Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. Most problems 5pts here.

Problem 82 Suppose $\{e_1, e_2, e_3\}$ is a frame at $p = (1,2,3)$ and $\{g_1, g_2, g_3\}$ is a frame at $q = (1,1,1)$. Furthermore, you're given

$$e_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle, \; e_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right\rangle, \; e_3 = \langle 0,0,1 \rangle$$

and $g_1 = \langle 0,\, 1,\, 0\rangle$, $g_2 = \langle 0,\, 0,\, 1\rangle$, $g_3 = \langle 1,\, 0,\, 0\rangle$. Find an isometry F which maps p to q and has a tangent map F_* for which $F_*(e_i) = g_i$ for $i = 1, 2, 3$.

Problem 83 Prove the Frenet Serret Equations for a nonstop, non-linear path in $s \mapsto \gamma(s) \in \mathbb{R}^3$:

$$\frac{dT}{dt} = \kappa v N, \qquad \frac{dN}{dt} = -\kappa v T + \tau v B, \qquad \frac{dB}{dt} = -\tau v N.$$

Recall we define $T = \frac{1}{v}\gamma'$ where $v = \|\gamma'\|$ and $N = \frac{1}{\|T'\|}T'$ and $B = T \times N$. You will also need to know the definitions of curvature $\kappa = \frac{1}{v}\|T'\|$ and torsion $\tau = -\frac{1}{v}N \bullet B'$.

Problem 84 Let α be a unit-speed, non-linear curve. Then α is a planar curve if and only if $\tau = 0$.

Problem 85 Generally two parametrized curves $\alpha : I \to \mathbb{R}^3$ and $\beta : I \to \mathbb{R}^3$ are **congruent** if there exists an isometry F for which $\beta = F \circ \alpha$. This problem asks for you to show if two unit-speed curves are congruent then they have the same curvature and (up to a sign) torsion. The converse is also true, in fact, arclength parametrized curves are congruent if and only if they have same curvature function and upto a sign the same torsion[1].

(a.) Let α be a nonlinear arclength parametrized curve with Frenet frame T, N, B and curvature κ and torsion τ. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^3$ is an Euclidean isometry. Show that $\bar{\alpha} = F \circ \alpha$ has Frenet frame $\bar{T}, \bar{N}, \bar{B}$ with curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ where

$$\bar{T} = F_*(T), \quad \bar{N} = F_*(N), \quad \bar{B} = \det(F_*)F_*(B), \quad \bar{\kappa} = \kappa, \quad \bar{\tau} = det(F_*)\tau$$

(b.) Let α be an arclength parametrized curve. Then α has constant curvature and torsion if and only if α is a helix.

Problem 86 Let A_{ik} be a p-form and B_{kj} be a q-form for $1 \le i \le m$, $1 \le k \le r$ and $1 \le j \le n$ then we say A is an $m \times r$ matrix of p-forms and B is a $r \times n$ matrix of q-forms. Then we say A and B are **multipliable** and define the $m \times n$-matrix of $(p+q)$-forms $A \wedge B$ by:

$$(A \wedge B)_{ij} = \sum_{k=1}^{r} A_{ik} \wedge B_{kj}.$$

[1]This is my Theorem 3.3.11 on page 72 of my Differential Geometry notes from 2021. You might need to assume the converse to argue part (b.).

Likewise, we denote the **exterior derivative of** A by dA which is defined to be the $m \times r$-matrix of $(p+1)$-forms given by $(dA)_{ik} = dA_{ik}$. Let A be a matrix of p-forms and B be a matrix of q-forms and suppose A, B are multipliable then show

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB.$$

Also, for a matrix of functions A for which $A^T A = I$ show $A \wedge dA^T \wedge A = -dA$

Problem 87 Suppose $W \in \mathfrak{X}(\mathbb{R}^3)$ and $v \in T_p\mathbb{R}^3$. The **covariant derivative** of W with respect to v at p is the tangent vector:

$$(\nabla_v W)(p) = W(p + tv)'(0) \in T_p\mathbb{R}^3.$$

If $V \in \mathfrak{X}(\mathbb{R}^3)$ and $V(p) = v_p$ then the assignment $p \to (\nabla_{v_p} W)(p)$ defines $\nabla_V W \in \mathfrak{X}(\mathbb{R}^3)$ and we say $\nabla_V W$ is the **covariant derivative** of W with respect to V. This derivative measures how W changes in the V-direction. A nice coordinate formula for the covariant derivative is simply:[2]

$$\nabla_V W = \sum_{j=1}^{3} V[W^j] U_j$$

where $U_i = \frac{\partial}{\partial x^i}$ for $i = 1, 2, 3$ and $W = \sum_j W^j U_j$.

(a.) $\nabla_{fV} W = f \nabla_V W$ for all smooth $f : \mathbb{R}^3 \to \mathbb{R}$

(b.) $\nabla_V (fW) = V[f]\, W + f \nabla_V W$ for all smooth $f : \mathbb{R}^3 \to \mathbb{R}$

(c.) $U[V \bullet W] = \nabla_U V \bullet W + V \bullet \nabla_U W$

(d.) If $\|V\| = 1$ then $\nabla_V (V) = 0$

Problem 88 If $\{E_1, E_2, E_3\}$ is an orthonormal frame for $\mathbb{R}^3$ and $V \in \mathfrak{X}(\mathbb{R}^3)$ then there exist functions f^1, f^2, f^3 called the **components of** V with respect to the frame $\{E_1, E_2, E_3\}$

$$V = f^1 E_1 + f^2 E_2 + f^3 E_3$$

Also we define $\omega_{ij}(p) \in (T_p\mathbb{R}^3)^*$ by

$$\omega_{ij}(p)(v) = (\, \nabla_v E_i \,) \bullet E_j(p)$$

for each $v \in T_p\mathbb{R}^3$. That is, ω_{ij} is a differential one-form on $\mathbb{R}^3$ defined by the assignment $p \mapsto \omega_{ij}(p)$ for each $p \in \mathbb{R}^3$ and it is known as a **connection form**. Suppose $V, W \in \mathfrak{X}(\mathbb{R}^3)$ where $V = \sum f^i E_i$ and $W = \sum_j g^j E_j$. Then notice $\sum_j \omega_{ij}(V) E_j = \nabla_V E_i$ hence

$$\nabla_V W = \nabla_V \left(\sum_i g^i E_i \right) = \sum_i (V[g^i] E_i + g^i \nabla_V E_i) = \sum_i \left(V[g^i] E_i + g^i \sum_j \omega_{ij}(V) E_j \right)$$

then relabling the first sum and exchanging the order of the second we derive

$$\nabla_V W = \sum_j \left(V[g^j] + \sum_i g^i \omega_{ij}(V) \right) E_j.$$

In short, finding the formulas for the connection forms allows us to capture the change in vector fields which is due to the change in the orthonormal frame over the space. Sorry for the lengthy preamble, now for the actual problem:

[2] perhaps I'll derive this in lecture, if not, it's on page 45 of my 2021 Differential Geometry notes

(a.) If $A_{ij} = E_i \bullet U_j$ then show $E_i = \sum_j A_{ij} U_j$ and $\theta^i = \sum_j A_{ij} dx^j$. We call A the **attitude matrix** of the frame E_1, E_2, E_3.

Notation: It is useful to use matrix notation to think about a column of one-forms, we can restate the proposition above as follows:

$$\theta = \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} \qquad \& \qquad d\xi = \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} \quad \Rightarrow \quad \theta = A d\xi.$$

Just to be explicit,

$$Ad\xi = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = \begin{bmatrix} A_{11}dx^2 + A_{12}dx^2 + A_{13}dx^3 \\ A_{21}dx^2 + A_{22}dx^2 + A_{23}dx^3 \\ A_{31}dx^2 + A_{32}dx^2 + A_{33}dx^3 \end{bmatrix} = \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \theta.$$

(b.) show $\omega = dA \wedge A^T$.

(c.) derive Cartan's Structure Equations for $\mathbb{R}^3$; $d\theta = \omega \wedge \theta$ and $d\omega = \omega \wedge \omega$.
To be explicit,

$$d\theta^i = \sum_j \omega_{ij} \wedge \theta^j \qquad \& \qquad d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Problem 89 Let α be a unit speed curve with $\kappa > 0$ and suppose E_1, E_2, E_3 is an orthonormal frame field on $\mathbb{R}^3$ such that the frame restricts to the Frenet frame T, N, B on α. Show the connection forms of the frame are precisely the curvature and torsion of the curve. Here you probably will find the identity $\nabla_{\alpha'(s)} W = \frac{d}{ds}(W \circ \alpha)(s)$ is helpful.

Problem 90 If $\phi = \sum_i f_i \theta^i$ where θ^i are dual to orthonormal frame E_i with connection forms ω_{ij} then $d\phi = \sum_j [df_j + \sum_i f_i \omega_{ij}] \wedge \theta^j$ (you can and should make use of Cartan's Structure Equations if helpful)

Problem 91 Let U be the unit-normal to a surface M in $\mathbb{R}^3$. Let $p \in M$, if $v \in T_pM$ then define the **shape operator** at p acting on v by $S_p(v) = -(\nabla_v U)(p)$. Show the following:

(a.) $S_p : T_pM \to T_pM$ and S_p is a linear transformation.

(b.) $S_p(v) \bullet w = v \bullet S_p(w)$ or equivalently, the matrix of S_p is symmetric

(c.) $S_p(v) \times S_p(w) = K(p) v \times w$ and $S_p(v) \times w + v \times S_p(w) = 2H(p) v \times w$ where K is the Gaussian curvature (defined by $K(p) = det(S_p)$) and H is the mean curvature (defined by $H(p) = trace(S_p)$)

(d.) If M is a surface with tangent vectors $v_1, v_2 \in T_pM$ such that $S_p(v_1) = 6v_1$ and $S_p(v_2) = 7v_2$ then find $K(p)$ and $H(p)$.

Problem 92 Use the E, F, G, L, M, N formulas for Gaussian curvature K and mean curvature H to find K and H for

(a.) the Helicoid $X(u, v) = \langle u \cos v, u \sin v, bv \rangle$ where $b \neq 0$ is a constant

(b.) the Cylinder $X(u, v) = \langle R \cos u, R \sin u, v \rangle$ where $R > 0$ is a constant