Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. Most problems 5pts here.

Problem 57 Let S be a mixed tensor field on a smooth manifold M such that $S_p : T_pM \times T_pM \times T_p^*M \to \mathbb{R}$ at each $p \in M$. If coordinate charts x and y have domains which overlap then express S in both coordinate charts and show how the components in x and y are related.

Problem 58 Let $T = (x^2 + y^2)dx \otimes dx + xydx \otimes dy$ define a tensor on $M = (0, \infty)^2$. Let $F(u, v, w) = (u + v, v + w)$ define a map from $(0, \infty)^3$ to $(0, \infty)^2$. Calculate the pull-back of T under F; that is, calculate F^*T.

Problem 59 (10pts) Let M be a smooth manifold and suppose $\mathcal{B}$ is the set of rank two smooth covariant tensor fields on M. In particular, if $T \in \mathcal{B}$ then $T_p : T_pM \times T_pM \to \mathbb{R}$ is a bilinear map and $T(X, Y) \in C^\infty(M)$ whenever $X, Y \in \mathfrak{X}(M)$. Prove the following assertions about $\mathcal{B}$:

(a.) any $T \in \mathcal{B}$ can be decomposed into the sum of a symmetric and antisymmetric tensor. Let us denote $\mathcal{B}_A$ for the antisymmetric tensors and $\mathcal{B}_S$ for the symmetric tensors in $\mathcal{B}$.

(b.) if $dx^i \wedge dx^j = dx^i \otimes dx^j - dx^j \otimes dx^i$ then $\gamma_A = \{dx^i \wedge dx^j \mid 1 \leq i < j \leq n\}$ is a point-wise basis for $\mathcal{B}_A$ on the domain of the chart x for M where $dim(M) = n$.

(c.) if $dx^i dx^j = dx^i \otimes dx^j + dx^j \otimes dx^i$ then $\gamma_S = \{dx^i dx^j \mid 1 \leq i \leq j \leq n\}$ is a is a point-wise basis for $\mathcal{B}_S$ on the domain of the chart x for M where $dim(M) = n$,

(d.) comment on the notations for the objects in this problem in terms of terminology from Chapter 12 of John Lee's text.

Problem 60 Suppose $T = \sum_{i,j=1}^{n} T_{ij} dx^i \otimes dx^j$ and suppose T is antisymmetric. If $T = \sum_{i<j} C_{ij} dx^i \wedge dx^j$ then how are the tensor components T_{ij} related to the form components C_{ij} ? Also, how would you define B_{ij} for which $T = \sum_{i,j=1}^{n} B_{ij} dx^i \wedge dx^j$?

Problem 61 SM Exercise 12.3 page 306. (establish property of $F \otimes G$ in the context of multilinear maps)

Problem 62 SM Exercise 12.15 page 315 (prove basic properties of the symmetric product)

Problem 63 SM Exercise 12.26, just part (b), page 320 (prove property of pull-back with respect to $\otimes$ of covariant tensor fields)

Problem 64 SM Problem 12-7 page 325 (breakdown of general covariant tensor into symmetric and antisymmetric is not possible)

Problem 65 SM Exercise 13.13 page 332. (on flatness)

Problem 66 SM Exercise 13.24 page 337. (on curve length)

Problem 67 SM Problem 13-7 page 345. (product of flat metrics flat)

Problem 68 SM Problem 13-7 page 345. (flat $\mathbb{T}^n$)