Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. 5pts per problem here.

Problem 35 SM Problem 7-4, page 171.

Problem 36 SM Problem 7-7, page 172.

Problem 37 SM Problem 7-9, page 172. (no proof of smoothness expected)

Problem 38 SM Problem 7-16, page 172. (no proof of smoothness expected)

Problem 39 SM Problem 8-3, page 199. (think about coordinate charts and open sets perhaps)

Problem 40 SM Problem 8-10, page 201.

Problem 41 SM Problem 8-16, page 201.

Problem 42 SM Problem 8-19, page 202.

Problem 43 SM Problem 8-20, page 202

Problem 44 SM Problem 8-27, page 203

Problem 45 SM Problem 8-28, page 203

Problem 46 SM Problem 8-29, page 203

P35 Problem 7-4, p. 171

Let $\det : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}$ denote the determinant function. Use Cor 3.25 $dF_p(v) = (F\circ\gamma)'(0)$ where $\gamma : J \subseteq \mathbb{R} \to M$ and $F : M \to N$ smooth where $\gamma(0) = p$, $\gamma'(0) = v$ to show,

(a.) For any $A \in \mathbb{R}^{n\times n}$, show that

$$\frac{d}{dt}\Big|_{t=0} \det(I_n + tA) = \text{trace}(A)$$

(Hint tells us to look at Appendix)

(b.) For $X \in GL(n,\mathbb{R})$ and $B \in T_X GL(n,\mathbb{R}) \equiv \mathbb{R}^{n\times n}$

Show that $d(\det)_X(B) = (\det X)\,tr(X^{-1}B)$

[Hint: $\det(X + tB) = \det(X)\det(I_n + tX^{-1}B)$]

(a.) $$\det(I + tA) = \sum_{i_1,\dots,i_n} \epsilon_{i_1\cdots i_n} (e_1 + t\,col_1(A))_{i_1} \cdots (e_n + t\,col_n(A))_{i_n}$$

$$= \sum_{i_1,\dots,i_n} \epsilon_{i_1\cdots i_n} \prod_{j=1}^{n} (e_j + t\,col_j(A))_{i_j}$$

Then differentiate and use the extended product rule,

$$\frac{d}{dt}\Big[\det(I+tA)\Big] = \sum_{i_1\cdots i_n} \epsilon_{i_1\cdots i_n} \sum_{k=1}^{n} \prod_{j\neq k}^{n} (e_j + t\,col_j(A))_{i_j} \frac{d}{dt}\Big[(e_k + t\,col_k(A))_{i_k}\Big]$$

$$= \sum_{i_1\cdots i_n} \epsilon_{i_1\cdots i_n} \sum_{k=1}^{n} \prod_{j\neq k}^{n} (e_j + t\,col_j(A))_{i_j} (col_k(A))_{i_k}$$

Therefore, notice we can pull-out the sum over k,

$$\frac{d}{dt}\Big|_{t=0} \det[I+tA] = \sum_{k=1}^{n}\Big[\sum_{i_1\cdots i_n} \epsilon_{i_1\cdots i_n} \prod_{j\neq k}^{n} (e_j + \underset{0}{t}\,col_j(A))_{i_j} (col_k(A))_{i_k}\Big]$$

$$= \sum_{k=1}^{n} \sum_{i_1\cdots i_n} \epsilon_{i_1\cdots i_n} (e_1)_{i_1} \cdots (e_{k-1})_{i_{k-1}} (col_k(A))_{i_k} \cdots (e_n)_{i_n}$$

$$= \sum_{k=1}^{n} \det[e_1|\cdots|e_{k-1}|col_k(A)|e_{k+1}|\cdots|e_n]$$

$$= \sum_{k=1}^{n} A_{kk} = \text{trace}(A).$$

P35 continued

(b.) Suppose $X \in GL(n,\mathbb{R})$ and $B \in T_X GL(n,\mathbb{R}) \equiv \mathbb{R}^{n\times n}$ then since det is continuous we find $\gamma(t) = X + tB$ takes values in $GL(n,\mathbb{R})$ for t sufficiently close to $t=0$ since $\gamma(0) = X$ and $\det(\gamma(0)) = \det(X) \neq 0$.

Notice $\gamma'(0) = B$ hence calculate,

$$d(\det_X)(\gamma'(0)) = (\det\circ\gamma)'(0)$$

To calculate the derivative on the RHS above,

$$\begin{aligned}(\det\circ\gamma)(t) &= \det(X + tB)\\ &= \det(X(I + tX^{-1}B))\\ &= \det(X)\det(I + tX^{-1}B)\end{aligned}$$

Consequently,

$$\begin{aligned}(\det\circ\gamma)'(0) &= \det(X)\frac{d}{dt}\Big|_{t=0}\det(I + tX^{-1}B)\\ &= \det(X)\,\mathrm{trace}(X^{-1}B) \quad \text{by (a.)}\end{aligned}$$

Therefore,

$$\underline{d(\det_X)(B) = \det(X)\,\mathrm{tr}(X^{-1}B)}\ .//$$

P36 Problem 7-7, p. 172

Prove Prop. 7.15: Let G be Lie group and let G_0 be its identity component. Then G_0 is <u>normal</u> subgroup of G and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0

The connected components partition G. Suppose G_0 is connected component of e. Consider $h \in G$ and notice $h \in hG_0$ and $h \in G_0 h$ as $h = eh = he$ hence $hG_0 = G_0 h$ as both hG_0 and $G_0 h$ are images of connected comp. containing e under L_h or R_h which are diffeomorphisms. We "know" a homeomorphism must map components to components. Thus $G_0 \trianglelefteq G$

If $H_0 \subseteq G_0$ is open then H_0 is open nbhd of e and thus, by Prop. 7.14, H_0 generates $G_0 \hookrightarrow H_0 = G_0$.

If $\tilde{G}$ is connected component of G then if $x_0 \in \tilde{G}$ notice $L_{x_0}(G_0) = x_0 G_0 = \tilde{G}$ hence $\tilde{G}$ diffeomorphic to G_0

Remark: this solution is probably missing abit topologically speaking, but I stop here.

P37 Problem 7-9

Show that $A \cdot [x] = [Ax]$ defines a smooth, transitive left-action of $Gl(n+1, \mathbb{R})$ on $\mathbb{R}P^n$

We need to verify that for all $A, B \in Gl(n+1, \mathbb{R})$ and $[x] \in \mathbb{R}P^n$,

$$A \cdot (B \cdot [x]) = (AB) \cdot [x]$$
$$I \cdot [x] = [x]$$

Recall $[x] = \{\lambda x \mid \lambda \in \mathbb{R}\} = \text{span}\{x\}$ where $x \neq 0$

Consider $I \cdot [x] = [Ix] = [x]$. Also,

$A \cdot (B \cdot [x]) = A \cdot ([Bx]) = [A(Bx)] = [(AB)x] = (AB) \cdot [x]$.

It remains to show the action is transitive.

Suppose $[x], [y] \in \mathbb{R}P^n$. It suffices to show

$\exists A \in Gl(n+1, \mathbb{R})$ for which $y = Ax$.

Notice $x, y \neq 0$ as $[x], [y] \in \mathbb{R}P^n$.

If $y = kx$ then $kI_{n+1} \in Gl(n+1, \mathbb{R})$ and we note

$(kI_{n+1}) \cdot [x] = [kIx] = [kx] = [y]$.

If $y \neq kx$ then we may extend $\{x, y\}$ to a basis $\beta = \{x, y, v_3, \dots, v_{n+1}\}$. Notice $\Phi_\beta(x) = e_1$ and $\Phi_\beta(y) = e_2$

Let $A = [\Phi_\beta^{-1} \circ L_B \circ \Phi_\beta]$ where $B = [e_2 | e_1 | e_3 | \cdots | e_{n+1}]$

then
$$\begin{aligned} Ax &= \Phi_\beta^{-1}(L_B(\Phi_\beta(x))) \\ &= \Phi_\beta^{-1}([e_2 | e_1 | e_3 | \cdots | e_{n+1}] e_1) \\ &= \Phi_\beta^{-1}(e_2) \\ &= y \end{aligned}$$

and note A is invertible since $\det A = \det \Phi_\beta^{-1} \det B \det \beta \neq 0$

Thus $A \cdot [x] = [Ax]$ is a transitive action.

$\boxed{P38}$ Problem 7-16, p. 172

$\boxed{\text{Prove } SU(2) \text{ is diffeomorphic } S^3}$

$$SU(2) = \{ A \in \mathbb{C}^{2\times 2} \mid A^{\dagger}A = I,\ \det(A) = 1 \}$$

$$A^{\dagger} = A^{-1} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\dagger} = \frac{1}{\underbrace{ad-bc}_{\det A = 1}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus $a^* = d$ and $b^* = -c$ or $c = -b^*$

Let $a = a_1 + i a_2$ and $b = b_1 + i b_2$

Then $d = a_1 - i a_2$ and $c = -b_1 + i b_2$

Therefore, $SU(2) = \left\{ \left[\begin{array}{c|c} a_1 + i a_2 & b_1 + i b_2 \\ \hline -b_1 + i b_2 & a_1 - i a_2 \end{array}\right] \;\middle|\; \underbrace{a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1}_{\det(A)} \right\}$

$$F\left(\left[\begin{array}{c|c} a_1 + i a_2 & b_1 + i b_2 \\ \hline -b_1 + i b_2 & a_1 - i a_2 \end{array}\right] \right) = (a_1, a_2, b_1, b_2)$$

defines map $F : SU(2) \longrightarrow S^3$ and the mapping is clearly smooth and invertible

$$F^{-1}(a_1, a_2, b_1, b_2) = \left[\begin{array}{c|c} a_1 + i a_2 & b_1 + i b_2 \\ \hline -b_1 + i b_2 & a_1 - i a_2 \end{array}\right]$$

Thus $SU(2)$ diffeomorphic to S^3.

P39 Problem 8-3, p. 199

Let M be nonempty positive-dim'l smooth manifold with or w/o boundary. Show $\mathfrak{X}(M)$ is infinitesimal-dim'l

Let (U, x) be chart on M then

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n} \in \mathfrak{X}(M)$$

However, if $f \in C^\infty(M)$ then $f \frac{\partial}{\partial x^1} \in \mathfrak{X}(M)$ as well. But, $\exists \infty$ly many LI $f \in C^\infty(M)$ for instance $f_\lambda(x) = e^{\lambda x^1}$ is LI to $f_\beta(x) = e^{\beta x^1}$ if $\lambda \neq \beta$ thus $\exists \infty$ly many functions $(\lambda, \beta \in \mathbb{R})$ in $C^\infty(M)$ which are LI. We can argue

$$S\left\{ e^{\lambda x^1} \frac{\partial}{\partial x^1} \;\middle|\; \lambda \in \mathbb{R} \right\} \subseteq \mathfrak{X}(M)$$

and S is LI. If $\lambda_1, \lambda_2, \dots, \lambda_n$ distinct,

$$\sum_{i=1}^{n} c_i e^{\lambda_i x^1} \frac{\partial}{\partial x^1} = 0$$

$$\Rightarrow \sum_{i=1}^{n} c_i e^{\lambda_i x^1} = 0$$

let's be lazier,

Then $\left(\frac{\partial}{\partial x^1} - \lambda_j \right) \sum_{i=1}^{n} c_i e^{\lambda_i x^1} = 0$

$$\Rightarrow \sum_{i=1}^{n} c_i (\lambda_i - \lambda_j) e^{\lambda_i x^1} = 0$$

$$S = \left\{ (x^1)^n \frac{\partial}{\partial x^1} \;\middle|\; n \in \mathbb{N} \right\} \subseteq \mathfrak{X}(M)$$

clearly LI since $\{ x^1, (x^1)^2, \dots, (x^1)^n, \dots \}$ is LI.

P40 Problem 8-10, p. 201

$M = (0,\infty)\times(0,\infty)$

$F : M \longrightarrow M$ defn by $F(x,y) = (xy,\ y/x)$

Show that F is a diffeomorphism and compute $F_* X$ and $F_* Y$ where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{and} \quad Y = y\frac{\partial}{\partial x}$$

Notice $J_F = \left[\frac{\partial F}{\partial x} \Big| \frac{\partial F}{\partial y} \right] = \begin{bmatrix} y & x \\ -y/x^2 & 1/x \end{bmatrix}$ has

$\det(J_F) = \frac{y}{x} + \frac{xy}{x^2} = \frac{2y}{x} \neq 0$ thus F is local diffeomorphism since it is smooth with $\text{rank}(J_F) = 2$ on M.

① If $F(x,y) = F(a,b)$ then $(xy,\ y/x) = (ab,\ b/a)$

hence $xy = ab$ and $\frac{y}{x} = \frac{b}{a} \Rightarrow ay = bx$

multiplying by a gives $axy = a^2 b \Rightarrow bx^2 = a^2 b$

$\Rightarrow x^2 = a^2$

$\Rightarrow \underline{x = a}$ (since $a, x > 0$)

then $xy = ab \Rightarrow xy = xb \Rightarrow \underline{y = b}$.

Thus F is injective.

② To show F onto we can for about the same trouble find formula for F^{-1},

$(xy,\ y/x) = (u, v)$ ← now solve for x & y

$$\left.\begin{aligned} xy &= u \\ \frac{y}{x} &= v \end{aligned}\right\} \quad \frac{xy}{y/x} = \frac{u}{v} \Rightarrow x^2 = \frac{u}{v} \therefore \underline{x = \sqrt{\frac{u}{v}}}.$$

Likewise $(xy)\left(\frac{y}{x}\right) = uv \Rightarrow y^2 = uv \therefore \underline{y = \sqrt{uv}}$.

P40 continued

$$F^{-1}(u,v) = \left(\sqrt{\tfrac{u}{v}}\,,\ \sqrt{uv}\right)$$

Let $(u,v) \in M$ then $F(F^{-1}(u,v)) = (u,v)$

since $F\left(\sqrt{\tfrac{u}{v}}, \sqrt{uv}\right) = \left(\sqrt{\tfrac{u}{v}}\sqrt{uv}, \dfrac{\sqrt{uv}}{\sqrt{\tfrac{u}{v}}}\right)$

$$= \left(\sqrt{u^2}, \sqrt{v^2}\right)$$

$$= (u, v) \quad \text{as } u,v > 0.$$

Thus F onto and we've shown F is smooth bijection, indeed F^{-1} is also clearly smooth on M.

Following Example 8.20: $(F_* X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})$

$$[dF_{(x,y)}] = \begin{bmatrix} y & x \\ -y/x^2 & 1/x \end{bmatrix} \qquad F^{-1}(u,v) = \left(\sqrt{\tfrac{u}{v}}, \sqrt{uv}\right)$$

$$[dF_{F^{-1}(u,v)}] = \begin{bmatrix} \sqrt{uv} & \sqrt{\frac{u}{v}} \\ -\sqrt{\frac{v^3}{u}} & \sqrt{\frac{v}{u}} \end{bmatrix}$$

$$X_{(x,y)} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \longrightarrow X_{F^{-1}(u,v)} = \sqrt{\tfrac{u}{v}}\frac{\partial}{\partial x} + \sqrt{uv}\frac{\partial}{\partial y}$$

$$Y_{(x,y)} = y\frac{\partial}{\partial x} \longrightarrow X_{F^{-1}(u,v)} = \sqrt{uv}\frac{\partial}{\partial x}$$

Therefore,

$$(F_* X)_{(u,v)} = \left(\sqrt{uv}\sqrt{\tfrac{u}{v}} + \sqrt{\tfrac{u}{v}}\sqrt{uv}\right)\frac{\partial}{\partial u}\Big|_{(u,v)} + \left(-\sqrt{\tfrac{v^3}{u}}\sqrt{\tfrac{u}{v}} + \sqrt{\tfrac{v}{u}}\sqrt{uv}\right)\frac{\partial}{\partial v}\Big|_{(u,v)}$$

Some cancellation,

$$\boxed{(F_* X) = 2u\frac{\partial}{\partial u}}$$

P40 continued

$$\underbrace{[F_* X]}_{\text{weights for } \left.\frac{\partial}{\partial u}\right|_{(u,v)} \text{ and } \left.\frac{\partial}{\partial v}\right|_{(u,v)}} = \left[dF_{F^{-1}(u,v)}\right]\left[X_{F^{-1}(u,v)}\right]$$

$$[F_* Y]_{(u,v)} = \begin{bmatrix} \sqrt{uv} & \sqrt{\frac{u}{v}} \\ -\sqrt{\frac{v^3}{u}} & \sqrt{\frac{v}{u}} \end{bmatrix}\begin{bmatrix} \sqrt{uv} \\ 0 \end{bmatrix} = \begin{bmatrix} uv \\ -v^2 \end{bmatrix}$$

$$(F_* Y)_{(u,v)} = uv \left.\frac{\partial}{\partial u}\right|_{(u,v)} - v^2 \left.\frac{\partial}{\partial v}\right|_{(u,v)}$$

$$\therefore \boxed{F_* Y = uv \frac{\partial}{\partial u} - v^2 \frac{\partial}{\partial v}}$$

How to calculate dF directly? $F^1 = xy$ and $F^2 = \frac{y}{x}$

$$dF_p\left(\left.\frac{\partial}{\partial x}\right|_p\right) = \frac{\partial F^1}{\partial x}(p) \left.\frac{\partial}{\partial u}\right|_{F(p)} + \frac{\partial F^2}{\partial x}(p) \left.\frac{\partial}{\partial v}\right|_{F(p)}$$

$$dF_p\left(\left.\frac{\partial}{\partial y}\right|_p\right) = \frac{\partial F^1}{\partial y}(p) \left.\frac{\partial}{\partial u}\right|_{F(p)} + \frac{\partial F^2}{\partial y}(p) \left.\frac{\partial}{\partial v}\right|_{F(p)}$$

Hence,

$$dF(\partial_x) = y \frac{\partial}{\partial u} - \frac{y}{x^2}\frac{\partial}{\partial v}$$

$$dF(\partial_y) = x \frac{\partial}{\partial u} + \frac{1}{x}\frac{\partial}{\partial v}$$

Then $dF(y\partial_x) = y\left(y\frac{\partial}{\partial u} - \frac{y}{x^2}\frac{\partial}{\partial v}\right) = y^2\frac{\partial}{\partial u} - \frac{y^2}{x^2}\frac{\partial}{\partial v}$

However, $u = xy$ and $v = y/x$ thus $x = \sqrt{\frac{u}{v}}$ and $y = \sqrt{uv}$

hence $y^2 = uv$ and $\frac{-y^2}{x^2} = \frac{-uv}{\frac{u}{v}} = -v^2$ $\therefore$ $\underline{F_*(y\partial_x) = uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}}$

P40 continued

(calculating via direct coordinate calculus)

$$F_* \mathbb{X} = dF(x\,\partial_x + y\,\partial_y)$$
$$= x\, dF(\partial_x) + y\, dF(\partial_y)$$
$$= x\left(y \frac{\partial}{\partial u} - \frac{y}{x^2}\frac{\partial}{\partial v}\right) + y\left(x \frac{\partial}{\partial u} + \frac{1}{x}\frac{\partial}{\partial v}\right)$$
$$= 2xy \frac{\partial}{\partial u}$$
$$= 2\sqrt{\frac{u}{v}}\sqrt{uv}\ \frac{\partial}{\partial u}$$
$$= \underline{2u \frac{\partial}{\partial u}}.$$

writing x & y in terms of u & v requires we know formula for $F^{-1}(u,v) = (x,y)$

I found the formulas by solving

$$F(x,y) = (u,v)$$

for x and y. Certainly we can calculate $dF(\mathbb{X})$ when $F: M \to N$ is not invertible, but if F is not 1-1 and $\mathbb{X} \in \mathfrak{X}(M)$ then it may not be the case that $F_*\mathbb{X} \in \mathfrak{X}(N)$.

P41 Problem 8-16, p. 201 Calculate $[X, Y]$

(a.) $X = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y} \qquad Y = \frac{\partial}{\partial y}$

$$[X, Y]f = X(Yf) - Y(X[f])$$

$$= (y\partial_z - 2xy^2\partial_y)\partial_y f - \partial_y(y\partial_z f - 2xy^2\partial_y f)$$

$$= \cancel{y\partial_z\partial_y f} - \cancel{2xy^2\partial_y^2 f} - \underbrace{\partial_z f - \cancel{y\partial_y\partial_z f}}_{\text{productrule}} + 2$$

$$+ 2x(2y)\partial_y f + \cancel{2xy^2\partial_y^2 f}$$

$$= -\partial_z f + 4xy\,\partial_y f$$

$$= \left[-\frac{\partial}{\partial z} + 4xy\frac{\partial}{\partial y}\right]f \quad \therefore \boxed{[X, Y] = 4xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z}}$$

Alternatively, since $\partial_x, \partial_y, \partial_z$ serves as "~~basis~~ frame" for $\mathcal{X}(\mathbb{R}^3)$

$$[X, Y] = ([X, Y]x)\frac{\partial}{\partial x} + ([X, Y]y)\frac{\partial}{\partial y} + ([X, Y]z)\frac{\partial}{\partial z}$$

$$[X, Y]x = X(Yx) - Y(Xx) = X(0) - Y(0) = 0.$$

$$[X, Y]y = X(Yy) - Y(Xy)$$

$$= X(1) - Y(-2xy^2)$$

$$= 0 + 4xy$$

$$[X, Y]z = X(Yz) - Y(Xz)$$

$$= X(0) - Y(y)$$

$$= -1.$$

Thus $\boxed{[X, Y] = 4xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z}}$

p41 continued

(b.) $\mathbb{X} = x\,\partial_y - y\,\partial_x$ $\qquad \mathbb{X}x = -y \qquad \mathbb{Y}x = 0$

$\mathbb{Y} = y\,\partial_z - z\,\partial_y$ $\qquad \mathbb{X}y = x \qquad \mathbb{Y}y = -z$

$\qquad \mathbb{X}z = 0 \qquad \mathbb{Y}z = y$

Then, $[\mathbb{X},\mathbb{Y}]x = \mathbb{X}(\mathbb{Y}x) - \mathbb{Y}(\mathbb{X}x) = -\mathbb{Y}(-y) = \mathbb{Y}(y) = -z$

$[\mathbb{X},\mathbb{Y}]y = \mathbb{X}(\mathbb{Y}y) - \mathbb{Y}(\mathbb{X}y) = \mathbb{X}(-z) - \mathbb{Y}(x) = 0$

$[\mathbb{X},\mathbb{Y}]z = \mathbb{X}(\mathbb{Y}z) - \mathbb{Y}(\mathbb{X}z) = \mathbb{X}(y) - \mathbb{Y}(0) = x$

Therefore, $\boxed{[\mathbb{X},\mathbb{Y}] = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}}$

Alternatively, to check my answer,

$$[\mathbb{X},\mathbb{Y}]f = \mathbb{X}(y\,\partial_z f - z\,\partial_y f) - \mathbb{Y}(x\,\partial_y f - y\,\partial_x f)$$

$$= (x\,\partial_y - y\,\partial_x)(y\,\partial_z f - z\,\partial_y f)$$

$$- (y\,\partial_z - z\,\partial_y)(x\,\partial_y f - y\,\partial_x f)$$

$$= x\,\partial_z f - z\,\partial_x f$$

$$= \left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right)f$$

(only keeping terms with <u>one</u> derivative since we've <u>proved</u> the 2nd order terms cancel)

Once more,

$$\boxed{[\mathbb{X},\mathbb{Y}] = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}}$$

(c.) $\mathbb{X} = x\,\partial_y - y\,\partial_x$ and $\mathbb{Y} = x\,\partial_y + y\,\partial_x$

$$[\mathbb{X},\mathbb{Y}]f = (x\,\partial_y - y\,\partial_x)(x\,\partial_y + y\,\partial_x)f - (x\,\partial_y + y\,\partial_x)(x\,\partial_y - y\,\partial_x)f$$

$$= x\,\partial_x f - y\,\partial_y f + y\,\partial_x f - y\,\partial_y f$$

$$= (x+y)\partial_x - 2y\,\partial_y f$$

$$\therefore \boxed{[\mathbb{X},\mathbb{Y}] = (x+y)\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}}$$

P42 Problem 8-19, p. 202

Show $\mathbb{R}^3$ with $\vec{V} \times \vec{W} = \sum_{i,j,k} \epsilon_{ijk} V_i W_j e_k$ is Lie Alg.

We know $(c\vec{V}_1 + \vec{V}_2) \times \vec{W} = c\,\vec{V}_1 \times \vec{W} + \vec{V}_2 \times \vec{W}$

and $\vec{W} \times (c\vec{V}_1 + \vec{V}_2) = c\,\vec{W} \times \vec{V}_1 + \vec{W} \times \vec{V}_2$ thus

the cross-product is <u>bilinear</u> map from $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

Moreover, $\vec{V} \times \vec{W} = -\vec{W} \times \vec{V}$ hence cross-prod. is antisymmetric.

To prove the Jacobi Identity it's helpful to have my Math 231 notes where it is proved

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

Thus,

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) =$$

$$= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} + (\vec{B} \cdot \vec{A})\vec{C} - (\vec{B} \cdot \vec{C})\vec{A} + (\vec{C} \cdot \vec{B})\vec{A} - (\vec{C} \cdot \vec{A})\vec{B}$$

$$= 0$$

Since $\vec{A} \cdot \vec{C} = \vec{C} \cdot \vec{A}$ and $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ and $\vec{B} \cdot \vec{C} = \vec{C} \cdot \vec{B}$.

Hence $(\mathbb{R}^3, \times)$ defines a Lie Algebra as ~~$\otimes$~~ $\times$ is bilinear, antisymmetric and satisfies <u>Jacobi</u> identity.

P43 Problem 8-20, p. 202

$A \subseteq \mathfrak{X}(\mathbb{R}^3)$ be subspace $A = \text{span}\{X, Y, Z\}$

$X = y\partial_z - z\partial_y$, $Y = z\partial_x - x\partial_z$, $Z = x\partial_y - y\partial_x$

Show A is Lie subalgebra of $\mathfrak{X}(\mathbb{R}^3)$ which is isomorphic to $\mathbb{R}^3$ with $\times$-product.

By properties of Lie Bracket we know $[X, X] = 0$ and $[Y, Y] = 0$ and $[Z, Z] = 0$. If we define $[\,,\,] : A \times A \longrightarrow A$ by linear extending from basis then bilinearity follows, really we just need to check closure of basis under bracket.

$$[X, Y] = [y\partial_z - z\partial_y,\ z\partial_x - x\partial_z]$$ see what I did?

$$= [y\partial_z,\ z\partial_x - x\partial_z] + [z\partial_x - x\partial_z,\ z\partial_y]$$
$$= y\partial_x - x\partial_y$$
$$= -Z$$

$$[X, Z] = [y\partial_z - z\partial_y,\ x\partial_y - y\partial_x]$$
$$= z\partial_x - x\partial_z$$
$$= Y$$

$$[Y, Z] = [z\partial_x - x\partial_z,\ x\partial_y - y\partial_x] = z\partial_y - y\partial_z = -X$$

Observe $[Y, X] = Z$ and $[X, Z] = Y$ and $[Z, Y] = X$

contrast to $\hat{x} \times \hat{y} = \hat{z}$ and $\hat{z} \times \hat{x} = \hat{y}$ and $\hat{y} \times \hat{z} = \hat{x}$

I'd like for $\Phi(\langle a, b, c\rangle) = aX + bY + cZ$ to serve as Lie alg. isomorphism, unfortunately,

P43 continued

$$\Phi(\hat{x} \times \hat{y}) = \Phi(\hat{z}) = Z$$

whereas

$$[\Phi(\hat{x}), \Phi(\hat{y})] = [X, Y] = -Z$$

Similarly for the rest of the basis, extrapolating,

$$\Phi(\vec{V} \times \vec{W}) = -[\Phi(\vec{V}), \Phi(\vec{W})]$$

which means Φ is not quite a Lie algebra homomorphism (it is bijective, though, so we might as well say isomorphism)

Here Φ is isomorphism of $\mathbb{R}^3$ to $\mathfrak{X}(\mathbb{R}^3)$ with the opposite bracket

$$[X, Y]_{opp} = [Y, X]$$

bilinear ✓
antisymmetric ✓
Jacobi ✓

all follow from corresp. traits for standard Lie Bracket.

The claim of the problem follows since

$$(\mathfrak{X}(M), [\,,\,]) \cong (\mathfrak{X}(M), [\,,\,]_{opp})$$

Lie Algebra isomorphism

(descends to $A \subseteq \mathfrak{X}(M)$ as well)

P44 Problem 8-27, p. 203

G & H Lie groups and suppose
$F: G \longrightarrow H$ a Lie group homomorphism
which is also a local diffeomorphism. Show
$F_*: \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is isomophism of Lie algebras.

Recall $\mathrm{Lie}(G) = \mathrm{LIVF}(G)$ where $X \in \mathrm{Lie}(G)$ means X is generated by its value at $e \in G$ and left-translation pushes X_e all around G,

$$X_g = dL_g(X_e) \quad \forall g \in G, \; X_e \in T_e G.$$

The push-forward under F of $X \in \mathrm{Lie}(G)$ is given by

$$\begin{aligned}
(F_*) X_g &= dF(X_g) \\
&= (dF)(dL_g(X_e)) \\
&= d(F \circ L_g)(X_e) \quad : \quad (F \circ L_g)(x) = F(gx) \\
& \qquad\qquad\qquad\qquad\qquad\qquad = F(g)F(x) \\
&= (dL_{F(g)} \circ dF)(X_e) \qquad\qquad = (L_{F(g)} \circ F)(x) \\
&= dL_{F(g)}(dF(X_e)) \\
&= dL_{F(g)}(Y_e) \\
&= Y_{F(g)}, \quad \text{so } F_* X = Y \in \mathrm{Lie}(H)
\end{aligned}$$

Then $F_*([X_1, X_2]) = [F_* X_1, F_* X_2]$ by Prop 8.30

(I'm just retracing some steps in proof of Thm 8.44 on pgs. 195 – 196)

continued

Problem 44 continued

Th$^{\underline{m}}$ 8.44 gives F_* is Lie algebra homomorphism ; $F_*([X_1, X_2]) = [F_* X_1, F_* X_2]$ for all $X_1, X_2 \in \text{Lie}(G)$. We're given $F: G \to H$ is local diffeomorphism thus $dF_g : T_g G \to T_{F(G)} H$ is isomorphism of vector spaces for each $g \in G$. Notice

$$\text{Lie}(G) \cong T_e G$$
$$\text{Lie}(H) \cong T_e H$$

Moreover, $dF_e : T_e G \to T_{F(e)} H = T_e H$ is an isomorphism. In Lee's notation,

$$(T_e G)^L = \text{Lie}(G)$$

$$\text{Lie}(G) = (T_e G)^L = \psi(T_e G)$$

ok, in summary,

$$\text{Lie}(G) = \psi(T_e G)$$
$$\text{Lie}(H) = \psi(T_e H) = \psi(dF_e(T_e G))$$

$$\psi^{-1}(\text{Lie}(H)) = dF_e(T_e G) = dF_e(\psi^{-1}(\text{Lie}\, G))$$

$$\text{Lie}(H) = (\psi \circ dF_e \circ \psi^{-1})(\text{Lie}\, G)$$

(any way, F_* isomorphism of Lie algebras Lie G and Lie H follows from Th$^{\underline{m}}$ 8.44 almost immediately)

P45 Problem 8-28, p. 203

$\det : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^*$ is Lie group homomorphism

Calculate its induced Lie algebra homomorphism, show it is the trace.

(use Prob. 7-4, $\left.\frac{d}{dt}\right|_{t=0} \det(I_n + tA) = tr(A)$

and $d(\det)_X(B) = \det(X)\, tr(X^{-1}B)$

$tr : \mathfrak{gl}(n,\mathbb{R}) \to \mathbb{R}$

$$(\det{}_*)_I (B) = \det(I)\, tr(I^{-1}B) = tr(B)$$

Hence $(\det{}_*)_I : T_I Gl(n,\mathbb{R}) \longrightarrow \mathbb{R}$ is the trace.

Here we use $T_e G = \mathfrak{g}$ rather than $LIVF(G)$.

P46 Problem 8-29, p. 203

Th^{m} 8.46 says $H \subseteq Gl(n,\mathbb{R}) \Rightarrow \mathfrak{h} \leq Lie(Gl(n,\mathbb{R})) \cong$

Under the isomorphism of Lie (H) with cannonically isomorphic subalgebra of $gl(n,\mathbb{R})$. Under this isomorphism,

$$\text{Lie}(SL(n,\mathbb{R})) \cong \mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \text{tr}\,A = 0\}$$

$$\text{Lie}(SO(n)) \cong \mathfrak{o}(n) = \{A \in \mathbb{R}^{n\times n} \mid A^T + A = 0\}$$

$$\text{Lie}(SL(n,\mathbb{C})) \cong \mathfrak{sl}(n,\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} \mid \text{tr}\,A = 0\}$$

$$\text{Lie}(U(n)) \cong \mathfrak{u}(n) = \{A \in \mathbb{C}^{n\times n} \mid A^* + A = 0\}$$

$$\text{Lie}(SU(n)) \cong \mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C})$$

I'll use Lie $(G) \cong T_e G$ and study curves through $e = I$. Let's begin,

① $SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A = 1\}$

Note $I \in SL(n,\mathbb{R})$ and by continuity of det $\exists\, \varepsilon > 0$ s.t. $\alpha(t) = I + tB \in SL(n,\mathbb{R})$ for all $t \in (-\varepsilon, \varepsilon)$ and $B = \alpha'(0) \in T_I SL(n,\mathbb{R})$. Using the identity

$$\left.\frac{d}{dt}\right|_{t=0} \det(I + tB) = \text{tr}\,B \qquad \text{(Problem 7-4)}$$

$$\left.\frac{d}{dt}\right|_{t=0} (1) = 0 = \text{tr}(B)$$

thus $T_I SL(n,\mathbb{R}) = \{B \in \mathbb{R}^{n\times n} \mid \text{tr}(B) = 0\}$
$= \mathfrak{sl}(n,\mathbb{R})$.

② $SO(n) = \{A \in \mathbb{R}^{n\times n} \mid A^TA = I \text{ and } \det(A) = 1\}$

Once more $I \in SO(n)$ and by continuity of the defining relations, $\alpha(t) = I + tB \in SO(n)$ $\forall t \in (-\varepsilon, \varepsilon)$ thus $\alpha'(0) = B \in T_I SO(n)$. Once more we find $tr(B) = 0$ since $\det(I + tB) = 1 \Rightarrow tr(B) = 0$.

Now we also have $\alpha(t)^T\alpha(t) = I$ thus

$$\left(\frac{d\alpha}{dt}\right)^T \alpha(t) + \alpha(t)^T \frac{d\alpha}{dt} = \frac{d}{dt}(I) = 0$$

Since $\alpha(0) = I$ and $\alpha'(0) = B$ then setting $t = 0$

$$B^T I + I^T B = 0 \Rightarrow B^T + B = 0$$

Notice $B_{ii} = -(B^T)_{ii} = -B_{ii} \Rightarrow B_{ii} = 0$ $\forall i = 1, 2, .., n$ thus $tr(B) = 0$ independent of the $\det(\alpha(t)) = 1$ condition, this is not surprising as $A^TA = I \Rightarrow \det A = \pm 1$. This is why the name

$$o(n) = \{B \in \mathbb{R}^{n\times n} / B^T + B = 0\}$$

is given, the same calculus gives $T_I O(n) = o(n)$

③ $SL(n, \mathbb{C}) = \{A \in \mathbb{C}^{n\times n} \mid \det(A) = 1\}$

The calculation of Problem 7-4, while done for $\mathbb{R}$, equally well goes for complex matrices.

$$\left.\frac{d}{dt}\right|_{t=0} \det(I + tB) = tr(B)$$

Thus $\alpha(t) = I + tB \in SL(n, \mathbb{C}) \Rightarrow \left.\frac{d}{dt}\right|_{t=0}(1) = tr(B)$

hence $tr(B) = 0$ $\therefore$ $T_I SL(n, \mathbb{C}) = \underbrace{sl(n, \mathbb{C})}_{\text{traceless } \mathbb{C}^{n\times n} \text{ matrices.}}$

(4) $U(n) = \{ A \in \mathbb{C}^{n\times n} \mid A^\dagger A = I \}$ $\quad A^\dagger \equiv (A^*)^T = (\overline{A})^T$

As usual $\exists \varepsilon > 0$ s.t. $\alpha(t) = I + tB$ $\forall t \in (-\varepsilon, \varepsilon)$ has $\alpha(t) \in U(n)$. Notice $\alpha(0) = I$ and $\alpha'(0) = B \in T_I U(n)$.

Since $\alpha(t) \in U(n)$,

$$\left(\overline{\alpha(t)}\right)^T \alpha(t) = I$$

$$\left(\frac{d\overline{\alpha}}{dt}\right)^T \alpha(t) + \overline{\alpha(t)}^T \frac{d\alpha}{dt} = \frac{d}{dt}(I) = 0$$

Setting $t = 0$,

$$(\overline{B})^T I + (\overline{I})^T B = 0 \hookrightarrow B^\dagger + B = 0$$

Thus $T_I U(n) = \{ B \in \mathbb{C}^{n\times n} \mid B^\dagger + B = 0 \} = u(n)$.

Remark: $B^\dagger + B = 0 \hookrightarrow B_{jj} = -\overline{B}_{jj}$ so the diagonal entries of $B \in u(n)$ are pure imaginary (not zero as was necessary for skew-symmetric B)

$B^\dagger = -B$ ← skew-Hermitian

(5) $SU(n) = \{ A \in \mathbb{C}^{n\times n} \mid A^\dagger A = I \ \& \ \det(A) = 1 \}$

$B \in T_I SU(n)$ clearly has $B^\dagger + B = 0$ by arguments already given in case (4). Now

$$\left.\frac{d}{dt}\right|_{t=0} \det(I + tB) = tr(B) \Rightarrow \underline{0 = tr(B)}.$$

Thus $T_I SU(n) = \{ B \in \mathbb{C}^{n\times n} \mid B^\dagger + B = 0 \text{ and } tr(B) = 0 \}$

$= su(n) = u(n) \cap sl(n, \mathbb{C})$.

$\boxed{P46}$ using matrix exponential

For $G \subseteq Gl(n, \mathbb{R})$ we can write

$\gamma(t) = e^{tB}$ and have $\gamma(0) = e^0 = I$ & $\gamma'(0) = B$

As we calculated

$$\frac{d}{dt}\left(e^{tB}\right) = Be^{tB}$$

Then the condition $\gamma(t) \in G$ when differentiated yields conditions describing $\mathfrak{g} = T_I G$ and by $\underline{t=1}$

$$\boxed{\det\left(e^{B}\right) = \exp(\mathrm{tr}(B))}$$ ← not an easy identity to derive.

So if $\det(e^B) = 1$ then $\exp(\mathrm{tr}(B)) = 1$ $\therefore$ $\underline{\mathrm{tr}(B) = 0}$.

(2) $\gamma(t) = e^{tB} \in SO(n)$

$$(e^{tB})^T e^{tB} = I$$

$\frac{d}{dt}$

$$(B^T e^{tB^T}) e^{tB} + (e^{tB})^T B e^{tB} = 0$$

$\underline{t=0}$ $\quad \underline{B^T + B = 0}$.

So, we have two options

1.) use $\alpha(t) = I + tB$ & Problem 7-4 to describe $T_I G$ for $G \subseteq Gl(n)$

2.) use $\gamma(t) = e^{tB}$ & $\det(e^B) = \exp(\mathrm{tr}(B))$