Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. 5pts per problem here.

Problem 16 Let M be a manifold and (x, U) a chart. Prove U is diffeomorphic to $x(U)$.

Problem 17 Show the chain-rule on a manifold is implied by the chain-rule for smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Problem 18 SM Problem 2-3, page 48.

Problem 19 SM Problem 2-4, page 48.

Problem 20 SM Exercise 3.5, page 54.

Problem 21 SM Exercise 3.7, page 56

Problem 22 SM Exercise 3.17, page 65

Problem 23 SM Problem 3-2, page 75

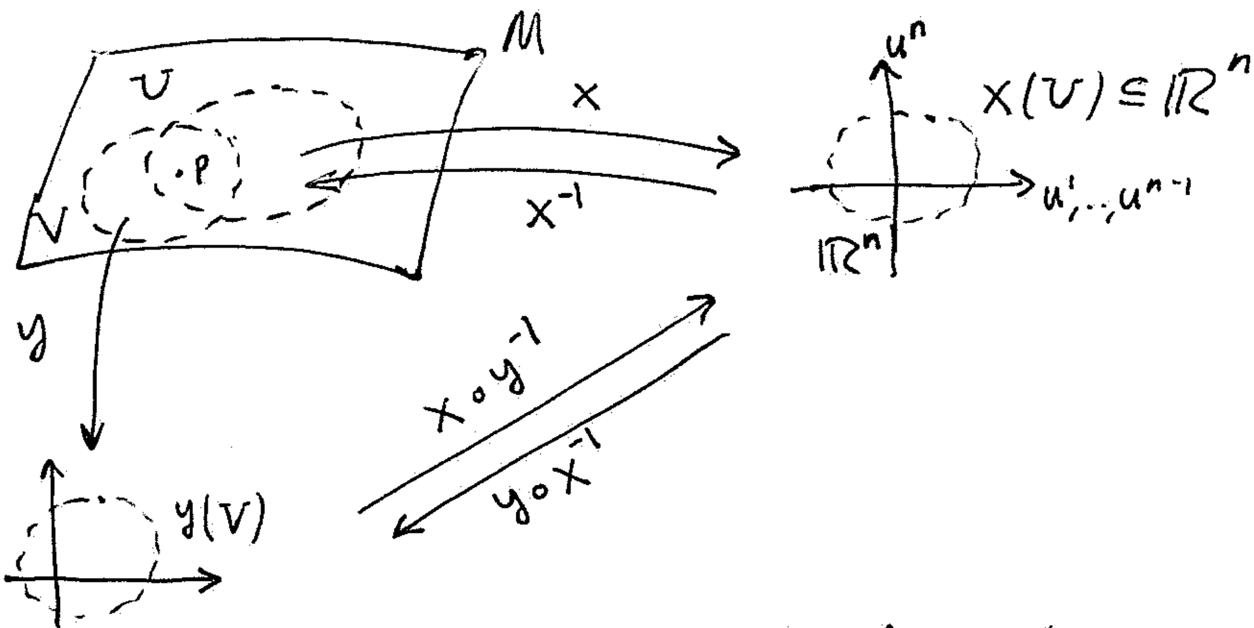
Problem 24 SM Problem 3-4, page 75

Problem 25 SM Problem 3-6, page 75

Problem 26 Explain the four different views of the tangent space to M and state the isomorphisms between them (you don't have to prove they're actually isomorphisms, nor do you have to state all possible isomorphisms, I'm just looking for 4 objects and 3 connecting isomorphisms)

MANIFOLDS MISSION 2 SOLUTION (from John Lee's text Smooth Manifolds) (or me)

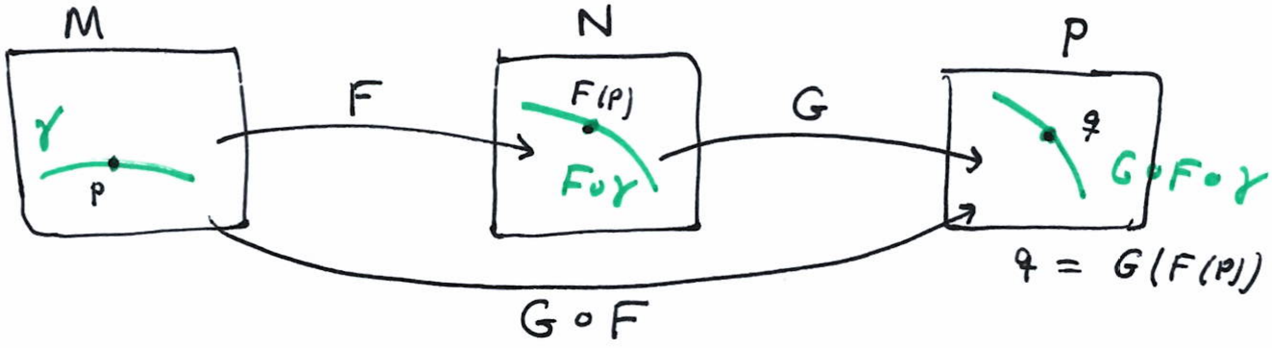
$\boxed{P16}$ Let M be a manifold and (x, U) a chart prove U is diffeomorphic to $x(U)$



Notice x is bijection and it's local coord. rep. $x \circ y^{-1}$ is smooth by defn of manifold. Likewise x^{-1} has local coord. rep. $y \circ x^{-1}$ which is smooth. But y was arbitrary chart, so all local coord. rep of x and x^{-1} are smooth, hence x is diffeomorphism onto its image $x(U)$.

Remark: charts are local diffeomorphisms from M to $\mathbb{R}^n$.

$\boxed{P17}$ see my posted pdf for the gory coordinate-based approach as I explained to Ernesto over an hour long conversation. Or, use the much easier



$[\gamma] \longleftrightarrow v = \gamma'(0)$ we can view tangent vectors as velocity vectors to curves and $[\gamma]$ denotes the equivalence class where $\gamma_1 \sim \gamma_2$ iff $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1'(0) = \gamma_2'(0)$. To understand $\gamma'(0)$ as derivation we define for $f \in C^\infty(M)$

$$\gamma'(0)[f] = (f\circ\gamma)'(0)$$

$$\gamma'(t) = \sum_{i=1}^{n} \frac{d(x^i\circ\gamma)}{dt} \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$$

$$\gamma'(0)f = \sum_{i=1}^{n} \frac{d(x^i\circ\gamma)}{dt}(0)\,\frac{\partial f}{\partial x^i}(\gamma(0)) = \left.\frac{d}{dt}\right|_{t=0}(f\circ\gamma)$$

Just reminding us of some formulas. We <u>defined</u>

$$(dF_p(v))(g) = v[g\circ F] \qquad \left(\begin{array}{l} g \in C^\infty(N) \\ g\circ F \in C^\infty(M) \end{array}\right)$$

One can prove (see Lee)

$$dF([\gamma]) = [F\circ\gamma]$$

Consider then,

$$d(G\circ F)([\gamma]) = [G\circ F\circ\gamma]$$

$$\begin{aligned}(dG\circ dF)([\gamma]) &= dG(dF([\gamma])) \\ &= dG([F\circ\gamma]) \\ &= [G\circ F\circ\gamma]\end{aligned}$$

Thus $d(G\circ F) = dG\circ dF$ (chain-rule for manifolds)

P17

$$F : M \longrightarrow N$$

$$dF\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \sum_{j=1}^{n} \frac{\partial(y^j \circ F)}{\partial x^i}(p)\, \frac{\partial}{\partial y^j}\bigg|_{F(p)}$$

using chart x at p for M
and chart y at $F(p)$ for N
If chart z at $G(F(p))$ for P then, using $q = G(F(p))$,

$$dG\left(\frac{\partial}{\partial y^j}\bigg|_{F(p)}\right) = \sum_{k=1}^{p} \frac{\partial(z^k \circ G)}{\partial y^j}(F(p))\, \frac{\partial}{\partial z^k}\bigg|_q$$

Thus,

$$\begin{aligned} dG\left(dF\left(\frac{\partial}{\partial x^i}\bigg|_p\right)\right) &= dG\left(\sum_{j=1}^{n} \frac{\partial(y^j \circ F)}{\partial x^i}(p)\, \frac{\partial}{\partial y^j}\bigg|_{F(p)}\right) \\ &= \sum_{j=1}^{n} \frac{\partial(y^j \circ F)}{\partial x^i}(p)\, dG\left(\frac{\partial}{\partial y^j}\bigg|_{F(p)}\right) \\ &= \sum_{k=1}^{p}\sum_{j=1}^{n} \frac{\partial(y^j \circ F)}{\partial x^i}(p)\, \frac{\partial(z^k \circ G)}{\partial y^j}(F(p))\, \frac{\partial}{\partial z^k}\bigg|_q \end{aligned}$$

Likewise,

$$d(G \circ F)\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \sum_{k=1}^{p} \frac{\partial(z^k \circ (G \circ F))}{\partial x^i}(p)\, \frac{\partial}{\partial z^k}\bigg|_q$$

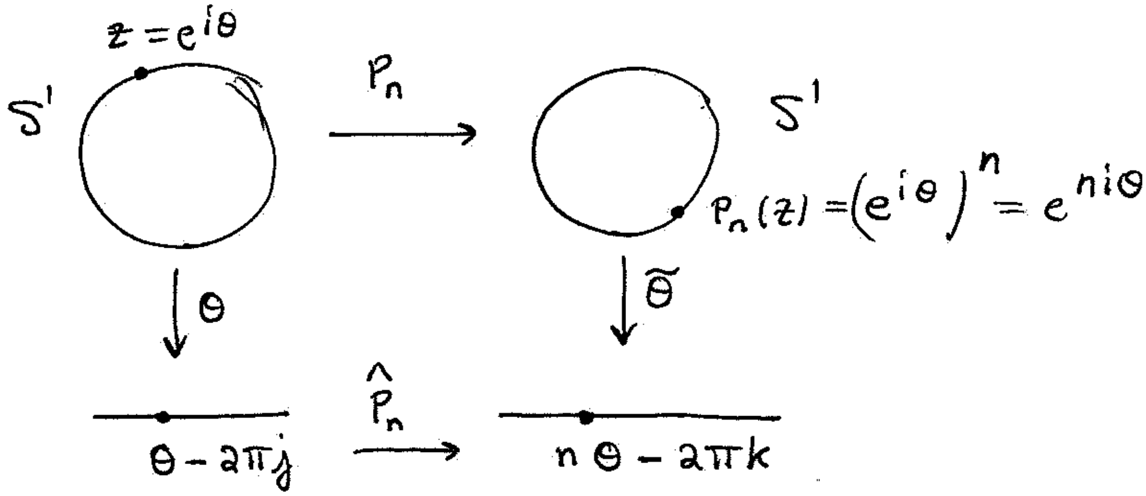
If we use the chain-rule proved last page and the theorem from Lee to connect formalisms of derivations and curves then it follows,

$$\boxed{\frac{\partial}{\partial x^i}\left[z^k \circ (G \circ F)\right] = \sum_{j=1}^{n} \frac{\partial(y^j \circ F)}{\partial x^i}(p)\, \frac{\partial(z^k \circ G)}{\partial y^j}(F(p))}$$

Seems to me this is the chain-rule for chart-derivatives.
(again, see the pdf on the website for a sol$^{\underline{n}}$ to question as asked)

P18) SM problem 2-3 pg. 48

(a.) $P_n : S^1 \longrightarrow S^1$, the <u>$n^{th}$ power map</u> $P_n(z) = z^n$



Here $j, k \in \mathbb{Z}$ are fixed and characterize the particular choice of angle chart on S^1, clearly $\hat{P}_n$ is smooth thus P_n is smooth.

<u>Remark:</u> if you went the ψ_N, ψ_S chart route for (a.) then $P_n(e^{i\theta}) = \cos(n\theta) + i\sin(n\theta)$ would be helpful.

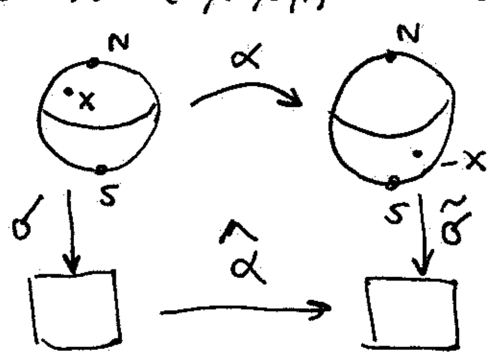
(b.) $\alpha : S^n \longrightarrow S^n$, the <u>antipodal map</u> $\alpha(x) = -x$
This is the restriction of a linear map on $\mathbb{R}^{n+1}$ to the submanifold S^n which by theory of later Chpt. is clearly smooth. That said, let's face the music here,

Following notation from Lee p. 30

$$\sigma : S^n - \{N\} \longrightarrow \mathbb{R}^n \qquad \sigma(x^1, .., x^{n+1}) = \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}$$

$$\tilde{\sigma}(x) = -\sigma(-x) \quad \forall x \in S^n - \{S\}$$

where $N = (0, .., 0, 1)$ and $S = (0, .., 0, -1) \in \mathbb{R}^{n+1}$



$x \in S^n - \{N\}$

$-x \in S^n - \{S\}$

$\hat{\alpha}(x) = \tilde{\sigma}(\alpha(\sigma^{-1}(x)))$

see over,

P18 continued

(b.) $$\hat{\alpha}(u) = \tilde{\sigma}(\alpha(\sigma^{-1}(u)))$$

$$= \tilde{\sigma}\left(\alpha\left(\frac{1}{|u|^2+1}(2u^1, \dots, 2u^n, |u|^2-1)\right)\right)$$

$$= \tilde{\sigma}\left(\frac{-1}{|u|^2+1}(2u^1, \dots, 2u^n, |u|^2-1)\right)$$

$$= -\sigma\left(\frac{1}{|u|^2+1}(2u^1, \dots, 2u^n, |u|^2-1)\right)$$

$$= \frac{-1}{1-\left(\frac{|u|^2-1}{|u|^2+1}\right)}\left(\frac{(2u^1, \dots, 2u^n)}{|u|^2+1}\right) \leftarrow \text{oops almost forgot!}$$

$$= \frac{-1}{|u|^2+1-(|u|^2-1)}(2u^1, \dots, 2u^n)$$

$$= -u \qquad (\text{funny, and smooth } u \mapsto -u)$$

I'll look at using σ in both S^n's as well,

$$(\sigma \circ \alpha \circ \sigma^{-1})(u) = \sigma(\alpha(\sigma^{-1}(u)))$$

$$= \sigma(-\sigma^{-1}(u))$$

$$= \sigma\left(\frac{-1}{|u|^2+1}(2u^1, \dots, 2u^n, |u|^2-1)\right)$$

$$= \sigma\left(\frac{-2u^1}{|u|^2+1}, \dots, \frac{-2u^n}{|u|^2+1}, \frac{1-|u|^2}{|u|^2+1}\right)$$

$$= \frac{1}{1-\left(\frac{1-|u|^2}{|u|^2+1}\right)}\left(\frac{-2u^1}{|u|^2+1}, \dots, \frac{-2u^n}{|u|^2+1}\right)$$

$$= \frac{1}{|u|^2+1-(1-|u|^2)}(-2u^1, \dots, -2u^n)$$

$$= \frac{-1}{|u|^2}u \qquad u \mapsto \frac{-u}{|u|^2} \text{ smooth for } u \neq 0.$$

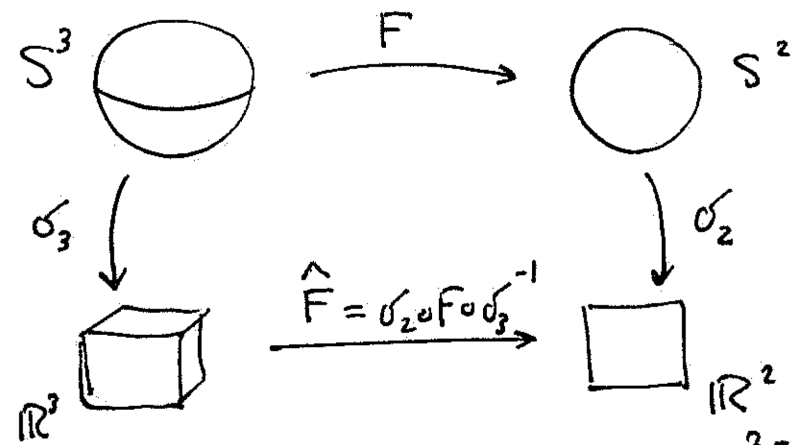
Notice $\sigma^{-1}(0) = S$ then $\alpha(\sigma^{-1}(0)) = -S = N \Rightarrow u \neq 0$ for this case.

P18 continued

(c.) $F : S^3 \longrightarrow S^2$ as defined by

$$F(z,w) = (z\bar{w} + w\bar{z},\ iw\bar{z} - iz\bar{w},\ z\bar{z} - w\bar{w})$$

where S^3 is viewed as subset of $\mathbb{C}^2 = \mathbb{R}^4$ in sense $S^3 = \{(z,w) \mid |z|^2 + |w|^2 = 1\}$



Note,
$F(\lambda(z,w)) = \lambda^2 F(z,w)$ for $\lambda \in \mathbb{R}$

$\lambda = a^2+b^2+c^2+1 \in \mathbb{R}$

$$\hat{F}(a,b,c) = \sigma_2\left(F\left(\frac{(2a,2b,2c,\overbrace{a^2+b^2+c^2-1}^{\lambda-2})}{a^2+b^2+c^2+1}\right)\right)$$

$$= \sigma_2\big(F(\lambda(2(a+ib),\ 2c + i(\lambda-2)))\big)$$

$$= \sigma_2\big(\lambda^2 F(\underbrace{2(a+ib)}_{z},\ \underbrace{2c+i(\lambda-2)}_{w})\big)$$

$$z\bar{w} = 2(a+ib)(2c - i(\lambda-2))$$
$$= 4ac + 2b(\lambda-2) + i(4bc - 2a(\lambda-2))$$

$$\bar{z}w = 4ac + 2b(\lambda-2) - i(4bc - 2a(\lambda-2))$$

$$z\bar{w} + w\bar{z} = 8ac + 4b(\lambda-2)$$
$$iw\bar{z} - iz\bar{w} = 8bc - 4a(\lambda-2)$$
$$z\bar{z} - w\bar{w} = 4(a^2+b^2) - (4c^2 + (\lambda-2)^2)$$

P18 continued

$$\hat{F}(a,b,c) = \sigma_2\left(\lambda^2 \left(8ac + 4b(\lambda-2),\ 8bc - 4a(\lambda-2),\ 4(a^2+b^2-c^2) - (\lambda-2)^2 \right)\right)$$

$$= \frac{\lambda^2}{1-\lambda^2\left[4(a^2+b^2-c^2)-(\lambda-2)^2\right]}\left(\underbrace{8ac+4b(\lambda-2)}_{\mathcal{B}},\ \underbrace{8bc-4a(\lambda-2)}_{\mathcal{A}}\right)$$

$$= \frac{\lambda^2}{1-\lambda^2\left[4(a^2+b^2-c^2-(\lambda^2-4\lambda+4))\right]}\left(\mathcal{B},\ \mathcal{A}\right)$$

I'll focus on the annoying denominator, [ ] term,

$$4(a^2+b^2-c^2-\lambda^2+4\lambda-4) =$$

$$= 4(5a^2+5b^2+3c^2-\lambda^2)$$

Recall,
$\lambda = a^2+b^2+c^2+1$
$4(\lambda-1) = 4(a^2+b^2+c^2)$

Ok, so,

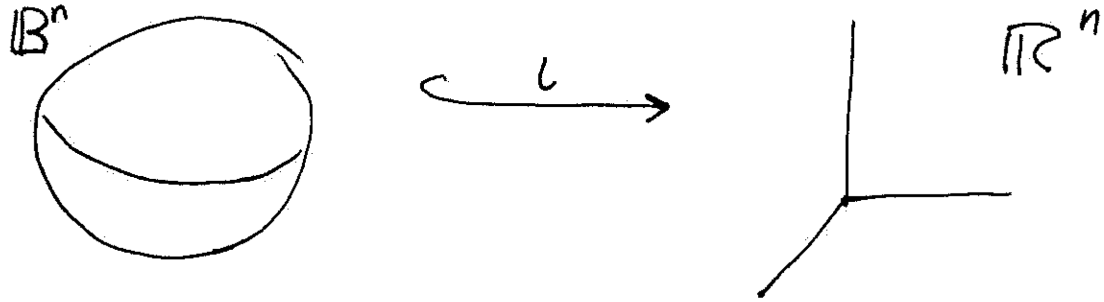
$$\hat{F}(a,b,c) = \frac{(1+a^2+b^2+c^2)^2}{1-(1+a^2+b^2+c^2)^2\left[4(5a^2+5b^2+3c^2-\lambda^2\right]}\left(\mathcal{B},\ \mathcal{A}\right)$$

provided the denominator is $\neq 0$ it is evident the expression above is smooth. Also, it is evident there is probably a better method to calculate (c). Need a manifestly complex coord. chart for S^3

P19 (2-4) $\overline{\mathbb{B}^n} \xhookrightarrow{\iota} \mathbb{R}^n$ is smooth
when $\mathbb{B}^n$ is regarded as manifold with boundary

$$\overline{\mathbb{B}^n} = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$$

$$\partial(\overline{\mathbb{B}^n}) = S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$$



$$\iota = \mathrm{Id}_{\mathbb{R}^n}\Big|_{\overline{\mathbb{B}^n}}$$

$\mathrm{Id}_{\mathbb{R}^n}$ is an extension of ι to an open set

$\therefore$ ι is smooth map on $\overline{\mathbb{B}^n}$ as closed subset of $\mathbb{R}^n$.

By Th^{m} 5.27 we're done using $S = \overline{\mathbb{B}^n} \subseteq M = N = \mathbb{R}^n$
(p. 112)

(But, sadly we're not in Chapter 5 yet!
So... I must invent nice coordinates ⤵)

P19 Warm-up calculation to construct n-spherical coords.

$x = r\cos\theta \qquad x^2+y^2=r^2$
$y = r\sin\theta$

$x = r\cos\theta = \rho\sin\phi\cos\theta$
$y = r\sin\theta = \rho\sin\phi\sin\theta$ $\Big\}\ x^2+y^2 = \rho^2\sin^2\phi$

$r=\rho\sin\phi \qquad z = \rho\cos\phi \qquad \Big|\ x^2+y^2+z^2=\rho^2$

$x_1^2 + x_2^2 = r_2^2$

$x_1^2 + x_2^2 + x_3^2 = r_3^2 \qquad r_2 = r_3\sin\phi_1$

$x_1 = r_3\sin\phi_1\cos\theta$
$x_2 = r_3\sin\phi_1\sin\theta$
$x_3 = r_3\cos\phi_1$

$x_1^2+x_2^2 = r_3^2\sin^2\phi_1$

$r_3 = r_4\sin\phi_2 \qquad r_3^2 = r_4^2\sin^2\phi_2$
$x_4 = r_4\cos\phi_2$ $\Big\}\ r_3^2 + x_4^2 = r_4^2 = x_1^2+\cdots+x_4^2$

$r_4 = r_5\sin\phi_3 \qquad r_4^2 = r_5^2\sin^2\phi_3$
$x_5 = r_5\cos\phi_3$ $\Big\}\ r_4^2 + x_5^2 = x_1^2+\cdots+x_5^2$

$x_1 = r_5\cos\theta\sin\phi_1\sin\phi_2\sin\phi_3$
$x_2 = r_5\sin\theta\sin\phi_1\sin\phi_2\sin\phi_3$
$x_3 = r_5\cos\phi_1\sin\phi_2\sin\phi_3$
$x_4 = r_5\cos\phi_2\sin\phi_3$
$x_5 = r_5\cos\phi_3$

$x_1^2+x_2^2 = r_5^2\sin^2\phi_1\sin^2\phi_2\sin^2\phi_3$
$x_1^2+x_2^2+x_3^2 = r_5^2\sin^2\phi_2\sin^2\phi_3$
$x_1^2+x_2^2+x_3^2+x_4^2 = r_5^2\sin^2\phi_2$
$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 = r_5^2$

P19 continued

n-spherical coordinates for $\mathbb{R}^n$

We define $r_n, \theta, \phi_1, \phi_2, \ldots, \phi_{n-2}$ implicitly by,

$$x_1 = r_n \cos\theta \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{n-2}$$
$$x_2 = r_n \sin\theta \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{n-2}$$
$$x_3 = r_n \cos\phi_1 \sin\phi_2 \cdots \sin\phi_{n-2}$$
$$x_4 = r_n \cos\phi_2 \sin\phi_3 \cdots \sin\phi_{n-2}$$
$$x_5 = r_n \cos\phi_3 \sin\phi_4 \cdots \sin\phi_{n-2}$$
$$\vdots$$
$$x_{n-1} = r_n \sin\phi_{n-3} \sin\phi_{n-2}$$
$$x_n = r_n \cos\phi_{n-2}$$

By construction,

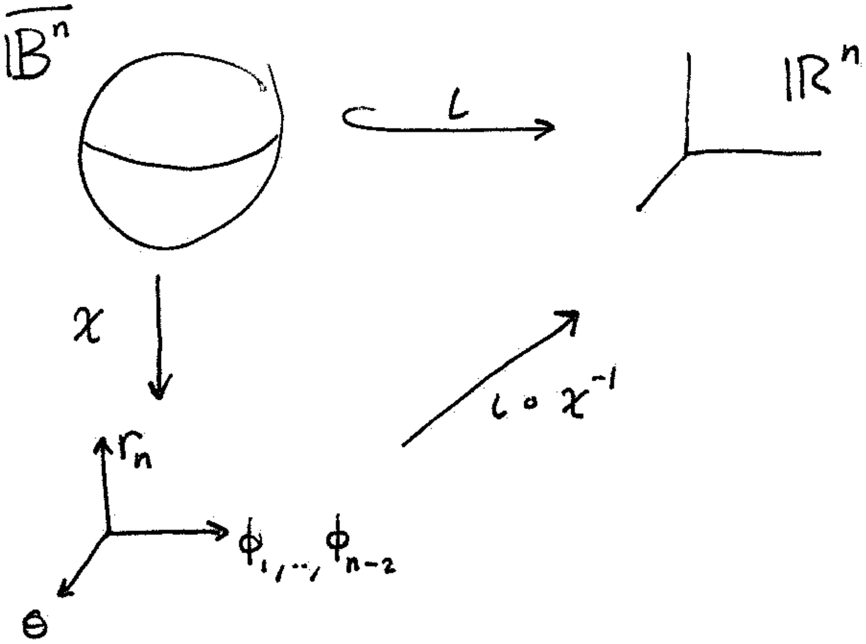
$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = r_n^2 \sin^2\phi_{n-2}$$

Thus $x_1^2 + \cdots + x_{n-1}^2 + x_n^2 = r_n^2 \sin^2\phi_{n-2} + r_n^2\cos^2\phi_{n-2} = r_n^2$

If we restrict $0 < \theta, \phi_i < \pi/2$ for $i = 1, 2, .., n-2$ then $\chi = (\theta, \phi_1, \phi_2, .., \phi_{n-2}, r_n)$ gives chart on $\mathbb{R}^n$ for points $x \in \mathbb{R}^n$ such that $x_i > 0 \;\; \forall i = 1, 2, .., n$.

Then $S^{n-1} = \{ x \in \mathbb{R}^n \mid r_n(x) = 1 \}$ and $\overline{\mathbb{B}^n} = \{ x \in \mathbb{R}^n \mid r_n(x) \leq 1 \}$ thus $\partial \overline{\mathbb{B}^n} = S^{n-1}$ and the n-spherical coordinate chart suitably restricted to avoid angle degeneracy yields chart on $\overline{\mathbb{B}}^n$

$\boxed{\text{P19}}$ continued



$$\iota \circ \chi^{-1}(\theta, \phi_1, \dots, \phi_{n-2}, r_n) = \iota\,(r_n \cos\theta \sin\phi_1 \cdots \sin\phi_{n-2}, \dots, r_n \cos\phi_{n-2})$$

$$= (r_n \cos\theta \sin\phi_1 \cdots \sin\phi_{n-2}, \dots, r_n \cos\phi_{n-2})$$

Thus $\iota \circ \chi^{-1}$ is clearly smooth $\therefore$ ι is smooth.

P20 Ex. 3.5 p. 54 of John Lee SM

Suppose M is smooth manifold w/o boundary and $p \in M$, $v \in T_pM$ and $f, g \in C^\infty(M)$

(a.) if f constant function then $vf = 0$

(b.) if $f(p) = g(p) = 0$ then $v(fg) = 0$

(a.) $f(p) = c$ $\forall p \in M$ then $f = c \cdot 1$ where $1(p) = 1$ $\forall p \in M$. Notice $1 \cdot 1 = 1$ as function on M hence by product rule

$$v(1) = v(1 \cdot 1) = 1(p)\, v(1) + v(1)\, 1(p)$$

Hence $v(1) = 2v(1)$ $\therefore$ $v(1) = 0$. Then

$$v(f) = v(c \cdot 1) = c\, v(1) = 0. //$$

(b.) $v(fg) = f(p)\, v(g) + g(p)\, v(f)$ by product rule thus $f(p) = 0$ and $g(p) = 0 \Rightarrow v(fg) = 0$ for $v \in T_pM$.

Remark: P20 makes nice test questions unlike say part (C) of P18 (2-~~3~~ part c of SM)

P21 Exercise 3.7 p. 56 of SM

$F : M \to N$ and $G : N \to P$ smooth maps on smooth manifolds and suppose $p \in M$,

(a.) $dF_p : T_pM \to T_{F(p)}N$ is linear

(b.) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G \circ F(p)}P$

(c.) $d(Id_M)_p = Id_{T_pM} : T_pM \to T_pM$

(d.) If F is diffeomorphism then dF_p is isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$

(a.) we defined $dF_p(v)(g) = v[g \circ F]$ $\forall v \in T_pM$ and $g \in C^\infty N$

Suppose $v, w \in T_pM$ and $c \in \mathbb{R}$. Let $g \in C^\infty N$,

$$\begin{aligned} dF_p(cv+w)(g) &= (cv+w)[g \circ F] \\ &= cv[g \circ F] + w[g \circ F] \\ &= c\,dF_p(v)(g) + dF_p(w)(g) \\ &= (c\,dF_p(v) + dF_p(w))(g) \end{aligned}$$

property of derivations, $g \circ F \in C^\infty M$ so $v, w \in T_pM$ naturally act here.

Thus $dF_p : T_pM \to T_{F(p)}N$ is linear.

(b.) did this in other problem!

(c.) $d(Id_M)_p(v)(f) = v[f \circ Id_M] = v[f]$ $\forall f \in C^\infty M$

thus $d(Id_M)_p(v) = v$ $\forall v \in T_pM \Rightarrow d(Id_M)_p = Id_{T_pM}$.

(d.) Given F and F^{-1} are smooth where $F : M \to N$ and $F^{-1} : N \to M$ and $F^{-1} \circ F = Id_M$ & $F \circ F^{-1} = Id_N$ hence by chain-rule we obtain

$$(dF^{-1})_{F(p)} \circ dF_p = d(Id_M)_p = Id_{T_pM}$$

$$(dF)_p \circ (dF^{-1})_{F(p)} = d(Id_N)_{F(p)} = Id_{T_{F(p)}N}$$

$\therefore \boxed{(dF_p)^{-1} = dF^{-1}_{F(p)}}$

P22 SM exercise 3.17 p. 65

Let (x,y) denote Cartesian coord. on $\mathbb{R}^2$. Verify $(\tilde{x},\tilde{y})$ given by $\tilde{x}=x$, $\tilde{y}=y+x^3$ are global smooth coordinates on $\mathbb{R}^2$. Let p be point $(1,0)\in\mathbb{R}^2$ (in standard coord) and show that $\frac{\partial}{\partial x}\Big|_p \neq \frac{\partial}{\partial \tilde{x}}\Big|_p$

$$\tilde{\varphi}(x,y) = (x,\ x^3+y) = (u,v) \quad \begin{matrix} \to x = u \\ \to x^3+y = v \\ y = v - x^3 = v-u^3 \end{matrix}$$

$$\tilde{\varphi}^{-1}(u,v) = (u,\ v-u^3)$$

Thus $\tilde{\varphi}$ is bijection on $\mathbb{R}^2$ and it is smooth since its coordinate formulas are all polynomial.

I'll use the chain-rule to express $\frac{\partial f}{\partial x}(p)$ in terms of $\frac{\partial f}{\partial \tilde{x}}(p)$ and $\frac{\partial f}{\partial \tilde{y}}(p)$, $\tilde{\varphi}(1,0)=(1,1)$

$$\frac{\partial f}{\partial x}(p) = \frac{\partial \tilde{x}}{\partial x}(p)\frac{\partial \bar{f}}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial \bar{f}}{\partial \tilde{y}} \leftarrow \bar{f} = f\circ\tilde{\varphi}^{-1}$$

$$= \frac{\partial x}{\partial x}\frac{\partial \bar{f}}{\partial \tilde{x}} + \frac{\partial}{\partial x}(y+x^3)\frac{\partial \bar{f}}{\partial \tilde{y}}$$

$$= \frac{\partial \bar{f}}{\partial \tilde{x}}(1,1) + 3x^2\frac{\partial \bar{f}}{\partial \tilde{y}}(1,1)$$

$$= \frac{\partial \bar{f}}{\partial \tilde{x}}(1,1) + 3\frac{\partial \bar{f}}{\partial \tilde{y}}(1,1)$$

So, with the usual notational abuses,

$$\boxed{\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}} + 3x^2\frac{\partial}{\partial \tilde{y}}}$$

P23 SM, Problem 3-2, p. 75

$M_1, M_2, \ldots, M_k$ smooth manifolds w/o boundary and define $\pi_j : M_1 \times M_2 \times \cdots \times M_k \longrightarrow M_j$ be projection onto M_j. For any point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$ the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \longrightarrow T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v))$$

is an isomorphism. Indeed still true if one of M_i is smooth with boundary

Suppose $f_j \in C^\infty(M_j)$ and $v \in T_p(M_1 \times \cdots \times M_k)$, let's calculate $d(\pi_j)_p(v)$ acting on $f_j \in C^\infty(M_j)$

$$d(\pi_j)_p(v)(f_j) = v\,[f_j \circ \pi_j] \qquad f \circ \pi_j : M_1 \times \cdots \times M_k \xrightarrow{\pi_j} M_j \xrightarrow{f_j} \mathbb{R}$$

Suppose $\alpha(v) = 0$

Then $d(\pi_j)_p(v) = 0 \quad \forall j = 1, 2, \ldots, k$

Let $x_1 \times x_2 \times \cdots \times x_k : U_1 \times U_2 \times \cdots \times U_k \subseteq M_1 \times M_2 \times \cdots \times M_k$ serve as a coordinate system at p then the typical coordinate function at p can be written

$$x^I = x^{i_1 i_2 \cdots i_k} = x_1^{i_1} \times x_2^{i_2} \times \cdots \times x_k^{i_k} :$$

$$d(\pi_j)_p \left(\frac{\partial}{\partial x^I} \right)(x^J)$$

(not wrong, but I make better notational choices) it's all about notation!

P23 continued

$$\underbrace{(X_1 \times \cdots \times X_k)}_{\mathcal{X}}(p_1, \dots, p_k) = (X_1(p_1), \dots, X_k(p_k))$$

$$= (x_1^1(p_1), x_1^2(p_1), \dots, x_1^{m_1}(p_1), \dots, X_k^1(p_k), \dots, X_k^{m_k}(p_k))$$

$$\mathcal{X}_i^j(p) = x_i^j(\pi_i(p))$$

$$\pi_i : M_1 \times \cdots \times M_k \longrightarrow M_i$$

$$x_i^j : M_i \longrightarrow \mathbb{R} \quad \text{or} \quad X_i : M_i \longrightarrow \mathbb{R}^{m_i}$$

Let $M = M_1 \times \cdots \times M_k$,

$$T_p M = \text{span}\left\{ \frac{\partial}{\partial \mathcal{X}_i^{j_i}} \,\middle|\, 1 \le i \le k,\ 1 \le j_i \le m_i \right\}$$

$$T_{p_i} M_i = \text{span}\left\{ \frac{\partial}{\partial x_i^{j_i}} \,\middle|\, 1 \le j_i \le m_i \right\}$$

$$\mathcal{X}_i^{j_i} = x_i^{j_i} \circ \pi_i$$

Let $V = \sum_{i=1}^{k} \sum_{j_i=1}^{m_i} V_{j_i}^i \frac{\partial}{\partial \mathcal{X}_i^{j_i}} \in T_p M$

and suppose $\alpha(V) = 0$ then $d(\pi_\ell)_p(V) = 0$ for $\ell = 1, 2, \dots, k$. By linearity of $d(\pi_\ell)_p$,

$$d(\pi_\ell)_p(V) = \sum_{i=1}^{k} \sum_{j_i=1}^{m_i} V_{j_i}^i \underbrace{d(\pi_\ell)_p\left(\frac{\partial}{\partial \mathcal{X}_i^{j_i}}\bigg|_p \right)}_{\delta_{i\ell} \frac{\partial}{\partial x_\ell^{j_i}}\big|_{p_\ell}} = 0$$

for why.

P23 continued

$$\mathcal{X}_i^{j_i} = X_i^{j_i} \circ \Pi_i \quad , \quad \Pi_\ell : M \longrightarrow M_\ell$$

$$d(\Pi_\ell)_p \left(\frac{\partial}{\partial \mathcal{X}_i^{j_i}}\Big|_p \right) = \sum_{a=1}^{m_\ell} \frac{\partial (X_\ell^a \circ \Pi_\ell)}{\partial \mathcal{X}_i^{j_i}} \frac{\partial}{\partial X_\ell^a}\Big|_{p_\ell}$$

$$= \sum_{a=1}^{m_\ell} \frac{\partial \mathcal{X}_\ell^a}{\partial \mathcal{X}_i^{j_i}} \frac{\partial}{\partial X_\ell^a}\Big|_{p_\ell}$$

$$= \sum_{a=1}^{m_\ell} \delta_{j_i}^a \, \delta_i^\ell \frac{\partial}{\partial X_\ell^a}\Big|_{p_\ell}$$

$$= \delta_i^\ell \frac{\partial}{\partial X_\ell^{j_i}}\Big|_{p_\ell} \in T_{p_\ell} M_\ell$$

Thus,

$$d(\Pi_\ell)_p(V) = \sum_{i=1}^{k} \sum_{j_i=1}^{m_i} V_{j_i}^i \, \delta_i^\ell \frac{\partial}{\partial X_\ell^{j_i}}\Big|_{p_\ell}$$

$$= \sum_{j_\ell=1}^{m_\ell} V_{j_\ell}^\ell \frac{\partial}{\partial X_\ell^{j_\ell}}\Big|_{p_\ell}$$

Therefore, $\alpha(V) = (d(\Pi_1)_p(V), \dots, d(\Pi_k)_p(V)) = (0, \dots, 0)$ implies $V_{j_\ell}^\ell = 0 \quad \forall \ell = 1, 2, \dots, k$ and $1 \le j_\ell \le m_\ell$

thus $V = 0$ as all its components are zero and hence α is one-to-one as α is linear.

$T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$ has basis given by

$$e_i \left.\frac{\partial}{\partial x_i^{j_i}}\right|_{p_i} \quad \text{for } i = 1, 2, \dots, k \text{ and } 1 \le j_i \le m_i$$

Since $d(\pi_\ell)_p(v) = \displaystyle\sum_{j_\ell=1}^{m_\ell} v_{j_\ell}^{\ell} \left.\frac{\partial}{\partial x_\ell^{j_\ell}}\right|_{p_\ell}$

we find

$$d(\pi_\ell)_p\left(\left.\frac{\partial}{\partial x_i^{j_i}}\right|_p\right) = \delta_i^\ell \left.\frac{\partial}{\partial x_\ell^{j_i}}\right|_{p_\ell}$$

Therefore,

$$\begin{aligned} \alpha\left(\left.\frac{\partial}{\partial x_i^{j_i}}\right|_p\right) &= \sum_{\ell=1}^{k} e_\ell \, d\pi_\ell\left(\left.\frac{\partial}{\partial x_i^{j_i}}\right|_p\right) \\ &= \sum_{\ell=1}^{k} \delta_i^\ell \, e_\ell \left.\frac{\partial}{\partial x_\ell^{j_i}}\right|_{p_\ell} \\ &= e_i \left.\frac{\partial}{\partial x_i^{j_i}}\right|_{p_i} \end{aligned}$$

Consequently α maps onto the basis for $\bigoplus_{i=1}^{k} T_{p_i}M_i$ which completes are argument that α is a bijection, indeed a linear bijection, an <u>isomorphism</u> of $T_p(M_1 \times \cdots \times M_k)$ and $T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$.

$\boxed{P24}$ SM, Problem 3-4, p. 75

Show TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$

Notice $\gamma(t) = (\cos t, \sin t)$ yields $\gamma'(t) = -\sin t \frac{\partial}{\partial x}\Big|_{\gamma(t)} + \cos t \frac{\partial}{\partial y}\Big|_{\gamma(t)}$

or simply $V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on S^1. This vector field V is nonvanishing on S^1.

Let $F : S^1 \times \mathbb{R} \longrightarrow TS^1$ be defined by

$$F((x,y),t) = \left((x,y),\ t\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)$$

Let $(p,v) \in TS^1$ then as $V = t_0\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ for some t_0 as

$x\frac{\partial}{\partial y}\Big|_p - y\frac{\partial}{\partial x}\Big|_p \in T_pS^1$ and $\dim(S^1) = 1$ we note

$$F(p, t_0) = (p, v) \quad \therefore \ F \text{ is surjective.}$$

Alternatively, $F(p,t) = \left(p,\ t\frac{d}{d\theta}\Big|_p\right)$ is clearly onto.

If $F(p_1, t_1) = F(p_2, t_2)$ then $\left(p_1, t_1\frac{d}{d\theta}\Big|_{p_1}\right) = \left(p_2, t_2\frac{d}{d\theta}\Big|_{p_2}\right)$

hence $p_1 = p_2$ and $t_1\frac{d}{d\theta}\Big|_{p_1} = t_2\frac{d}{d\theta}\Big|_{p_1} \Rightarrow \underline{t_1 = t_2}$.

(<u>Remark</u>: $\exists$ other formalisms to show F is bijection.)

$$\begin{tikzcd}
\boxed{S^1 \times \mathbb{R}} \arrow[r, "F"] \arrow[d, "\psi = \theta \times Id"'] & \boxed{TS^1} \arrow[d, "\tilde{\theta} = \theta \times d\theta"] \\
\boxed{\mathbb{R}^2} \arrow[r, "\hat{F} = \tilde{\theta} \circ F \circ \psi^{-1}"] & \boxed{\mathbb{R}^2}
\end{tikzcd}$$

$$\begin{aligned}
\hat{F}(\theta, t) &= \tilde{\theta}\left(F(\psi^{-1}(\theta,t))\right) \\
&= \tilde{\theta}\left(F((\cos\theta, \sin\theta), t)\right) \\
&= \tilde{\theta}\left((\cos\theta, \sin\theta), t\frac{\partial}{\partial\theta}\right) = (\theta, t).
\end{aligned}$$

Hence F smooth and similar calculation shows F^{-1} smooth $\therefore$ F diffeomorphism.

p25 3-6, p. 75

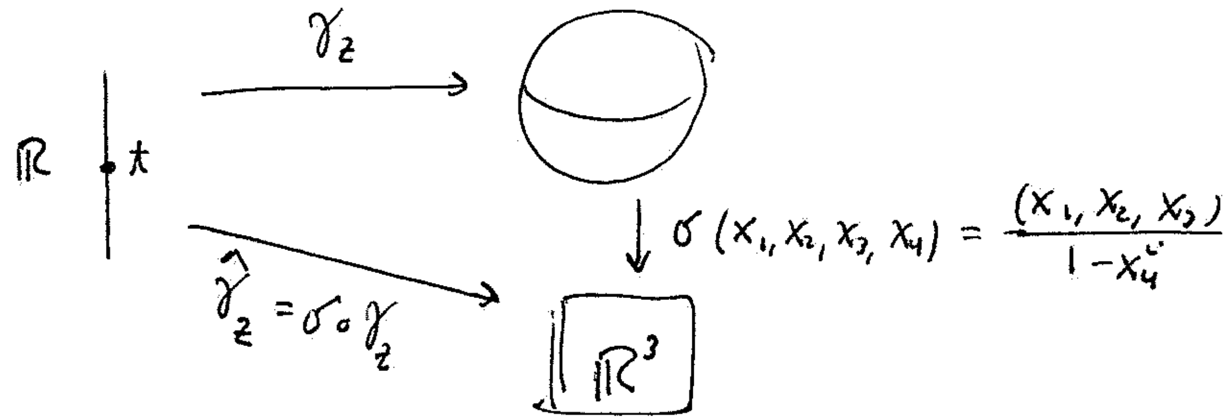
$S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2 = \mathbb{R}^4, \quad z = (z^1, z^2) \in S^3$

$\gamma_z : \mathbb{R} \to S^3$ by $\gamma_z(t) = (e^{it} z^1, e^{it} z^2)$

We wish to show γ_z smooth with $\gamma_z'(t) \neq 0 \ \forall t$.

$$\gamma_z'(t) = (i e^{it} z^1, i e^{it} z^2) = i e^{it}(z^1, z^2) \neq 0$$

given $(z^1, z^2) \in S^3$ where $|z^1|^2 + |z^2|^2 = 1$ hence $(z^1, z^2) \neq 0$.



Let $z \in \mathbb{C}^2$ have $z = (z^1, z^2)$ and $z^1 = x^1 + iy^1$ and $z^2 = x^2 + iy^2$ thus

$$\hat{\gamma}_z(t) = \sigma\big((\cos t + i \sin t)(x^1 + iy^1), (\cos t + i\sin t)(x^2 + iy^2)\big)$$

$$= \sigma\big((\cos t)x^1 - \sin t\, y^1 + i(\cos t\, y^1 + \sin t\, x^1),$$

$$(\cos t)x^2 - (\sin t)y^2 + i(\cos t\, y^2 + \sin t\, x^2)\big)$$

$$= \frac{(x^1 \cos t - y^1 \sin t,\ x^1 \sin t + y^1 \cos t,\ x^2 \cos t - y^2 \sin t)}{1 - y^2 \cos t - x^2 \sin t}$$

Thus $t \mapsto \hat{\gamma}_z(t)$ is smooth from $\mathbb{R}$ to $\mathbb{R}^3$ and it follows γ_z is smooth from $\mathbb{R}$ to S^3 since its local coord. rep. is smooth.

P26

$$T_pM = \left\{ v : C^\infty(M) \longrightarrow \mathbb{R} \;\middle|\; \underbrace{v[f+cg] = v[f] + cv[g]}_{\text{linearity}} \;\&\; \underbrace{v[fg] = f(p)v[g] + g(p)v[f]}_{\text{product rule}} \right\}$$

$$\mathcal{C}_p = \{ [\gamma] \mid \gamma : J \subseteq \mathbb{R} \to M,\ \gamma(0) = p \} = \mathcal{V}_p M$$ (pg. 72 John Lee text Smooth Manifolds)

$$\gamma_1 \sim \gamma_2 \iff \gamma_1(0) = \gamma_2(0) \text{ and } \gamma_1'(0) = \gamma_2'(0)$$

$$[\gamma_1] = \{ \gamma : J \subseteq \mathbb{R} \to M \mid \gamma(0) = \gamma_1(0) \text{ and } \gamma_1'(0) = \gamma'(0) \}$$

($\sim$ forms equivalence relation on set of all curves through $p \in M$ of the form $\gamma : J \subseteq \mathbb{R} \to M$ where $0 \in J$ and $\gamma(0) = p$)

$$\gamma'(t_0)f = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right)f = \left.\frac{d}{dt}\right|_{t_0}(f \circ \gamma) = (f \circ \gamma)'(t_0)$$

$$\gamma'(t_0) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(t_0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)}$$

Isomorphism of T_pM and $\mathcal{V}_pM$

$$\Psi : \mathcal{V}_pM \longrightarrow T_pM$$

$$\Psi([\gamma]) = \gamma'(0)$$

Note if $[\gamma_1] = [\gamma_2]$ then $\gamma_1 \sim \gamma_2$ and $\gamma_1'(0) = \gamma_2'(0)$ by def^{n} of $\sim$

Thus Ψ is well-defined. To prove Ψ is an isomorphism of vector spaces we need to explain how $\mathcal{V}_pM$ is a vector space.

$$\begin{aligned}\Psi(c[\gamma_1] + [\gamma_2]) &= \Psi([\alpha]) \quad \text{where } \alpha'(0) = c\gamma_1'(0) + \gamma_2'(0) \\ &= \alpha'(0) \\ &= c\gamma_1'(0) + \gamma_2'(0) \\ &= c\,\Psi([\gamma_1]) + \Psi([\gamma_2])\end{aligned}$$

It remains to show Ψ is a bijection.

If $v \in T_pM$ then, using chart x at p, $v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p$ where $v^i = v(x^i)$.

construct $\gamma(t) = x^{-1}(x(p) + t\langle v^1, v^2, \dots, v^n\rangle)$ thus $(x \circ \gamma)(t) = x(p) + t\langle v^1, \dots, v^n\rangle$

Hence $\Psi([\gamma]) = \gamma'(0) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(0) \frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p = v.$

So Ψ is surjective.

Why is $\Psi : \mathcal{V}_pM \longrightarrow T_pM$ injective?

$$\Psi([\gamma]) = \gamma'(0)$$

Suppose $\Psi([\gamma_1]) = \Psi([\gamma_2])$

$$\gamma_1'(0) = \gamma_2'(0)$$

$$\Rightarrow [\gamma_1] = [\gamma_2]$$

$$\Rightarrow \Psi \text{ one-to-one}$$

$\Psi^{-1}(V) = [\gamma_V]$ where $\underbrace{\gamma_V(t) = x^{-1}(x(p) + t\langle V^1, \dots, V^n \rangle)}$

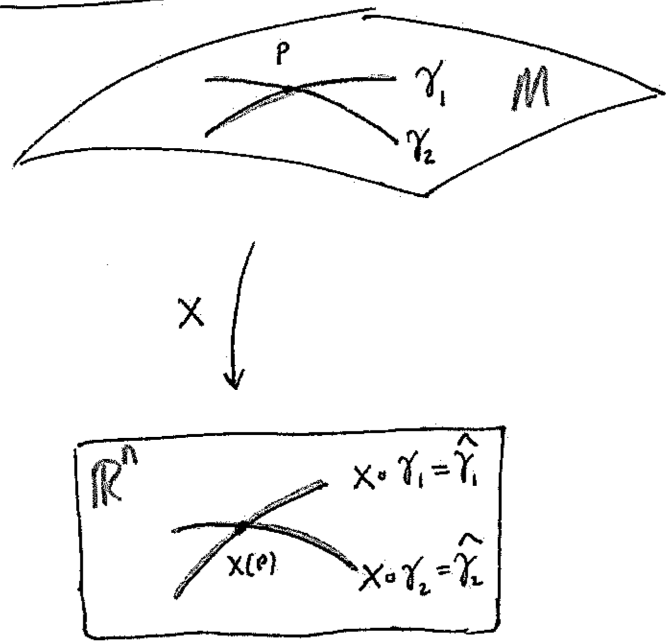
$$(x \circ \gamma_V)(t) = x(p) + t\vec{V}$$

$$\gamma_V'(0) = V$$

$\boxed{\text{Def}^n/ \quad c[\gamma_V] + [\gamma_W] = [\gamma_{cV+W}]}$

this notation makes $\mathcal{V}_pM$ vector space structure easier to sort through

VECTOR SPACE $\mathcal{T}_p M$ (I)



$$\hat{\gamma}_1(0) = \hat{\gamma}_2(0) = X(P)$$

$$\hat{\gamma}_1'(0) = \sum \frac{d(u^i \circ \hat{\gamma}_1)}{dt}(0) \left.\frac{\partial}{\partial u^i}\right|_{X(P)} = \sum_{i=1}^{n} \frac{d\hat{\gamma}_1^i}{dt}(0) \left.\frac{\partial}{\partial u^i}\right|_{X(P)}$$

$$\hat{\gamma}_2'(0) = \sum_{i=1}^{n} \frac{d\hat{\gamma}_2^i}{dt}(0) \left.\frac{\partial}{\partial u^i}\right|_{X(P)}$$

$$\hat{\alpha}(t) = X(P) + \left(c\left\langle \frac{d\hat{\gamma}_1^1}{dt}(0), \dots, \frac{d\hat{\gamma}_1^n}{dt}(0)\right\rangle + \left\langle \frac{d\hat{\gamma}_2^1}{dt}(0), \dots, \frac{d\hat{\gamma}_2^n}{dt}(0)\right\rangle\right)t$$

Lift the line $\hat{\alpha}$ back up to M,

$$\alpha = X^{-1} \circ \hat{\alpha}$$

Has $\alpha(0) = X^{-1}(\hat{\alpha}(0)) = X^{-1}(X(P)) = P$.

Also, by construction,

$$\alpha'(0) = \sum_{i=1}^{n} \frac{d(X^i \circ \alpha)}{dt}(0) \left.\frac{\partial}{\partial x^i}\right|_p$$

$$= \sum_{i=1}^{n} \frac{d}{dt}\left(X^i \circ X^{-1} \circ \hat{\alpha}\right) \left.\frac{\partial}{\partial x^i}\right|_p$$

$$= \sum_{i=1}^{n} \frac{d\hat{\alpha}^i}{dt} \left.\frac{\partial}{\partial x^i}\right|_p \qquad (X^i \circ X^{-1})(u) = u^i$$

$$= \sum_{i=1}^{n} \left(c\,\frac{d\hat{\gamma}_1^i}{dt}(0) + \frac{d\hat{\gamma}_2^i}{dt}(0)\right) \left.\frac{\partial}{\partial x^i}\right|_p$$

$$= c\,\gamma_1'(0) + \gamma_2'(0)$$

(II) $\mathcal{V}_p M = \{ [\gamma] \mid \gamma : \text{dom}\,\gamma \subseteq \mathbb{R} \to M,\ \gamma(0) = p \}$

If we consider $[\gamma_1], [\gamma_2] \in \mathcal{V}_p M$ then construct

$\hat{\alpha}(t) = x(p) + t(c\vec{v}_1 + \vec{v}_2)$ where $(\vec{v}_1)^i = \gamma_1'(0)x^i$ and $(\vec{v}_2)^i = \gamma_2'(0)x^i$

where we suppose $(U, x) \in \mathcal{A}_M$ and $p \in U$. Then

$\boxed{\text{Def}^{\underline{n}}/\ c[\gamma_1] + [\gamma_2] = [\alpha] \text{ where } \alpha = x^{-1} \circ \hat{\alpha}}$

To see this is well-defined notice $\alpha'(0) = c\gamma_1'(0) + \gamma_2'(0)$
hence if $[\gamma_1] = [\gamma_3]$ and $[\gamma_2] = [\gamma_4]$ we'd construct α_2 for
which $\alpha_2'(0) = c\gamma_3'(0) + \gamma_4'(0) = c\gamma_1'(0) + \gamma_2'(0)$ thus $[\alpha] = [\alpha_2]$
as $\alpha(0) = p$ and $\alpha_2(0) = p$ as constructed previous page.

I invite the reader to check on the vector space axioms for $\mathcal{V}_p M$.

$T_p^{physics} M = \{ [(p, \vec{v}, x)] \mid p \in M, \vec{v} \in \mathbb{R}^n, x \text{ chart at } p \}$

$\underbrace{(p, v^i, x) \sim (\bar{p}, \bar{v}^i, \bar{x})}_{\text{equivalent}}$ if $\cancel{\bar{v}^i = \sum \frac{\partial x^j}{\partial \bar{x}^i}}$

iff $p = \bar{p}$ and $\bar{v}^i = \sum_{j=1}^{n} \frac{\partial \bar{x}^i}{\partial x^j} v^j$

(recall, $v = v^i \frac{\partial}{\partial x^i}\Big|_p = \bar{v}^i \frac{\partial}{\partial \bar{x}^i}\Big|_p \Rightarrow \bar{v}^j = v^i \frac{\partial \bar{x}^j}{\partial x^i}$.)

$$\boxed{\begin{array}{l} \text{Def}^n / \; c[(p, \vec{v}, x)] + [(p, \vec{w}, x)] = [(p, c\vec{v} + \vec{w}, x)] \\ \text{for } [(p, \vec{v}, x)], [(p, \vec{w}, x)] \in T_p^{physics} M \text{ where } c \in \mathbb{R}. \end{array}}$$

If $(p, v^i, x) \sim (\bar{p}, \bar{v}^i, \bar{x})$ and $(p, w^i, x) \sim (\bar{p}, \bar{w}^i, \bar{x})$

then $c[(\bar{p}, \bar{v}^i, \bar{x})] + [(\bar{p}, \bar{w}^i, \bar{x})] = [(p, c\bar{v}^i + \bar{w}^i, \bar{x})]$

Notice $c\bar{v}^i + \bar{w}^i = \sum_j c \frac{\partial \bar{x}^i}{\partial x^j} v^j + \sum_j \frac{\partial \bar{x}^i}{\partial x^j} w^j = \sum_j \frac{\partial \bar{x}^i}{\partial x^j} (c v^j + w^j)$

thus $(\bar{p}, c\bar{v}^i + \bar{w}^i, \bar{x}) \sim (p, cv^i + w^i, x)$ hence $T_p^{physics} M$ has well-defined addition & scalar multiplication. We can prove $T_p^{physics} M$ is a vector space over $\mathbb{R}$.

Isomorphisms with $T_p^{PHYSICS}M$ and T_pM

$$\Phi : T_pM \longrightarrow T_p^{PHYSICS}M$$

$$\Phi(v) = \Phi\left(v^i \frac{\partial}{\partial x^i}\Big|_p\right) = [(p, v^i, x)]$$

If $v = \bar{v}^i \frac{\partial}{\partial \bar{x}^i}\Big|_p$ then notice $\Phi\left(\bar{v}^i \frac{\partial}{\partial \bar{x}^i}\Big|_p\right) = [(p, \bar{v}^i, \bar{x})]$

But, $(p, v^i, x) \sim (p, \bar{v}^i, \bar{x})$ as $\bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^j} v^j$ as before.

Thus Φ is well-defined.

$$\begin{aligned}
\Phi(cv+w) &= \Phi\left((cv^i+w^i)\frac{\partial}{\partial x^i}\Big|_p\right) \\
&= [(p, cv^i+w^i, x)] \\
&= c[(p, v^i, x)] + [(p, w^i, x)] \\
&= c\,\Phi(v) + \Phi(w).
\end{aligned}$$

$$\frac{\partial \bar{x}^i}{\partial x^k} v^k \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial}{\partial x^l}\Big|_p = \frac{\partial x^l}{\partial x^k} \frac{\partial}{\partial x^l}\Big|_p v^k = v^k \frac{\partial}{\partial x^k}\Big|_p$$

$$\Phi^{-1}([(p, v^i, x)]) = v^i \frac{\partial}{\partial x^i}\Big|_p$$

If $[(p, v^i, x)] = [(p, \bar{v}^i, \bar{x})]$ note $\Phi^{-1}([(p, \bar{v}^i, \bar{x})]) = \bar{v}^i \frac{\partial}{\partial \bar{x}^i}\Big|_p = v^i \frac{\partial}{\partial x^i}\Big|_p$