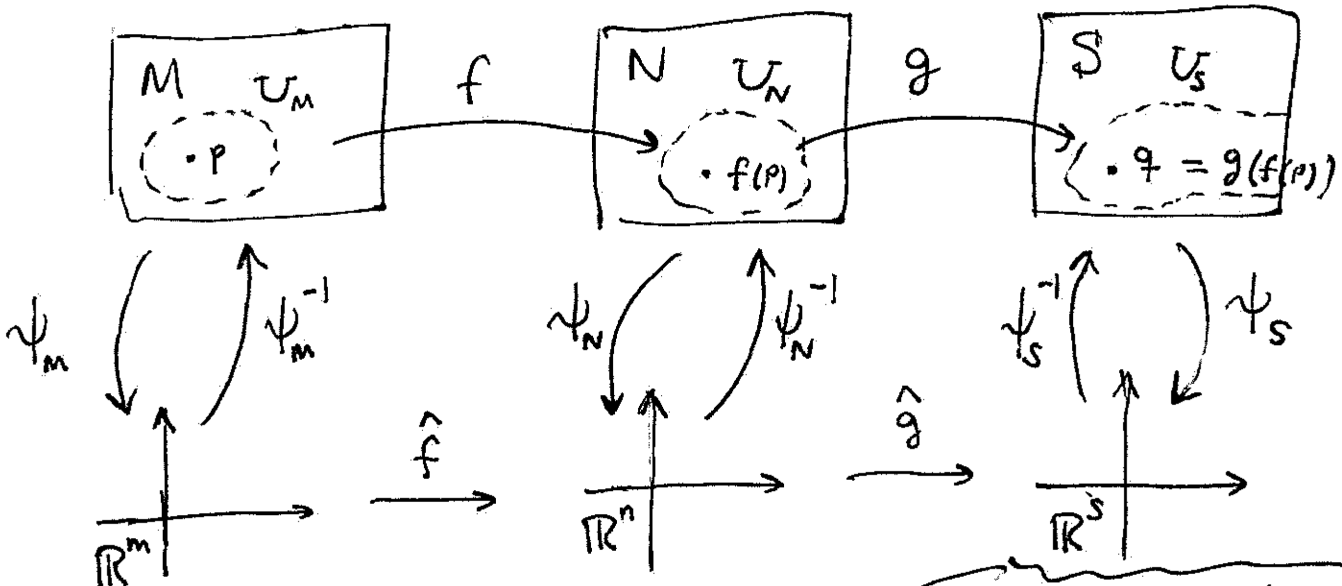
I'm done, I leave other overlaps to reader.

P8) Exercise 1.54

Show composition of C^r maps is C^r

$$f : M \longrightarrow N \qquad g : N \longrightarrow S$$

Suppose f and g are C^r maps. Consider $g \circ f : M \longrightarrow S$



Remark: a picky sol^{n} requires attention to chart domains which would need appropriate restriction to fit context.

$$\begin{aligned} \widehat{g \circ f} &= \psi_S \circ (g \circ f) \circ \psi_M^{-1} \\ &= \left(\psi_S \circ g \circ \psi_N^{-1}\right) \circ \left(\psi_N \circ f \circ \psi_M^{-1}\right) \\ &= \hat{g} \circ \hat{f} \end{aligned}$$

Hence $\widehat{g \circ f}$ is C^r as both $\hat{g}$ and $\hat{f}$ are r-times differentiable maps on Euclidean space and the chain rule from Advanced Calculus implies

$$J_{\widehat{g \circ f}} = \underbrace{J_{\hat{g}}\, J_{\hat{f}}}_{}$$

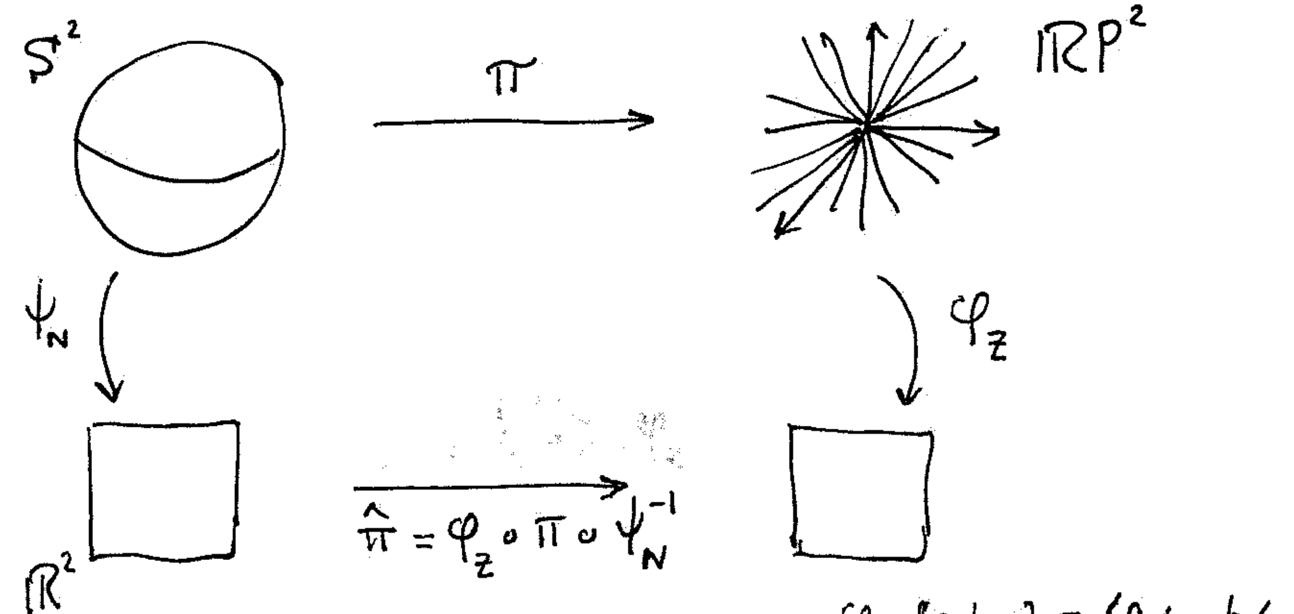
r-continuous partial derivative components $\Rightarrow$ same true for LHS components.

P9 ex. 1.56

$\pi : S^2 \longrightarrow \mathbb{R}P^2$
given by mapping point (x, y, z) to line from $(0,0,0)$ to (x,y,z)

$$\pi(x,y,z) = [x, y, z]$$

show π is smooth (check $\hat{\pi}$ for some choice of charts)



$$\varphi_Z [a,b,c] = (a/c,\ b/c)$$

$$\psi_N^{-1}(x) = \frac{1}{1+\|x\|^2}\left(2x_1, 2x_2, 1-\|x\|^2\right)$$

$$\hat{\pi}(u,v) = \varphi_Z\left(\pi\left(\frac{1}{1+u^2+v^2}(2u, 2v, 1-u^2-v^2)\right)\right)$$

$$= \varphi_Z\,[2u,\ 2v,\ 1-u^2-v^2]$$

$[k(a,b,c)] = [a,b,c]$

$$= \left(\frac{2u}{1-u^2-v^2},\ \frac{2v}{1-u^2-v^2}\right)$$

- Consider the context, dom $\hat{\pi}$ includes (u,v) for which $\pi(\psi_N^{-1}(u,v)) \in \mathrm{dom}(\varphi_Z) = U_Z = \{[x, y, 1]\}$

apparently this means $1 - \|(u,v)\|^2 \neq 0 \Rightarrow 1 \neq u^2+v^2 \Rightarrow$ $\hat{\pi}$ smooth.

P10 Exercise 1.60 (Example actually, but would make nice exercise!)

$r_\theta : S^2 \longrightarrow S^2$

$$r_\theta(x,y,z) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ z)$$

is diffeomorphism.

$$(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 + z^2 = x^2+y^2+z^2 = 1$$

Thus $r_\theta(x,y,z) \in S^2$ for $(x,y,z) \in S^2$. In fact,

$$r_\theta(x,y,z) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

thus r_θ is a linear map for any choice of θ and is thus smooth on $\mathbb{R}^3$, indeed smooth when restricted to S^2. The inverse of r_θ is $r_{-\theta}$ and you can easily check $r_\theta \circ r_{-\theta} = r_{\theta-\theta} = r_0 = Id$.

(r_θ is the rotation by angle θ about the z-axis, it's geometrically obvious it maps S^2 to S^2)

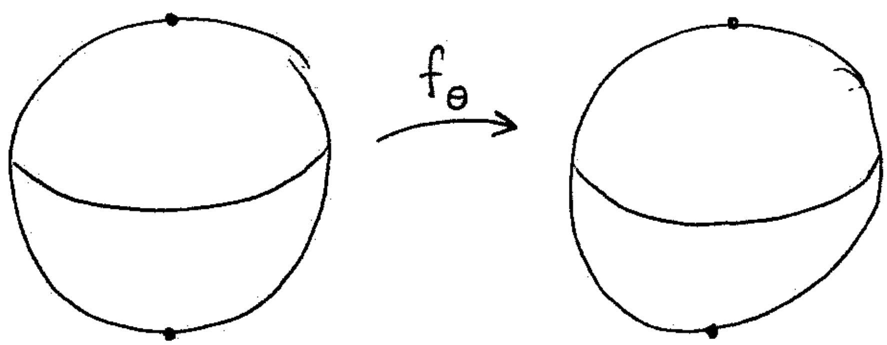
P11 Exercise 1.61

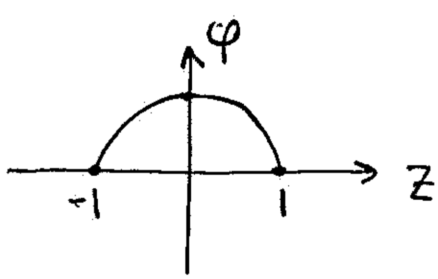
Suppose $0 < \theta < 2\pi$. Consider the map $f: S^2 \to S^2$ given by $f_\theta(x,y,z) = (x\cos((1-z^2)\theta) - y\sin((1-z^2)\theta),$
$x\sin((1-z^2)\theta) + y\cos((1-z^2)\theta), z)$

Is this map a diffeomorphism? Try to picture it.

$$f_\theta(x,y,z) = \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \varphi = (1-z^2)\theta$$

Can check $A^TA = I$ thus $\|f_\theta(x,y,z)\|^2 = \|(x,y,z)\|^2$ so at least $f: S^2 \to S^2$ is well-defined.





For $\theta = \pi$ the map f_θ
- fixes $(0,0,\pm 1)$
- rotates equator by π

Actually, $f_\theta(0,0,\pm 1) = (0,0,\pm 1)$ $\forall \theta \in (0, 2\pi)$.

f_θ^{-1} = same map, but rotate by $-\varphi$ $(\varphi = (1-z^2)\theta)$

$$f_\theta^{-1}(x,y,z) = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

P11 continued

Is it a diffeomorphism?

$$(\psi_N \circ f_\theta \circ \psi_S^{-1})(x,y) =$$

$$= \psi_N\left(f_\theta\left(\frac{1}{1+x^2+y^2}(2x,\ 2y,\ x^2+y^2-1)\right)\right)$$

$$= \psi_N\left(f_\theta\left(\underbrace{\frac{2x}{1+x^2+y^2}}_{u},\ \underbrace{\frac{2y}{1+x^2+y^2}}_{v},\ \underbrace{\frac{x^2+y^2-1}{1+x^2+y^2}}_{w}\right)\right)$$

$$= \psi_N\left(\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}\right) \qquad \varphi = (1-w^2)\theta$$

$$= \psi_N\left(\left(\underbrace{u\cos\varphi - v\sin\varphi}_{x_1},\ \underbrace{u\sin\varphi + v\cos\varphi}_{x_2},\ \underbrace{w}_{x_3}\right)\right)$$

$$= \psi_N(x_1, x_2, x_3)$$

$$= \frac{1}{1+x_3}(x_1,\ x_2)$$

$$= \frac{1}{1+w}\left(u\cos\varphi - v\sin\varphi,\ u\sin\varphi + v\cos\varphi\right)$$

$$= \frac{1}{1+\frac{x^2+y^2-1}{1+x^2+y^2}}\left(\frac{2x}{1+x^2+y^2}\cos\varphi - \frac{2y}{1+x^2+y^2}\sin\varphi,\ \frac{2x}{1+x^2+y^2}\sin\varphi + \frac{2y}{1+x^2+y^2}\cos\varphi\right)$$

$$= \frac{1}{1+x^2+y^2+x^2+y^2-1}\left(2x\cos\varphi - 2y\sin\varphi,\ 2x\sin\varphi + 2y\cos\varphi\right)$$

$$= \frac{1}{x^2+y^2}\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \qquad \varphi = \left(1-\left(\frac{x^2+y^2-1}{1+x^2+y^2}\right)^2\right)\theta$$

If $x=y=0$ is in domain of $\hat{f}_\theta$ then not smooth. Is it?

P12 Problem 4 from p. 51 of Jeffrey Lee's MDG

Let M_1, M_2, M_3 be smooth manifolds

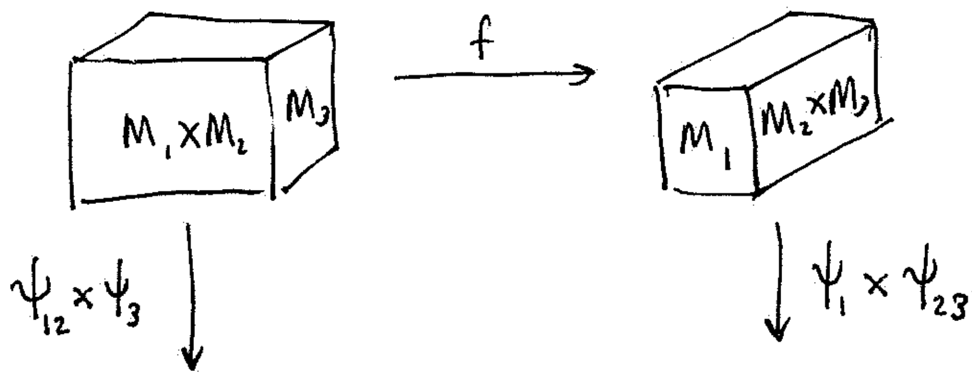
(a.) show $(M_1 \times M_2) \times M_3 \cong M_1 \times (M_2 \times M_3)$

(b.) show $f : M \longrightarrow M_1 \times M_2$ is C^∞ iff $Pr_1 \circ f : M \longrightarrow M_1$ and $pr_2 \circ f : M \longrightarrow M_2$ are both C^∞

(a.) $f(((P_1, P_2), P_3)) = (P_1, (P_2, P_3))$ is a <u>diffeomorphism</u> (claim)

with inverse $f^{-1}(P_1, (P_2, P_3)) = ((P_1, P_2), P_3)$.

Why is f smooth? (I've shown f is bijection)



$$
\begin{aligned}
(\psi_{12} \times \psi_3)((P_1, P_2), P_3) &= (\psi_{12}(P_1, P_2), \psi_3(P_3)) \\
&= ((x_1(P_1), x_2(P_2)), x_3(P_3)) \\
&= (x_1(P_1), x_2(P_2), x_3(P_3)) \\
&= (\psi_1 \times \psi_{23})(P_1, (P_2, P_3))
\end{aligned}
$$

usual identification

Sorry, I'm not sure (a.) can be understood w/o delving deeply into set theory & the very construction of ordered pairs... I'm mostly interested in (b.)

P12 continued

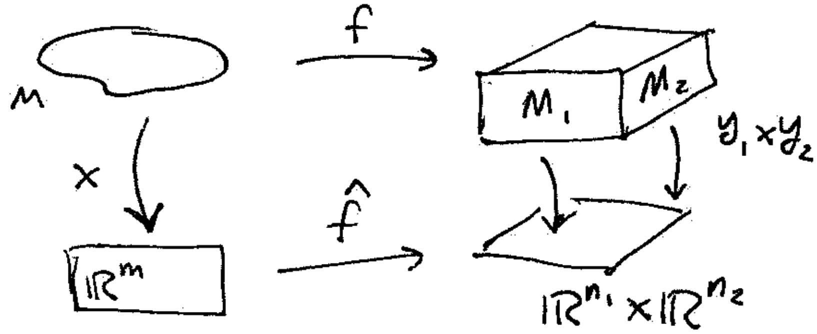
$\Rightarrow$] Suppose $f: M \longrightarrow M_1 \times M_2$ is C^∞

then if x is a chart on M at p and

$y_1 \times y_2$ is a chart on $M_1 \times M_2$ at $f(p)$ then

$\hat{f} = (y_1 \times y_2) \circ f \circ x^{-1}$

is smooth from subset of $\mathbb{R}^m$ to $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$



Observe

$$\hat{f}(u) = (y_1 \times y_2)(f(x^{-1}(u))) \qquad f = (f_1, f_2)$$

$$= (y_1(f_1(x^{-1}(u))),\ y_2(f_2(x^{-1}(u)))) \qquad (*)$$

is smooth (by assumption). However,

$$(pr_1 \circ f)(p) = pr_1(f_1(p), f_2(p)) = f_1(p)$$

$$(pr_2 \circ f)(p) = f_2(p)$$

Then the local coordinate rep. of $pr_j \circ f$ is

$$y_1 \circ (pr_1 \circ f) \circ x^{-1}(u) = y_1(f_1(x^{-1}(u)))$$

$$y_2 \circ (pr_2 \circ f) \circ x^{-1}(u) = y_2(f_2(x^{-1}(u)))$$

which are smooth from $*$. Ok, to be more logical, if we look at any local coord. rep of $pr_1 \circ f$ or $pr_2 \circ f$ then we can see these as a component of a local coord. rep of $f = (f_1, f_2)$ based on the same charts.

$\Leftarrow$] Same calculation, smoothness of each comp. function in Euclidean space $\Rightarrow$ smoothness of map.

P13 For product manifold $M \times N$ we have
$pr_1 : M \times N \longrightarrow M$ and $pr_2 : M \times N \longrightarrow N$
$pr_1(x, y) = x$ $\quad$ $pr_2(x, y) = y$
Suppose $f_1 : P \longrightarrow M$ and $f_2 : P \longrightarrow N$ are smooth.
Show $(f_1, f_2) : P \longrightarrow M \times N$ given by $(f_1, f_2)(p) = (f_1(p), f_2(p))$
is the unique smooth map s.t $pr_1 \circ (f_1, f_2) = f_1$
and $pr_2 \circ (f_1, f_2) = f_2$

Let $h = (f_1, f_2) : P \longrightarrow M \times N$ as given above

consider chart x on P with chart $y \times z$ on $M \times N$
such that p_0 in $\mathrm{dom}(x) \subseteq P$ whereas
$f(p_0) = (q_1, q_2)$ where q_1 in $\mathrm{dom}(y) \subseteq M$ and $q_2 \in \mathrm{dom}(z) \subseteq N$
hence $y \times z$ is chart on $M \times N$ about $f(p_0) = q$

$$
\begin{aligned}
\hat{h} &= (y \times z) \circ (f_1, f_2) \circ x^{-1} \\
&= (y \times z) \circ (f_1 \circ x^{-1}, f_2 \circ x^{-1}) \\
&= y(f_1 \circ x^{-1}), z(f_2 \circ x^{-1})) \\
&= (y \circ f_1 \circ x^{-1}, z \circ f_2 \circ x^{-1})
\end{aligned}
$$

This is smooth since f_1 & f_2 are given to be smooth.
Moreover, $pr_1 \circ (f_1, f_2) = f_1$ and $pr_2 \circ (f_1, f_2) = f_2$.

Uniqueness?

If $\gamma : P \longrightarrow M \times N$ is smooth s.t. $pr_1 \circ \gamma = f_1$
and $pr_2 \circ \gamma = f_2$ then
$$\gamma(p_0) = (pr_1(\gamma(p_0)), pr_2(\gamma(p_0))) = (f_1(p_0), f_2(p_0)) = f(p_0).$$

$\boxed{P14}$ Problem 16 from Jeffrey Lee's MDG p. 52

Show that if S is regular k-dim'l submanifold of an n-manifold M, then we may cover S by special single-slice charts from the atlas of M which are of form

$$x : U \longrightarrow V_1 \times V_2 \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$$

with $x(U \cap S) = V_1 \times \{0\}$ for some open sets $V_1 \subseteq \mathbb{R}^k$ and $V_2 \subseteq \mathbb{R}^{n-k}$. Show we may arrange V_1 & V_2 to both be Euclidean balls or cubes.

By def$^{\underline{n}}$ of regular k-dim'l submanifold, at each point in S there exists chart on M for which $(U, x) \in \mathcal{A}_M$ and

$$x(U \cap S) = x(U) \cap (\mathbb{R}^k \times \{c\})$$

Let $\bar{x}(p) = x(p) - (0, c)$ where $c \in \mathbb{R}^{n-k}$ then $\bar{x}(U \cap S) = \bar{x}(U) \cap (\mathbb{R}^k \times \{0\})$. Since U open and $\bar{x}$ is homeomorphism we have $\bar{x}(U)$ open thus $\pi_1(\bar{x}(U)) = V_1 \subseteq \mathbb{R}^k$ open and $\bar{x}(U \cap S) = V_1 \times \{0\}$

Finally, we may shrink $\pi_1(\bar{x}(U)) = V_1$ and $\pi_2(\bar{x}(U)) = V_2$ to be ball or cube centered at point in question since V_1 & V_2 are open sets in Euclidean space.

<u>Remark</u>: sorry, I'm not too pleased with the last couple problems. It's hard to locate the problem, I'll try to give you more clear tasks in future. $\boxed{P15}$ is nice though ↲

[P15] Let $f, g \in C^\infty(M)$ and suppose (U, x) is chart on M

(a.) Show $\frac{\partial}{\partial x^i}\left[f+g\right] = \frac{\partial f}{\partial x^i} + \frac{\partial g}{\partial x^i}$

Let $p \in U$ and denote Cartesian coordinates for $\mathbb{R}^m$ by $u^1, u^2, \dots, u^m$. We define,

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}\bigg|_{x(p)} (f \circ x^{-1})(u)$$

Thus,

$$\begin{aligned}
\frac{\partial}{\partial x^i}\left[f+g\right](p) &= \frac{\partial}{\partial u^i}\bigg|_{x(p)} (f+g)\circ x^{-1}(u) \\
&= \frac{\partial}{\partial u^i}\bigg|_{x(p)} \left[(f\circ x^{-1})(u) + (g\circ x^{-1})(u)\right] \\
&= \frac{\partial}{\partial u^i}\bigg|_{x(p)} f\circ x^{-1}(u) + \frac{\partial}{\partial u^i}\bigg|_{x(p)} g\circ x^{-1}(u) \\
&= \frac{\partial f}{\partial x^i}(p) + \frac{\partial g}{\partial x^i}(p)
\end{aligned}$$

Since $p \in U$ was arbitrary the result follows.

(b.)
$$\begin{aligned}
\frac{\partial}{\partial x^i}\left[fg\right](p) &= \frac{\partial}{\partial u^i}\bigg|_{x(p)} (fg)\circ x^{-1}(u) \\
&= \frac{\partial}{\partial u^i}\bigg|_{x(p)} (f\circ x^{-1})(u)\,(g\circ x^{-1})(u) \\
&= \left[\frac{\partial}{\partial u^i}\bigg|_{x(p)} (f\circ x^{-1})(u)\right](g\circ x^{-1})(x(p)) + \\
&\quad + (f\circ x^{-1})(x(p))\,\frac{\partial}{\partial u^i}\bigg|_{x(p)} (g\circ x^{-1})(u) \\
&= \frac{\partial f}{\partial x^i}(p)\,g(p) + f(p)\,\frac{\partial g}{\partial x^i}(p) \quad .//
\end{aligned}$$