Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. 5pts per problem here.

Problem 27 Let $F(x, y, z) = (x^2 - y^2 + z^2, y + z, x + 2y + z)$ define a map on $\mathbb{R}^3$. Where is this map a local diffeomorphism ?

Problem 28 SM Exercise 4.10, page 80.

Problem 29 SM Problem 4-5, page 96.

Problem 30 SM Problem 5-1, page 123.

Problem 31 SM Problem 5-6, page 123.

Problem 32 SM Problem 5-7, page 123.

Problem 33 SM Problem 5-10, page 123.

Problem 34 Show $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$ is a proper embedded submanifold of $\mathbb{R}^{n\times n}$.

$\boxed{P27}$ $F(x,y,z) = (x^2 - y^2 + z^2,\ y + z,\ x + 2y + z)$ defines map on $\mathbb{R}^3$
where is F a local diffeomorphism ?

$dF_p(h) = J_F(p)h$ making the $T_p\mathbb{R}^3 = \mathbb{R}^3$ identification

Then $dF_p : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is surjective & injective iff $\det J_F(p) \neq 0$.

$$J_F = [\partial_x F \,|\, \partial_y F \,|\, \partial_z F] = \begin{bmatrix} 2x & -2y & 2z \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\det(J_F) = 2x(1-2) + 2y(0-1) + 2z(0-1)$$

$$\det(J_F) = -2x - 2y - 2z$$

Thus F is local diffeomorphism at each point
$p \in \{ (x,y,z) \in \mathbb{R}^3 \mid x+y+z \neq 0 \}$

(Everywhere except the plane $x+y+z=0$)

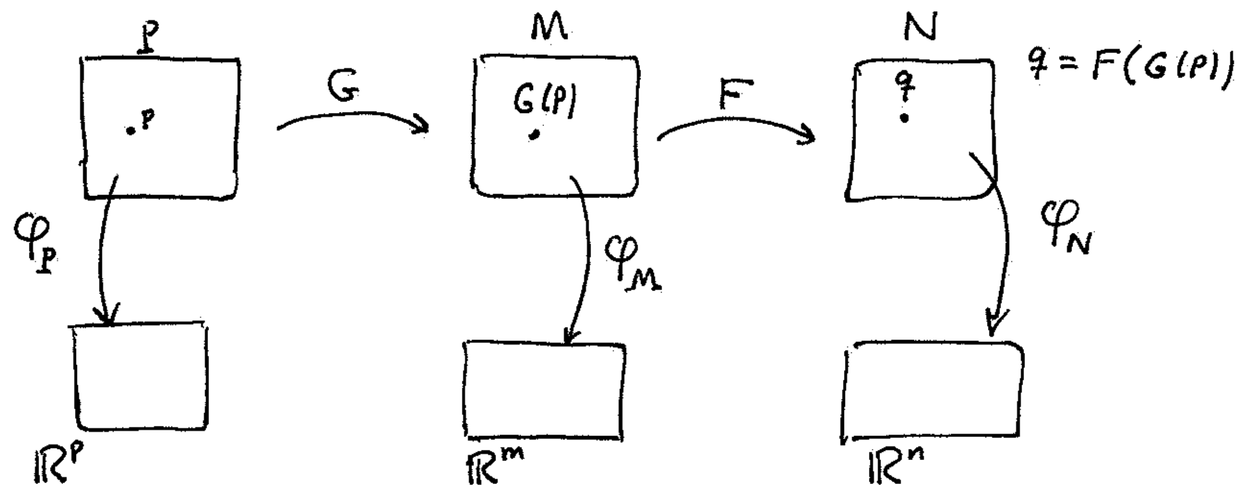
P28 Exercise 4.10, p.80

Suppose M, N, P are smooth manifolds with or w/o boundary and $F: M \longrightarrow N$ is local diffeomorphism.

(a.) If $G: P \longrightarrow M$ is continuous then G smooth $\Leftrightarrow$ $F \circ G$ smooth

(b.) If F is surjective and $G: N \longrightarrow P$ is any map then G is smooth $\Leftrightarrow$ $G \circ F$ smooth.

(a.) Suppose G is smooth. We seek to show $F \circ G$ smooth given $F: M \to N$ is local diffeomorphism. I'll begin with a picture



Let φ_P and φ_N be charts defined near $p \in P$ and $q \in N$. If necessary, shrink $\text{dom}(\varphi_N)$ such that $\exists\, \varphi_M$ chart at $G(p) \in M$ for which $\hat{F} = \varphi_N \circ F \circ \varphi_M^{-1}$ is smooth. Since G is continuous $G^{-1}(\text{dom}\, \varphi_M)$ is open in P hence we can shrink the $\text{dom}(\varphi_P)$ to coincide with $G^{-1}(\text{dom}\, \varphi_M)$. With these restrictions in mind,

$$\widehat{F \circ G} = \varphi_N \circ (F \circ G) \circ \varphi_P^{-1} = \Big(\underbrace{\varphi_N \circ F \circ \varphi_M^{-1}}_{\substack{\text{smooth map} \\ \text{from } \mathbb{R}^m \text{ to } \mathbb{R}^m \\ \text{as } F \text{ local diff.}}}\Big) \circ \Big(\underbrace{\varphi_M \circ G \circ \varphi_P^{-1}}_{\substack{\text{smooth map} \\ \text{from } \mathbb{R}^p \longrightarrow \mathbb{R}^m}}\Big)$$

Thus $\widehat{F \circ G}$ is smooth hence $F \circ G$ smooth as p was an arbitrary point in P.

P28 continued

(u.) $\boxed{\Leftarrow}$ Suppose $F \circ G$ smooth and G continuous where F is local diff. We seek to show G smooth.
Once more use the diagram ↘ to guide logic.
If $p \in P$ and φ_P is chart at p and φ_M is chart at $G(p)$ then observe,

$$
\begin{tikzcd}
\underset{\cdot\, p}{P} \arrow[r, "G"] \arrow[d, bend right, "\varphi_P"'] & \underset{\cdot\, G(p)}{M} \arrow[r, "F"] \arrow[d, "\varphi_M"] & \underset{\cdot\, q \,=\, F(G(p))}{N} \arrow[d, bend left, "\varphi_N"] \\
\mathbb{R}^p \arrow[r, "\hat{G}"] & \mathbb{R}^m \arrow[r, shift left, "\hat{F}"] & \mathbb{R}^n \arrow[l, shift left, "\hat{F}^{-1}"]
\end{tikzcd}
$$

$\hat{F}^{-1}$ ← exists since F local diffeom.
$\hat{F} = \varphi_N \circ F \circ \varphi_M^{-1}$

$$\underbrace{\hat{G} = \varphi_M \circ G \circ \varphi_P^{-1}}_{\text{need to show this is smooth}}$$

$$
\begin{aligned}
\widehat{F \circ G} &= \varphi_N \circ F \circ G \circ \varphi_P^{-1} \\
&= \varphi_N \circ F \circ \varphi_M^{-1} \circ \varphi_M \circ G \circ \varphi_P^{-1} \\
&= \hat{F} \circ \hat{G}
\end{aligned}
$$

Then $\hat{G} = \underbrace{\hat{F}^{-1}}_{\text{smooth}} \circ \underbrace{(\widehat{F \circ G})}_{\text{smooth}} \quad \therefore \quad G$ smooth.

<u>Remark</u>: continuity of G allows us to shrink dom(φ_P) to work in concert with dom(φ_M), which in turn must be made small enough to use local diffeomorphism prop. for F.

p28 continued

(6.) If $F: M \to N$ is surjective local diffeomorphism and $G: N \to P$ is any map then G smooth $\Leftrightarrow$ $G \circ F$ smooth

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$\Rightarrow$]

$$\begin{tikzcd} M \ni p \arrow[r, "F"] \arrow[d, "\varphi_M"] & N \ni q \arrow[r, "G"] \arrow[d, "\varphi_N"] & P \ni G(q) \arrow[d, "\varphi_P"] \\ \mathbb{R}^m \arrow[r, "\hat{F}"] & \mathbb{R}^n \arrow[r, "\hat{G}"] & \mathbb{R}^p \end{tikzcd}$$

Let $q \in N = F(M)$ hence $\exists\ p \in M$ s.t. $F(p) = q$.
If φ_N is chart at $q \in N$ and φ_P is chart at $G(q) \in P$ then G smooth implies $\hat{G} = \varphi_P \circ G \circ \varphi_N^{-1}$ smooth.
Consider $p \in M$ and $G(F(p)) = G(q) \in P$ and study charts φ_M at p and φ_P at $G(F(p))$.

$$\widehat{G \circ F} = \varphi_P \circ (G \circ F) \circ \varphi_M^{-1}$$
$$= \underbrace{\varphi_P \circ G \circ \varphi_N^{-1}}_{\text{smooth}} \circ \underbrace{\varphi_N \circ F \circ \varphi_M^{-1}}_{\text{smooth}} \quad \therefore\ \widehat{G \circ F} \text{ smooth} \Rightarrow \underline{G \circ F \text{ smooth}}.$$

$\Leftarrow$] Given $G \circ F$ smooth we can calculate, making the appropriate restrictions,

$$\widehat{G \circ F} = \hat{G} \circ \hat{F} \Rightarrow \hat{G} = \widehat{G \circ F} \circ \hat{F}^{-1}$$

since F is local diffeomorphism $\hat{F}$ can be made invertible with smooth inverse $\hat{F}^{-1}$ $\therefore$ $\hat{G}$ smooth $\Rightarrow$ $\underline{G \text{ smooth}}$. //

P29) Problem 4-5, p. 96

Let $\mathbb{C}P^n$ denote complex projective space as defined in Problem 1-9.

(a.) Show $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}P^n$ is surjective smooth submersion

(b.) Show $\mathbb{C}P^1$ is diffeomorphic to S^2 (See P6 of Mission 1)

(a.) to show π smooth we need to show $\hat{\pi}$ smooth where $\hat{\pi}$ is local coord. rep. However $\mathbb{C}^{n+1} - \{0\}$ has a global coord. chart if we identify $\mathbb{C}^{n+1} - \{0\}$ with $\mathbb{R}^{2n+2} - \{0\}$ hence we need only check

$$\hat{\pi} = \varphi \circ \pi \quad \text{for all } \varphi \text{ in some atlas for } \mathbb{C}P^n$$

From our previous work in Mission 1 etc From Jeff. Lee

Atlas for $\mathbb{C}P^n$? $V_k = \pi(U_k)$

$$U_k = \{ (z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} - \{0\} \mid z^k \neq 0 \}$$

$$V_k = \pi(U_k) = \{ [z] \mid z \in U_k \}$$

$$[z] = \{ \lambda z \mid \lambda \in \mathbb{C} \}$$

k^{th} entry

If $[z] \in V_k$ then $[z] = [z^1, \dots, 1, \dots, z^{n+1}]$

$$\varphi_k : V_k \longrightarrow \mathbb{C}^n \text{ by } \varphi_k[z] = \left(\frac{z^1}{z^k}, \dots, \frac{z^{k-1}}{z^k}, \frac{z^{k+1}}{z^k}, \dots, \frac{z^{n+1}}{z^k} \right)$$

Hence consider $(\varphi_k \circ \pi)(z) = \varphi_k[z]$ (for $z \neq 0$)

$$z \xmapsto{\varphi_k \circ \pi} \left(\frac{z^1}{z^k}, \dots, \frac{z^{k-1}}{z^k}, \frac{z^{k+1}}{z^k}, \dots, \frac{z^{n+1}}{z^k} \right)$$

Hence $\hat{\pi}$ smooth for every local coord. rep. for an atlas of $\mathbb{C}P^n$ $\therefore$ π smooth.

P29 continued (this proof suboptimal, need complex notation for best formulation)

(a.) wish to show $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}\mathbb{P}^n$ is a submersion; for $z_0 \neq 0$, show $d\pi_{z_0}$ is onto $T_{[z_0]}\mathbb{C}\mathbb{P}^n$. Consider the curve in $\mathbb{C}^{n+1}$ with velocity at $t=0$ of e_j for $j \neq k$ (we'll suppose $[z_0] \in V_k$ hence $(z_0)^k \neq 0$),

$$\alpha_j(t) = te_j + e_k \qquad \alpha_j'(0) = e_j \equiv \frac{\partial}{\partial x^j}\Big|_{z_0}$$

Likewise,

$$\beta_j(t) = ite_j + e_k \qquad \beta_j'(0) = ie_j \equiv \frac{\partial}{\partial y^j}\Big|_{z_0}$$

Consider then, $d\pi(\gamma'(0)) = (\pi \circ \gamma)'(0)$. Calculate,

$$(\pi \circ \alpha_j)(t) = \pi(te_j + e_k) = [(0, \dots, t, \dots, \underset{\uparrow k^{th}\text{ entry}}{1}, \dots, 0)]$$

$$\varphi_k(\pi(\alpha_j(t))) = (0, \dots, t, \dots, 0)$$

$$(\varphi_k \circ \pi \circ \alpha_j)'(0) = (0, \dots, 1, \dots, 0)$$

Likewise

$$\varphi_k(\pi(\beta_j(t))) = \varphi_k[(0, \dots, it, \dots, \overset{k^{th}\text{ entry}}{1}, \dots 0)]$$
$$= (0, \dots, it, \dots, 0)$$

$$(\varphi_k \circ \pi \circ \beta_j)'(0) = (0, \dots, i, \dots, 0)$$

Thus as we cycle through pushing forward these basis for $\mathbb{C}^{n+1} - \{0\}$ (modulo e_k and ie_k) we find $(2n)$ - LI outputs $\Longrightarrow d\pi_{z_0}$ surjective.//

P 30 Problem 5-1, p. 123

Consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\Phi(x,y,s,t) = (x^2+y,\ x^2+y^2+s^2+t^2+y)$$

Show that $(0,1)$ is regular value of Φ, and that the level set $\Phi^{-1}(0,1)$ is diffeomorphic to S^2

$$J_\Phi = [\partial_x \Phi \mid \partial_y \Phi \mid \partial_s \Phi \mid \partial_t \Phi]$$
$$= \begin{bmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y & 2s & 2t \end{bmatrix}$$

We need $\text{rank}(J_\Phi) = 2$ for $d\Phi$ to be a surjection.

$(x,y,s,t) \in \Phi^{-1}(0,1)$ gives that

$$x^2+y = 0 \quad \text{and} \quad x^2+y^2+s^2+t^2+y = 1$$

Hence $x^2+y^2+s^2+t^2+y = 1 \Rightarrow \underline{y^2+s^2+t^2 = 1}$. suggests diffeomorphism to S^2

Thus for $(x,y,s,t) \in \Phi^{-1}(0,1)$ we cannot have $y = s = t = 0$ $*$

$$\det\begin{bmatrix} 2x & 1 \\ 2x & 2y \end{bmatrix} = 4xy - 2x = 2x(2y-1)$$

$$\det\begin{bmatrix} 1 & 0 \\ 2y & 2s \end{bmatrix} = 2s \qquad \det\begin{bmatrix} 1 & 0 \\ 2y & 2t \end{bmatrix} = 2t \qquad \det\begin{bmatrix} 2x & 0 \\ 2x & 2t \end{bmatrix} = 2xt$$

If any one of these 2×2 determinants is non zero for $\Phi^{-1}(0,1)$ then that suffices to show $\text{rank}(J_\Phi) = 2$,

- If both $s=0$ and $t=0$ then $y \neq 0$ by $*$, but $x^2 = -y \neq 0$ hence $x \neq 0$. From $y^2+s^2+t^2=1$ and $s=t=0$ we get $y^2=1 \Rightarrow \underline{y = \pm 1}$. However, $x^2 = -y \Rightarrow \underline{y = -1}$.
  Hence $x^2 = 1 \therefore x = \pm 1$ and we find $2x(2y-1) \neq 0$.
- If either $s \neq 0$ or $t \neq 0$ then $2s \neq 0$ or $2t \neq 0$. Hence in all cases $\Phi^{-1}(0,1)$ gives J_Φ with rank two.
  Thus $(0,1)$ is $\underline{\text{regular}}$ $\underline{\text{value}}$ of Φ

P30 continued

$$\Phi^{-1}(0,1) = \{(x,y,s,t) \mid y^2+s^2+t^2=1,\ \cancel{x=\sqrt{-y}}\ \overset{x^2+y=0}{} \}$$

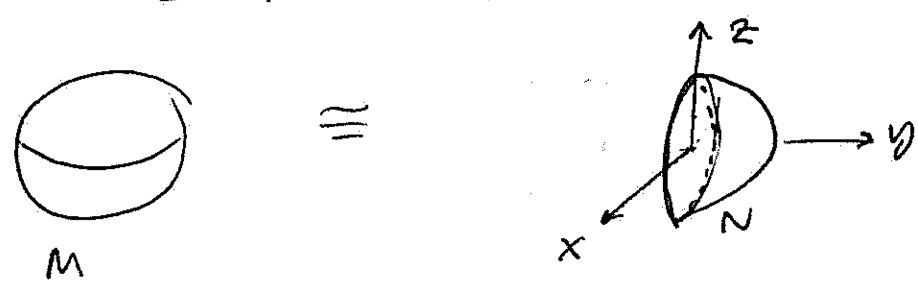
If $x^2+y=0$ then $x^2=-y \geq 0 \Rightarrow \underline{y \leq 0}$.

I don't see how we have all of S^2 if y is not free to range over $-1 \leq y \leq 1$.

Ok, so returning to def^n of Φ,

$$x^2+y^2+s^2+t^2+y = 1 \quad \& \quad x^2+y=0$$

Nope. I don't see it. Unless



But $\partial M = \phi$ whereas $\partial N \neq \phi$ (disk, vertical on $y=0$)

$\boxed{P31}$ Problem 5-6, p. 123

Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dim'l submanifold and let $UM \subseteq T\mathbb{R}^n$ be the unit tangent bundle
$$UM = \{ (x,v) \in T\mathbb{R}^n \mid x \in M, v \in T_x M, \|v\| = 1 \}$$
Prove UM is embedded $(2m-1)$-dim'l submanifold of $T\mathbb{R}^n$

Consider $F : T\mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $F(x,v) = v \cdot v$

Or, better yet, consider $G : TM \longrightarrow \mathbb{R}$ defined by
$$G(x,v) = v \cdot v = (v^1)^2 + (v^2)^2 + \cdots + (v^n)^2$$

Notice $G = F|_{TM}$ so we might as well study F since TM is submanifold of $T\mathbb{R}^n$ (Th$^{\underline{m}}$ 5.27 says restriction to submanifold of smooth map is smooth)

$\varphi = (\varphi^1, \dots, \varphi^n, \varphi^{n+1}, \dots, \varphi^{2n})$ coord. map on $T\mathbb{R}^n$ which picks off coord. of x and v
$$\varphi(x,v) = (x^1, x^2, \dots, x^n, v^1, v^2, \dots, v^n)$$

Better notation $\varphi = (x, y)$
$$x(x,v) = x \in \mathbb{R}^n$$
$$y(x,v) = v \in \mathbb{R}^n$$

Oh, then $F(x,y) = (y_1)^2 + \cdots + (y_n)^2$ hence,
$$J_F = [dF] = \left[\frac{\partial F}{\partial x^1} \middle| \cdots \middle| \frac{\partial F}{\partial x^n} \middle| \frac{\partial F}{\partial y^1} \middle| \cdots \middle| \frac{\partial F}{\partial y^n} \right]$$
$$J_F(a,b) = [0 | \cdots | 0 | 2b_1 | 2b_2 | \cdots | 2b_n]$$

$U(M) = G^{-1}\{1\}$ and $\operatorname{rank}(dG_p) = 1 \quad \forall p \in U(M)$

thus $U(M)$ is embedded submanifold of codimension 1 of TM.

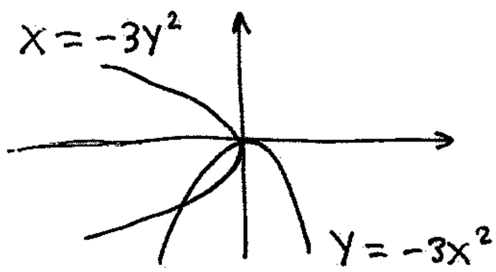
Since $\dim(TM) = 2\dim(M) = 2m \Rightarrow \underline{\dim(UM) = 2m-1}$.

P32 Problem 5-7, p. 123 of John Lee

Consider $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,y) = x^3 + xy + y^3$.
Which level sets of F are embedded submanifolds of $\mathbb{R}^2$?
For each level set, prove either that it is or that it is not an embedded submanifold

Consider $J_F = \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right] = \left[3x^2 + y, \; x + 3y^2 \right]$

If $3x^2 + y = 0$ and $x + 3y^2 = 0$ then $y = -3x^2$ and $x = -3y^2$



We should expect two critical points for F.

$$y = -3x^2 = -3(-3y^2)^2$$

Then $y = -27y^4$ hence either $\underline{y = 0}$ or $y^3 = \frac{-1}{27}$ $\therefore$ $y = \sqrt[3]{\frac{-1}{27}} = \frac{-1}{3}$.

Every point except $(0,0)$ and $(-1/3, -1/3)$ is regular for F. Notice $F(0,0) = 0^3 + 0(0) + 0^3 = 0$ and

$$F(-1/3, -1/3) = \left(\tfrac{-1}{3}\right)^3 + \left(\tfrac{-1}{3}\right)\left(\tfrac{-1}{3}\right) + \left(\tfrac{-1}{3}\right)^3 = \frac{-2}{27} + \frac{1}{9} = \frac{1}{27}.$$

Let $C = F^{-1}\{\alpha\}$ where $\alpha \neq 0, \frac{1}{27}$ then every point in C is regular for F hence C is embedded submanifold by the Regular Level Set Th$^{\underline{m}}$ (Cor 5.14)

P33 Problem 5-10, p. 123

For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \{ (x,y) \mid y^2 = x(x-1)(x-a) \}$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold?

Let $F(x,y) = x(x-1)(x-a) - y^2 = x^3 - (a+1)x^2 + ax - y^3$

Calculate $\frac{\partial F}{\partial x} = 3x^2 - 2(a+1)x + a$ and $\frac{\partial F}{\partial y} = -3y^2$

then we find critical point of F requires both

$3x^2 - 2(a+1)x + a = 0$ and $-3y^2 = 0$ then

we find $y = 0$ and $x = \dfrac{2(a+1) \pm \sqrt{4(a+1)^2 - 12a}}{6}$

Notice if $(x,y) \in M_a$ and $y = 0$ then $0 = x(x-1)(x-a)$ which implies $(0,0)$, $(1,0)$, $(a,0)$ are the points on M_a which lie on the x-axis. We should find if $\frac{\partial F}{\partial x} = 0$ for any of these points,

$$\frac{\partial F}{\partial x}(0,0) = a$$

$$\frac{\partial F}{\partial x}(1,0) = 3 - 2(a+1) + a = 1 - 2a$$

$$\frac{\partial F}{\partial x}(a,0) = 3a^2 - 2(a+1)a + a = a^2 - a = a(a-1)$$

Thus M_a is embedded submanifold for $a \neq 0, \frac{1}{2}, 1$.

(I leave the question about immersed submanifold to the reader)

$\boxed{P34}$ Show $SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$ is a proper embedded submanifold of $\mathbb{R}^{n\times n}$ (Example 2.5.5 of Conlon's Diff. Manifolds)

Consider $\det : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ is polynomial in the standard coord. on $\mathbb{R}^{n\times n}$, $\det(A) = \sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2 \cdots i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$

thus det is a smooth map. We need to show that $\det : Gl(n) \longrightarrow \mathbb{R}$ has constant rank 1. Consider the linear map, for $A \in Gl(n)$,

$$d(\det)_A : T_A(Gl(n)) \longrightarrow T_{\det(A)}(\mathbb{R})$$

we seek a curve $\gamma : (-\epsilon, \epsilon) \longrightarrow Gl(n)$ with $\gamma(0) = A$ and $\gamma(t) \in Gl(n)$ for $|t|$ sufficiently small.

Let $\gamma(t) = \left[\begin{array}{c} (1+t)\,\mathrm{row}_1(A) \\ \hline \mathrm{row}_2(A) \\ \hline \vdots \\ \hline \mathrm{row}_n(A) \end{array}\right]$ then $\gamma(0) = A \in Gl(n)$

and since $Gl(n)$ is open it follows $\gamma(t) \in Gl(n)$ for $|t| < \varepsilon$ for some $\varepsilon > 0$. Notice

$$\det(\gamma(t)) = (1+t)\det(A) \neq 0 \quad \text{for} \quad -\varepsilon < t < \varepsilon$$

Then $d(\det)_A(\gamma'(0)) = \frac{d}{dt}\Big|_{t=0} (1+t)\det A = \det(A) \neq 0$

thus $\mathrm{rank}(d(\det_A)) = 1$ for $A \in Gl(n)$. Then

$$SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\} = \det^{-1}\{1\}$$

is an embedded regular submanifold since the det. map is regular everywhere in $\det^{-1}\{1\}$.