Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. 5pts per problem here.

Problem 47 SM Problem 9-3, page 245 (you pick two)

Problem 48 SM Problem 9-17, page 247

Problem 49 SM Problem 9-18, page 247

Problem 50 SM Exercise 11.3, page 273

Problem 51 SM Exercise 11.5, page 274

Problem 52 SM Exercise 11.7, page 274

Problem 53 SM Exercise 11.21, page 281 (don't do all of it, pick an interesting part)

Problem 54 SM Problem 11-5, page 300 (I suspect this is easy with the right theorem in mind)

Problem 55 SM Problem 11-7, page 300 (pullback calculation)

Problem 56 SM Problem 11-14, page 301 (exactness of vector fields, I hope the calculation here is not too horrible)

P47 Problem 9-3, p. 245 find the flow

(a.) $V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

I'll begin by calculating the integral curves,

$$V_{\gamma(t)} = \frac{d\gamma}{dt} = \frac{dx}{dt}\frac{\partial}{\partial x}\Big|_{\gamma(t)} + \frac{dy}{dt}\frac{\partial}{\partial y}\Big|_{\gamma(t)} = y\frac{\partial}{\partial x}\Big|_{\gamma(t)} + \frac{\partial}{\partial y}\Big|_{\gamma(t)}$$

We face

$$\frac{dx}{dt} = y \quad \text{and} \quad \underbrace{\frac{dy}{dt} = 1}_{\text{solve this } 1^{st}} \hookrightarrow \underline{y = y_0 + t}.$$

Then $\frac{dx}{dt} = y_0 + t \Rightarrow \underline{x = x_0 + ty_0 + \frac{1}{2}t^2}$.

$$\boxed{\tau_t(x,y) = (x + ty + \tfrac{1}{2}t^2,\ y + t)} \leftarrow \text{flow of } V.$$

(b.) $W = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$

$$\frac{dx}{dt} = x \quad \text{and} \quad \frac{dy}{dt} = 2y$$

$$x = x_0e^t \quad \text{and} \quad y = y_0e^{2t}$$

$$\boxed{\tau_t(x,y) = (xe^t,\ ye^{2t})} \leftarrow \text{flow of } W$$

(c.) $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

$$\frac{dx}{dt} = x \quad \text{and} \quad \frac{dy}{dt} = -y$$

$$x = x_0e^t \qquad y = y_0e^{-t}$$

$$\boxed{\tau_t(x,y) = (xe^t,\ ye^{-t})}$$

P47 continued

(d.) $\boxed{Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}}$

$$\frac{dx}{dt} = y \quad \text{and} \quad \frac{dy}{dt} = x$$

$$\frac{d^2y}{dt^2} = y$$

$$y = y_0 \cosh t + C_2 \sinh t$$

$$x = \frac{dy}{dt} = y_0 \sinh t + C_2 \cosh t \hookrightarrow x(0) = C_2 = x_0$$

Thus
$$x = y_0 \sinh t + x_0 \cosh t$$
$$y = y_0 \cosh t + x_0 \sinh t$$

$$\boxed{\tau_t(x,y) = (x\cosh t + y\sinh t,\ x\sinh t + y\cosh t)}$$

P48 Problem 9-17 For each k-tuple of vector fields on $\mathbb{R}^3$ either find coordinates (s^1, s^2, s^3) in nbhd of $(1,0,0)$ s.t. $V_i = \partial/\partial s^i$ for $i = 1, 2, \dots k$, or explain where there are none. (use Example 9.47 as model calculation)

(a.) $k=2$ $V_1 = \frac{\partial}{\partial x}$ and $V_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

Here $[V_1, V_2] = 0$ since partial derivatives commute.

$$\eta_t(x,y) = (x+t, y) \text{ flow of } V_1$$

$$\theta_t(x,y) = (x+t, y+t) \text{ flow of } V_2$$

$$\Phi(s,t) = (\eta_t \circ \theta_s)(1,0) = \eta_t(1+s, s) = (1+s+t, s)$$

P48 continued

(a.) $\Phi(s,t) = (1+s+t, s) = (x,y)$

Solve for s & t

$x = 1+s+t \longleftrightarrow x = 1+y+t \therefore \boxed{\begin{aligned} t &= x-y-1 \\ s &= y \end{aligned}}$

$y = s$

Setting $\boxed{s^1 = x-y-1 \text{ and } s^2 = y}$ yields

$$\frac{\partial}{\partial s^1} = \underbrace{\frac{\partial x}{\partial s^1}}_{1}\frac{\partial}{\partial x} + \underbrace{\frac{\partial y}{\partial s^1}}_{0}\frac{\partial}{\partial y} \qquad \left(\begin{aligned} x &= 1+s^2+s^1 \\ y &= s^2 \end{aligned}\right)$$

$$\frac{\partial}{\partial s^1} = \frac{\partial}{\partial x} \checkmark$$

$$\frac{\partial}{\partial s^2} = \frac{\partial x}{\partial s^2}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s^2}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \checkmark$$

(b.) $V_1 = (x+1)\partial_x - (y+1)\partial_y$ & $V_2 = (x+1)\partial_x + (y+1)\partial_y$

$$\begin{aligned}[V_1, V_2] &= [(x+1)\partial_x - (y+1)\partial_y, (x+1)\partial_x + (y+1)\partial_y] \\ &= (x+1)\partial_x - (y+1)\partial_y - (x+1)\partial_x + (y+1)\partial_y \\ &= 0\end{aligned}$$

(aww man, I was hoping this was $\neq 0$ since then there is nothing to do)

$\underline{V_1}$ $\frac{dx}{dt} = x+1$ & $\frac{dy}{dt} = -y-1$ $\qquad$ $\underline{V_2}$ $\frac{dx}{dt} = x+1$, $\frac{dy}{dt} = y+1$

P48 continued

V_1 $\quad \frac{dx}{dt} = x+1 \qquad \frac{dy}{dt} = -(y+1)$

$\frac{dx}{dt} - x = 1 \qquad \frac{dy}{dt} + y = -1$

$x(t) = c_1 e^t - 1 \qquad y(t) = c_2 e^{-t} - 1$

$x(0) = c_1 - 1 = x_0 \qquad y(0) = c_2 - 1 = y_0$

$c_1 = x_0 + 1 \qquad c_2 = y_0 + 1$

$\eta_t(x,y) = \left((x+1)e^t - 1,\ (y+1)e^{-t} - 1\right)$ flow of V_1

V_2 $\quad \frac{dx}{dt} = x+1$ and $\frac{dy}{dt} = y+1$

$x = (x_0+1)e^t - 1$ and $y = (y_0+1)e^t - 1$

$\Theta_t(x,y) = \left((x+1)e^t - 1,\ (y+1)e^t - 1\right)$ flow of V_2

$$\begin{aligned}\Phi(s,t) = (\eta_t \circ \Theta_s)(1,0) &= \eta_t(2e^s - 1,\ e^s - 1) \\ &= (2e^s e^t - 1,\ e^s e^{-t} - 1)\end{aligned}$$

- I bet $[V_1, V_2] = 0$ was miscalculated.

$$\begin{aligned}[V_1, V_2] &= [(x+1)\partial_x, (y+1)\partial_y] - [(y+1)\partial_y, (x+1)\partial_x] \\ &= 2[(x+1)\partial_x, (y+1)\partial_y] \\ &= 0.\end{aligned}$$

(nope, I miscalculated flows 1st time be careful with initial conditions)

$\Phi(s,t) = (2e^{s+t} - 1,\ e^{s-t} - 1)$ next solve $(x,y) = \Phi(s,t)$ for s and t ↓

P48 continued

$$x = 2e^{s+t} - 1 \Rightarrow e^{s+t} = \tfrac{1}{2}(x+1)$$
$$y = e^{s-t} - 1 \Rightarrow e^{s-t} = y+1$$

$$\pm\begin{pmatrix} s+t = \ln\left(\tfrac{1}{2}(x+1)\right) \\ s-t = \ln(y+1) \end{pmatrix}$$

$$\boxed{\begin{aligned} s &= \tfrac{1}{2}\left[\ln\left(\tfrac{1}{2}(x+1)\right) + \ln(y+1)\right] = s^2 \\ t &= \tfrac{1}{2}\left[\ln\left(\tfrac{1}{2}(x+1)\right) - \ln(y+1)\right] = s^1 \end{aligned}}$$

Let me check,

$$x = 2e^{s^2+s^1} - 1$$
$$y = e^{s^2-s^1} - 1$$
$$e^{s^2+s^1} = \tfrac{1}{2}(x+1)$$
$$e^{s^2-s^1} = y+1$$

$$\frac{\partial}{\partial s^1} = \frac{\partial x}{\partial s^1}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s^1}\frac{\partial}{\partial y}$$
$$= 2e^{s^2+s^1}\frac{\partial}{\partial x} - e^{s^2-s^1}\frac{\partial}{\partial y}$$
$$= \underline{(x+1)\frac{\partial}{\partial x} - (y+1)\frac{\partial}{\partial y} = V_1} \checkmark$$

$$\frac{\partial}{\partial s^2} = \frac{\partial x}{\partial s^2}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s^2}\frac{\partial}{\partial y}$$
$$= 2e^{s^2+s^1}\frac{\partial}{\partial x} + e^{s^2-s^1}\frac{\partial}{\partial y}$$
$$= \underline{(x+1)\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial y} = V_2} \checkmark$$

P48) continued,

(c.) $V_1 = x\,\partial_y - y\,\partial_x$ $\qquad [V_1, V_2] = [x\partial_y - y\partial_x,\ y\partial_z - z\partial_y]$

$V_2 = y\,\partial_z - z\,\partial_y$ $\qquad = x\,\partial_z - z\,\partial_x$

$V_3 = z\,\partial_x - x\,\partial_z$ $\qquad \neq 0$ hooray.

If I miscalculated here, my apologies $(\eta_{s^1} \circ \theta_{s^2} \circ \tau_{s^3})(1,0,0) = \Phi(s^1, s^2, s^3)$

etc...

P49) Consider $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ and $Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$ find flow θ of X and ψ of Y and show the flows do not commute explicitly (Prob 9-18)

X] $\frac{dx}{dt} = x$ & $\frac{dy}{dt} = -y$

$x = x_0 e^t \qquad y = y_0 e^{-t}$

$\theta_t(x,y) = (xe^t, ye^{-t})$

Y] $\frac{dx}{dt} = y$ & $\frac{dy}{dt} = x$

$x = x_0\cosh t + y_0 \sinh t$
$y = x_0 \sinh t + y_0 \cosh t$

$\psi_s(x,y) = (x\cosh s + y\sinh s,\ x\sinh s + y\cosh s)$

$$(\theta_t \circ \psi_s)(x,y) = \theta_t(x\cosh s + y\sinh s,\ x\sinh s + y\cosh s)$$
$$= ((x\cosh s + y\sinh s)e^t,\ (x\sinh s + y\cosh s)e^{-t})$$

contrast to

$$(\psi_s \circ \theta_t)(x,y) = \psi_s(xe^t,\ ye^{-t})$$
$$= (xe^t\cosh s + ye^{-t}\sinh s,\ xe^t\sinh s + ye^{-t}\cosh s)$$

Notice $(\theta_0 \circ \psi_0)(x,y) = (x,y) = (\psi_0 \circ \theta_0)(x,y)$

yet clearly $(x\sinh s + y\cosh s)e^{-t} \neq xe^t\sinh s + ye^{-t}\cosh s$

(forgive me if I don't prove ↑ in detail)

P50 Exercise 11.3 p. 273

Suppose V and W are vector spaces and $A : V \to W$ a linear map. Define $A^* : W^* \to V^*$ the dual map or transpose of A by $(A^*\omega)(v) = \omega(Av)$ for $\omega \in W^*$ and $v \in V$.

Show $A^*\omega \in V^*$ and A^* is linear map.

Let $x, y \in V$ and $c \in \mathbb{R}$,

$$\begin{aligned}(A^*\omega)(cx+y) &= \omega(A(cx+y)) \\ &= \omega(cA(x) + A(y)) && A \text{ linear map} \\ &= c\,\omega(A(x)) + \omega(A(y)) && \omega \in W^* \\ &= c(A^*\omega)(x) + (A^*\omega)(y) && : \text{Def}^{\underline{n}} \text{ of } A^*\end{aligned}$$

Thus $A^*\omega \in V^*$ as it is linear on V with values in $\mathbb{R}$ by construction. //

Let $\omega_1, \omega_2 \in W^*$ and let $c \in \mathbb{R}$. Let $v \in V$

$$\begin{aligned}(A^*(\omega_1 + c\omega_2))(v) &= (\omega_1 + c\omega_2)(Av) && : \text{def}^{\underline{n}} \text{ of } A^* \\ &= \omega_1(Av) + c\,\omega_2(Av) && : \text{def}^{\underline{n}} \text{ of } + \text{ in } W^* \\ &= (A^*\omega_1)(v) + c(A^*\omega_2)(v) && : \text{def}^{\underline{n}} \text{ of } A^* \\ &= (A^*\omega_1 + cA^*\omega_2)(v) && \forall v \in V\end{aligned}$$

Thus, $\underline{A^*(\omega_1 + c\omega_2) = A^*\omega_1 + cA^*\omega_2}$

Hence A^* is linear on W^* as claimed. //

$\boxed{P51}$ Exercise 11.5 p. 274

Prop 11.4 (prove it)

$(A \circ B)^* = B^* \circ A^*$

$(Id_V)^* : V^* \to V^*$ is identity map on V^*

Suppose $A : V \to W$ and $B : U \to V$ are linear maps then $A \circ B : U \xrightarrow{B} V \xrightarrow{A} W$ is a linear map. Let $\omega \in W^*$ and $x \in U$ and study

$$
\begin{aligned}
\big((A \circ B)^* \omega\big)(x) &= \omega\big((A \circ B)(x)\big) \\
&= \omega\big(A(B(x))\big) \\
&= \big(A^*(\omega)\big)\big(B(x)\big) \\
&= \big(B^*(A^*(\omega))\big)(x) \\
&= \big((B^* \circ A^*)(\omega)\big)(x)
\end{aligned}
$$

Thus $(A \circ B)^* = B^* \circ A^*$ as claimed.

Next, let $\omega \in V^*$ and $x \in V$

$$\big((Id_V)^* \omega\big)(x) = \omega\big(Id_V(x)\big) = \omega(x)$$

Hence $Id_V^*(\omega) = \omega \quad \forall \omega \in V^*$

$\therefore \underline{Id_V^* = Id_{V^*}}$ // $\qquad (Id_V)^* = Id_{V^*}$

$\boxed{\text{PS2}}$ Exercise 11.7, p. 274

Let V be a vector space,

(a.) For any $v \in V$, show $\xi(v)(w)$ depends linearly on w thus $\xi(v) \in V^{**}$

(b.) Show that $\xi : V \longrightarrow V^{**}$ is linear

Def^{n}/ $\xi(v)(w) = w(v)$ for all $v \in V$ and $w \in V^*$

(a.) Let $w_1, w_2 \in V^*$ and $c \in \mathbb{R}$. Let $x \in V$,

$$\begin{aligned}
\xi(x)(cw_1 + w_2) &= (cw_1 + w_2)(x) && \text{Def}^{n} \text{ of } \xi \\
&= cw_1(x) + w_2(x) && \text{Def}^{n} \text{ of } + \\
&= c\,\xi(x)(w_1) + \xi(x)(w_2)
\end{aligned}$$

Hence $\xi(x) : V^* \longrightarrow \mathbb{R}$ is linear and so $\xi(x) \in V^{**}$

(b.) Let $x, y \in V$ and $c \in \mathbb{R}$ and let $w \in V^*$

$$\begin{aligned}
\xi(cx+y)(w) &= w(cx+y) && w : V \longrightarrow \mathbb{R} \text{ linear} \\
&= c\,w(x) + w(y) \\
&= c\,\xi(x)(w) + \xi(y)(w) \\
&= (c\,\xi(x) + \xi(y))(w) && \forall w \in V^*
\end{aligned}$$

Therefore, $\xi(cx+y) = c\,\xi(x) + \xi(y)$ and so $\xi : V \longrightarrow V^{**}$ is linear as claimed. //

$\boxed{\text{PS3}}$ Exercise 11.21, p. 281

$df_p(v) = vf$ for $v \in T_pM$

Prop. 11.20 for $f, g \in C^\infty M$

(a.) $d(af + bg) = a\,df + b\,dg$ where a, b constants.

(b.) $d(fg) = f\,dg + g\,df$

(c.) $d\left(\frac{f}{g}\right) = \frac{1}{g^2}(g\,df - f\,dg)$ where $g \neq 0$

(d.) If $J \subseteq \mathbb{R}$ is interval containing the image of f and $h : J \longrightarrow \mathbb{R}$ is smooth fnct. then

$$d(h \circ f) = (h' \circ f)\,df$$

(prove these)

(e.) If f constant then $df = 0$

(a.) $$\begin{aligned} d(af + bg)_p(v) &= v[af + bg] \qquad v \in T_pM \\ &= a\,vf + b\,vg \\ &= a\,df_p(v) + b\,dg_p(v) \\ &= (a\,df_p + b\,dg_p)(v) \Rightarrow \underline{d(af+bg) = a\,df + b\,dg} \end{aligned}$$ //

(b.) $$\begin{aligned} d(fg)_p(v) &= v[fg] \qquad v \in T_pM \text{ is } \underline{\text{derivation}} \\ &= f(p)\,vg + g(p)\,vf \\ &= f(p)\,dg_p(v) + g(p)\,df_p(v) \\ &= (f\,dg + g\,df)_p(v) \Rightarrow \underline{\text{(b) true}}. \end{aligned}$$

(c.) Let $h = \frac{f}{g}$ then $f = hg$ and for $g \neq 0$ calculate

$$\underbrace{df = g\,dh + h\,dg}_{\text{by (b.)}} \Rightarrow dh = \frac{1}{g}\left(df - h\,dg\right) = \frac{g\,df - f\,dg}{g^2}.$$

PS3 continued

(d.) Consider $h : J \to \mathbb{R}$ where $f \in C^\infty(M)$ has image $(f) \subseteq J$ hence $h \circ f \in C^\infty(M)$

$$d(h \circ f)_p(V) = V(h \circ f)$$
$$= \left(\sum_{i=1}^{m} V^i \frac{\partial}{\partial x^i}\bigg|_p \right)(h \circ f)$$

Recall,

$$\frac{\partial}{\partial x^i}\bigg|_p g = \left(\frac{\partial}{\partial u^i} (g \circ x^{-1})(u) \right)\bigg|_{u = x(p)}$$

$$\frac{\partial}{\partial x^i}\bigg|_p h \circ f = \left(\frac{\partial}{\partial u^i} (h \circ f \circ x^{-1})(u) \right)\bigg|_{u = x(p)}$$
$$= \frac{\partial}{\partial u^i}\Big((h \circ (f \circ x^{-1}))(u) \Big)\bigg|_{u = x(p)}$$
$$= h'(f(x^{-1}(x(p)))) \cdot \frac{\partial}{\partial u^i}\left[(f \circ x^{-1})(u) \right]\bigg|_{u = x(p)}$$
$$= h'(f(p)) \frac{\partial f}{\partial x^i}(p)$$

Hence,

$$d(h \circ f)_p(V) = \sum_{i=1}^{m} h'(f(p)) V^i \frac{\partial f}{\partial x^i}(p)$$
$$= h'(f(p)) Vf$$
$$= h'(f(p)) df_p(V)$$
$$= ((h' \circ f) df)_p(V)$$

Thus $d(h \circ f)_p = (h' \circ f) df_p$.

(e.) easy. I leave to the reader. $d(1 \cdot 1) = 1 d(1) + 1 d(1)$ etc...

P 54 Problem 11-5 p. 300

For any smooth M, show T^*M is trivial vector bundle iff TM is trivial

Recall Corollary 10.20 from p. 259

- smooth vector bundle is smoothly trivial iff it admits a global frame.

TM trivial $\Rightarrow$ $(E_1, \dots, E_n)$ global frame for TM exists

$\Rightarrow$ $(\varepsilon^1, \dots, \varepsilon^n)$ dual to global frame gives global frame for T^*M

$\Rightarrow$ T^*M trivial by Cor 10.20

Conversely,

T^*M trivial $\Rightarrow$ $(\varepsilon^1, \dots, \varepsilon^n)$ global coframe for T^*M exists.

$\Rightarrow$ $(E_1, \dots, E_n)$ global frame for TM dual to the coframe on T^*M exists.

$\Rightarrow$ TM trivial.

PSS Problem 11-7 p. 300 / $F: M \to N$, $\omega \in \mathcal{X}^*(N)$

calculate $F^*\omega$ for the following,

(a.) $M = N = \mathbb{R}^2$

$F(s,t) = (st, e^t)$, $F: \mathbb{R}_{s,t} \to \mathbb{R}_{x,y}$

$\omega = x\,dy - y\,dx$ (basically $x = st$, $y = e^t$)

$$
\begin{aligned}
F^*\omega &= (F^*x)\, d(F^*y) - (F^*y)\, d(F^*x) \\
&= (x \circ F)\, d(y \circ F) - (y \circ F)\, d(x \circ F) \\
&= st\, d(e^t) - e^t\, d(st) \\
&= ste^t dt - e^t (t\,ds + s\,dt) \\
&= (ste^t - se^t)\,dt - te^t ds \\
&= \boxed{(st - s)e^t dt - te^t ds}
\end{aligned}
$$

(b.) $F(\theta, \varphi) = (\underbrace{(\cos\varphi + 2)\cos\theta}_{x}, \underbrace{(\cos\varphi + 2)\sin\theta}_{y}, \underbrace{\sin\varphi}_{z})$

$\omega = z^2 dx$

$$
\begin{aligned}
F^*\omega &= (z^2 \circ F)\, d(x \circ F) \\
&= (\sin\varphi)^2\, d((\cos\varphi + 2)\cos\theta) \\
&= \sin^2\varphi\, (-\sin\varphi\cos\theta\, d\varphi - (\cos\varphi + 2)\sin\theta\, d\theta) \\
&= \boxed{-\sin^3\varphi\cos\theta\, d\varphi - \sin^2\varphi(\cos\varphi + 2)\sin\theta\, d\theta}
\end{aligned}
$$

$\boxed{PS5}$ continued

(c.) $\mathcal{M} = \{ (s,t) \in \mathbb{R}^2 \mid s^2+t^2<1 \}$ & $N = \mathbb{R}^3 - \{0\}$

$$F(s,t) = (s, t, \sqrt{1-s^2-t^2})$$

$$(x \circ F)(s,t) = s$$

$$(y \circ F)(s,t) = t$$

$$(z \circ F)(s,t) = \sqrt{1-s^2-t^2}$$

$$\omega = (1-x^2-y^2)\,dz$$

$$F^*\omega = (1-s^2-t^2)\,d\sqrt{1-s^2-t^2}$$

$$= (1-s^2-t^2)\frac{1}{2\sqrt{1-s^2-t^2}}\,d(1-s^2-t^2)$$

$$= \frac{1}{2}\sqrt{1-s^2-t^2}\,(-2s\,ds - 2t\,dt)$$

$$= \boxed{\left(-\sqrt{1-s^2-t^2}\right)(s\,ds + t\,dt)}$$

PS6 Problem 11-14, p. 301

$$\omega = \frac{-4z\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2x\,dz}{x^2+1}$$

$$\eta = \frac{-4xz\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2\,dz}{x^2+1}$$

(a.) set-up and evaluate the line integral of each covector field along the line segment from $(0,0,0)$ to $(1,1,1)$

(b.) determine if ω or η is exact

(c.) for each exact form recalculate (a.) using potential.

(a.) $x = y = z = t$ for $0 \le t \le 1$ parametrizes C

$$\int_C \omega = \int_0^1 \left(\frac{-4t}{(t^2+1)^2} + \frac{2t}{t^2+1} + \frac{2t}{t^2+1} \right) dt$$

$$= \int_0^1 \left(\frac{-4t + 4t(t^2+1)}{(t^2+1)^2} \right) dt$$

$$= \int_0^1 \frac{4t^3\,dt}{t^4 + 2t^2 + 1}$$

$$\int \frac{4x^3\,dx}{(x^2+1)^2} = \int \frac{4x^2 \cdot x\,dx}{(x^2+1)^2} = \int \frac{(w-1)\,2\,dw}{w}$$

$w = x^2+1$
$x^2 = w-1$
$dw = 2x\,dx$
$2\,dw = 4x\,dx$

$$= \int \left(2 - \frac{2}{w}\right) dw$$

$$= 2(w - \ln w) + c$$

$$= 2(x^2+1 - \ln(x^2+1)) + c$$

$$\int_C \omega = \left[2(t^2+1) - 2\ln(t^2+1)\right]\Big|_0^1 = 4 - 2\ln(2) - 2$$

$$= \boxed{2(1 - \ln(2))}$$

PS6 continued

(a.) $$\int_C \eta = \int_0^1 \left(\frac{-4t^2}{(t^2+1)^2} + \frac{2t}{t^2+1} + \frac{2}{t^2+1} \right) dt$$

$$= \int_0^1 \left(\frac{-4t^2 + 2t(t^2+1) + 2(t^2+1)}{(t^2+1)^2} \right) dt$$

$$= \int_0^1 \left(\frac{4t^3 - 2t^2 + 2t + 2}{(t^2+1)^2} \right) dt$$

$$= \int_0^1 \frac{4t^3\,dt}{(t^2+1)^2} + 2\int_0^1 \frac{-t^2+t+1}{(t^2+1)^2} dt$$

$$= 2(1-\ln(2)) - 2\int_0^1 \frac{t^2 dt}{(t^2+1)^2} + 2\int_0^1 \frac{(1+t)\,dt}{(t^2+1)^2}$$

$$\int \frac{x^2 dx}{(x^2+1)^2} = \int \frac{\tan^2\theta \cdot \sec^2\theta\, d\theta}{(\sec^2\theta)^2} = \int \frac{\tan^2\theta\, d\theta}{\sec^2\theta} = \int \sin^2\theta\, d\theta$$

$$\int_0^1 \frac{2t^2 dt}{(t^2+1)^2} = \int_0^{\pi/4} (1-\cos(2\theta))d\theta = \frac{\pi}{4} - \frac{1}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{1}{2}$$

$t = \tan\theta$

$$\int_0^1 \frac{2(1+t)dt}{(t^2+1)^2} = \int_0^{\pi/4} \frac{2}{\sec^4\theta}(1+\tan\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} 2(\cos^2\theta + \cos\theta\sin\theta)\, d\theta$$

$$= \int_0^{\pi/4} (1 + \cos(2\theta) + \sin(2\theta))\, d\theta$$

$$= \left(\frac{\pi}{4} + \frac{1}{2}\right) + \frac{1}{2} = \frac{\pi}{4} + 1$$

$$\int_C \eta = 2(1-\ln(2)) - \left(\frac{\pi}{2} - \frac{1}{2}\right) + \frac{\pi}{4} + 1$$

$$= \boxed{\frac{7}{2} - 2\ln(2) - \frac{\pi}{4}}$$

PS6 continued

(b.) determine if ω or η is exact

ω]

$$\frac{\partial f}{\partial x} = \frac{-4z}{(x^2+1)^2} \hookrightarrow \frac{\partial^2 f}{\partial z \partial x} = \frac{-4}{(x^2+1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{y^2+1}$$

$$\frac{\partial f}{\partial z} = \frac{2x}{x^2+1} \hookrightarrow \frac{\partial^2 f}{\partial x \partial z} = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}$$

Thus f s.t. $\omega = df$ d.n.e.

η]

$$\frac{\partial f}{\partial x} = \frac{-4xz}{(x^2+1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{y^2+1} \Rightarrow f = \ln(y^2+1) + C_1(x, z)$$

$$\frac{\partial f}{\partial z} = \frac{2}{x^2+1} \Rightarrow f = \frac{2z}{x^2+1} + C_2(x, y)$$

Looks like $\boxed{f = \frac{2z}{x^2+1} + \ln(y^2+1) \text{ gives } df = \eta}$

Thus η is exact.

(c.) $$\int_c \eta = \int_c df = \frac{2(1)}{1^2+1} + \ln(1+1) - 0 = \boxed{1 + \ln(2)}$$

so apparently
I have some
miscalculation
in (a.)!