Show work where appropriate with proper notation, thanks!

Problem 1 (5pts) Calculate the sum of the series $S = a_0 + a_1 + a_2 + \cdots$ given that for $n = 1, 2, 3 \ldots$:

$$\sum_{j=0}^{n} a_j = 3 - \frac{1}{n}$$

$$\sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} \left(3 - \underbrace{\frac{1}{n}}_{0}\right) = \boxed{3.}$$

Problem 2 (30pts) Find if the series below converge or diverge. (support answer with work)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n-1}$

$$a_n = \frac{n+1}{3n-1} = \frac{1 + 1/n}{3 - 1/n} \longrightarrow \frac{1}{3} \neq 0$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n-1}$ diverges by n^{th} term test.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{(2n+1)!}$ try ratio test,

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{3^n} \qquad 2(n+1)+1 = 2n+1+2 = 2n+3$$

$$= \frac{3(2n+1)!}{(2n+3)(2n+2)(2n+1)!}$$

$$= \frac{3}{(2n+3)(2n+2)} \longrightarrow 0 \text{ as } n \to \infty$$

$\therefore$ the $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{(2n+1)!}$ converges by ratio test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n\sqrt{n}}$

Observe $-1 \leq \sin(n) \leq 1 \Rightarrow 1 \leq 2 + \sin(n) \leq 3$

thus $\dfrac{2+\sin(n)}{n\sqrt{n}} < \dfrac{3}{n\sqrt{n}} = \dfrac{3}{n^{1.5}}$. Observe $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ converges by $P = 1.5$ series test $\therefore 3\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges and by Direct Comparison Test we find $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n\sqrt{n}}$ converges.

Problem 3 (10pts) Find the sum of the series $\sum_{k=0}^{\infty} \frac{2^{3k+1}}{7^{2k}}$. $= \frac{2}{1} + \frac{2^4}{49} + \cdots$

geometric, $a = 2$, $r = 8/49$.

$$\sum_{k=0}^{\infty} \frac{2^{3k+1}}{7^{2k}} = \sum_{k=0}^{\infty} 2\left(\frac{(2^3)^k}{(7^2)^k}\right) = \sum_{k=0}^{\infty} 2\left(\frac{8}{49}\right)^k = \frac{2}{1 - 8/49}$$

$$= \frac{98}{49-8}$$

$$= \boxed{\frac{98}{41}}$$

Problem 4 (5pts) Calculate the sum the series $\sum_{k=0}^{\infty} \frac{(\ln(2))^k}{k!}$

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \therefore \quad \exp(\ln(2)) = \boxed{\sum_{k=0}^{\infty} \frac{(\ln(2))^k}{k!} = 2}$$

Problem 5 (15pts) If $f(x) = \sum_{k=0}^{\infty} \underbrace{\frac{2^k}{k+1}(x-3)^k}_{a_k}$ then find the IOC and ROC for $f(x)$

as $k \to \infty$

$$\frac{|a_{k+1}|}{|a_k|} = \left(\frac{2^{k+1}|x-3|^{k+1}}{k+2}\right) \cdot \left(\frac{(k+1)}{2^k|x-3|^k}\right) = 2|x-3|\left(\frac{k+1}{k+2}\right) \to 2|x-3|$$

Thus $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 2|x-3| < 1 \Rightarrow |x-3| < \frac{1}{2}$ $\therefore$ $\underline{\text{ROC} = \frac{1}{2}}$.

<u>End points</u>:

<u>$x = 2.5$</u> $f(2.5) = \sum_{k=0}^{\infty} \frac{2^k}{k+1}(-0.5)^k = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}}$ since $(-0.5)^k = \frac{(-1)^k}{2^k}$.

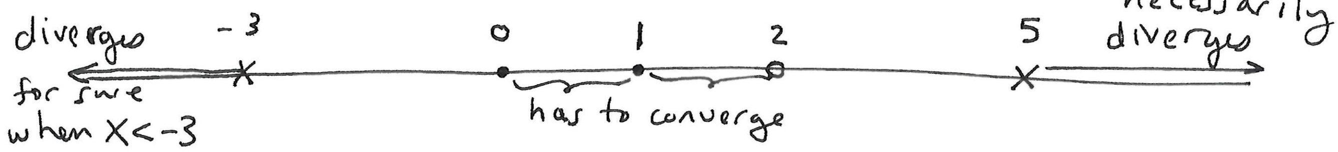
converges by A.S.T. as $\frac{1}{k+1} \to 0$ and is a positive, decreasing sequence.

<u>$x = 3.5$</u> $f(3.5) = \sum_{k=0}^{\infty} \frac{\cancel{2^k}}{k+1} \cancel{(0.5)}^k$

$= \sum_{k=0}^{\infty} \frac{1}{k+1}$ ← diverges, is $P=1$ series.

In summary, $\boxed{\text{IOC} = [2.5, 3.5) \text{ and ROC} = 1/2}$

Problem 6 (10pts) Suppose $f(x) = \sum_{k=0}^{\infty} c_k(x-1)^k$ converges at $x = 0$ and diverges at $x = 5$. Determine the largest and smalllest IOC for $f(x)$.



largest IOC $= [-3, 5)$ smallest IOC $= [0, 2)$

Problem 7 (10pts) Use the geometric series to find a power series in x for $f(x) = \dfrac{x^2}{3-x^2}$ and state the IOC for the power series.

$$f(x) = \frac{x^2}{3(1 - x^2/3)} = \sum_{n=0}^{\infty} \frac{x^2}{3}\left(\frac{x^2}{3}\right)^n ,$$

$a = \frac{x^2}{3}$, $r = \frac{x^2}{3}$

(need $|r| = |\frac{x^2}{3}| < 1$

aka $|x|^2 < 3$

$\hookrightarrow |x| < \sqrt{3}$

$\therefore x \in (-\sqrt{3}, \sqrt{3})$

$\therefore$ $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^{2n+2} \quad \text{for } x \in \underbrace{(-\sqrt{3}, \sqrt{3})}_{\text{IOC}}}$

Problem 8 (15pts) Use geometric series techniques to find the power series for $f(x) = \ln(3+5x)$ in centered at $x_o = 1$ in Σ notation.

$$\frac{df}{dx} = \frac{5}{3+5x} = \frac{5}{3+5(x-1)+5} = \frac{5}{8+5(x-1)} = \frac{5}{8(1+\frac{5}{8}(x-1))}$$

Thus, $\dfrac{df}{dx} = \sum_{n=0}^{\infty} \frac{5(-1)^n}{8}\left[\frac{5}{8}(x-1)\right]^n = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1}}{8^{n+1}} (x-1)^n.$

Hence, $\underline{f(x) = C + \sum_{n=0}^{\infty} \frac{5^{n+1}(-1)^n}{(n+1)8^{n+1}} (x-1)^{n+1}}$ * by $\int \frac{df}{dx} dx = f(x) + c.$

Note, $f(1) = \ln(3+5) = \ln(8) = C$ as * is zero for $x = 1$.

Consequently, $\boxed{f(x) = \ln(8) + \sum_{n=0}^{\infty} \frac{5^{n+1}(-1)^n}{(n+1)8^{n+1}} (x-1)^{n+1}}$

Problem 9 (10pts) Find the first three non-trivial terms in the Taylor series centered at $x_o = 1$ for $f(x) = \tan(x)$

$$\therefore \quad f(1) = \tan(1)$$

$$f'(x) = \sec^2(x) \qquad \therefore \quad f'(1) = \sec^2(1)$$

$$f''(x) = 2\sec(x)\sec(x)\tan(x) \qquad \therefore \quad f''(1) = 2\sec^2(1)\tan(1)$$

Thus, as $f(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \cdots$ we find,

$$\boxed{\tan(x) = \tan(1) + \sec^2(1)(x-1) + \sec^2(1)\tan(1)(x-1)^2 + \cdots}$$

Problem 10 (10pts) Calculate the first 3 terms in a power series solution for the indefinite integral $\int \frac{x^{11}}{\cos(x^4)}\,dx$

$$\frac{1}{\cos(x^4)} = \frac{1}{1 - \frac{1}{2}x^8 + \frac{1}{24}x^{16} + \cdots}$$

as $\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \cdots$

set $\theta = x^4$.

$$\begin{array}{r|l} & 1 + \frac{1}{2}x^8 + \frac{5}{24}x^{16} + \cdots \\ \hline 1 - \frac{1}{2}x^8 + \frac{1}{24}x^{16} + \cdots & 1 \\ & 1 - \frac{1}{2}x^8 + \frac{1}{24}x^{16} \\ \hline & \frac{1}{2}x^8 - \frac{1}{24}x^{16} \\ & \frac{1}{2}x^8 - \frac{1}{4}x^{16} \\ \hline & \frac{5}{24}x^{16} \\ & \vdots \end{array}$$

$$\hookrightarrow \frac{1}{\cos(x^4)} = 1 + \frac{x^8}{2} + \frac{5x^{16}}{24} + \cdots$$

$$\int \frac{x^{11}}{\cos(x^4)}\,dx = \int x^{11}\left(1 + \frac{1}{2}x^8 + \frac{5}{24}x^{16} + \cdots\right)dx$$

$$= \int\left(x^{11} + \frac{1}{2}x^{19} + \frac{5}{24}x^{27} + \cdots\right)dx$$

$$= \boxed{\frac{1}{12}x^{12} + \frac{1}{40}x^{20} + \frac{5}{672}x^{28} + \cdots + C}$$

Problem 11 (5pts) Find an exact fraction which possesses the following decimal expansion:

$$0.123123123123123123123123123\ldots = \frac{123}{1000} + \frac{123}{1000}\left(\frac{1}{1000}\right) + \frac{123}{1000}\left(\frac{1}{1000}\right)^2 + \cdots$$

$$= \frac{123/1000}{1 - 1/1000} \quad \text{-(geometric series)-}$$

$$= \frac{123}{1000-1}$$

$$= \boxed{\frac{123}{999}} \quad \text{aka} \quad \boxed{\frac{41}{333}}$$

Problem 12 (5pts) Calculate the value of the series $\sum_{n=0}^{\infty}(-1)^n\frac{1}{(3n)!}$ to within 0.001 of the sum. (make sure by an appropriate test that your choice for S_n is sufficient to insure an accuracy desired)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(3n)!} = 1 - \frac{1}{3!} + \frac{1}{6!} - \frac{1}{9!} + \frac{1}{(12)!} + \cdots \quad \text{alternating series.}$$

Observe $\frac{1}{6!} = \frac{1}{720} = 0.001389$ whereas $\frac{1}{(9)!} = 2.76 \times 10^{-6}$

So certainly $\sum_{n=0}^{3} \frac{(-1)^n}{(3n)!} = 1 - \frac{1}{6} + \frac{1}{720} =$ $\boxed{\frac{601}{720} \text{ is within } 0.001 \text{ of the true total}}$

(by alternating series estimation Thm) $\rightarrow$

$\underline{0.83472\ldots}$ is sure.

Problem 13 (10pts) Let $f(x) = x^2e^x + \cos(x^4) + x^{100}$. Find $f^{(56)}(0)$.

$$f(x) = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{8j} + x^{100} \qquad \left((x^4)^{2j} = x^{8j} \right)$$

and $f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m$ so we can compare coefficients of x^{56} to obtain $\frac{f^{(56)}(0)}{(56)!}$ $\left(\begin{array}{l}\text{need } n = 54 \\ \text{or } j = 7\end{array}\right)$

$$\frac{f^{(56)}(0)}{(56)!} = \frac{1}{54!} - \frac{1}{(14)!} \qquad (\text{the } x^{100} \text{ is irrelevant here})$$

$$\therefore \boxed{f^{(56)}(0) = (56)!\left[\frac{1}{(54)!} - \frac{1}{(14)!}\right]} \approx \frac{(56)!}{(14)!}$$