§2.5 #6d) Binet's formula,

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

Assume this true for all $k \le n$. Consider $f_{n+1} = f_n + f_{n-1}$. Then $n, n-1 \le n$ thus

$$
\begin{aligned}
f_{n+1} &= f_n + f_{n-1} && : \text{def}^{\underline{n}} \text{ of fibonacci} \\
&= \frac{\alpha^n - \beta^n}{\alpha-\beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} && : \text{(strong) induction hyp.} \\
&= \frac{\alpha^n - \beta^n}{\alpha-\beta} + \frac{\alpha^n}{\alpha(\alpha-\beta)} - \frac{\beta^n}{\beta(\alpha-\beta)} && : \ \alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2} \\
&= \frac{1}{\alpha-\beta}\left[\alpha^n - \beta^n + \frac{1}{\alpha}\alpha^n - \frac{1}{\beta}\beta^n\right] && \text{typo in text here.} \\
&= \frac{1}{\alpha-\beta}\left[\alpha^n\left[1+\frac{1}{\alpha}\right] - \beta^n\left[1+\frac{1}{\beta}\right]\right] && \alpha - \beta = \sqrt{5} \\
&= \frac{1}{\alpha-\beta}\left[\alpha^n\left[1+\frac{2}{1+\sqrt{5}}\right] - \beta^n\left[1+\frac{2}{1-\sqrt{5}}\right]\right] \\
&= \frac{1}{\alpha-\beta}\left[\alpha^n\left[\frac{1+\sqrt{5}+2}{1+\sqrt{5}}\right] - \beta^n\left[\frac{1-\sqrt{5}+2}{1-\sqrt{5}}\right]\right] \\
&= \frac{1}{\alpha-\beta}\left[\alpha^n\left[\frac{3+\sqrt{5}}{1+\sqrt{5}}\left(\frac{1-\sqrt{5}}{1-\sqrt{5}}\right)\right] - \beta^n\left[\frac{3-\sqrt{5}}{1-\sqrt{5}}\left(\frac{1+\sqrt{5}}{1+\sqrt{5}}\right)\right]\right] \\
&= \frac{1}{\alpha-\beta}\left[\alpha^n\left[\frac{3-5-2\sqrt{5}}{\cancel{6}-4}\right] - \beta^n\left[\frac{3+2\sqrt{5}-5}{-4}\right]\right] \\
&= \frac{1}{\alpha-\beta}\left[\alpha^n\left[\frac{1}{2}+\frac{\sqrt{5}}{2}\right] - \beta^n\left[\frac{-2+2\sqrt{5}}{-4}\right]\right] \\
&= \frac{1}{\alpha-\beta}\left[\alpha^n\left[\frac{1+\sqrt{5}}{2}\right] + \beta^n\left[\frac{1-\sqrt{5}}{2}\right]\right] \\
&= \frac{1}{\alpha-\beta}\left[\alpha^n\alpha - \beta^n\beta\right] \\
&= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}
\end{aligned}
$$

yep, I bet you can cut this in half.

§2.4#8n) ~~[illegible]~~ Assume inductively $P(n)$ true, $\exists k \in \mathbb{Z}$,

$10^n + 3\cdot 4^{n+2} + 5 = 9k$

Want to find $k_2 \in \mathbb{Z}$ such that $10^{n+1} + 3\cdot 4^{n+1+2} + 5 = 9k_2$

$$10^{n+1} = 10\cdot 10^n = 10^n + 9\cdot 10^n$$

$$3\cdot 4\cdot 4^{n+2} = 12\cdot 4^{n+2} = 3\cdot 4^{n+2} + 9\cdot 4^{n+2}$$

Then,

$$\begin{aligned}
10^{n+1} + 3\cdot 4^{n+1+2} + 5 &= 10^n + 9\cdot 10^n + 3\cdot 4^{n+2} + 9\cdot 4^{n+2} \\
&= 9(10^n + 4^{n+2}) + \underbrace{10^n + 3\cdot 4^{n+2}}_{\text{induction hypothesis}} \\
&= 9(10^n + 4^{n+2}) + 9k \\
&= 9\underbrace{(10^n + 4^{n+2} + k)}_{k_2}.
\end{aligned}$$

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§2.5#2) Let $a_1 = 2$, $a_2 = 4$, $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$. Prove that $a_n = 2^n$ for $n \in \mathbb{N}$.

Try PMI) Assume $a_n = 2^n$. Consider a_{n+1} ah nevermind we should use PCI here because it links $n+2$ to $n+1$ and n

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Try PCI) Let $S = \{n \mid a_n = 2^n\}$ notice $1 \in S \Rightarrow 2 \in S$.

Suppose $k \in S$ for all $k \leq n$. Consider a_{n+1}.

$$\begin{aligned}
a_{n+1} &= 5a_{n+1-1} - 6a_{n-1} && \text{: by def}^{\text{n}}\text{ of } a_n. \\
&= 5a_n - 6a_{n-1} && \text{: note now } n, n-1 \leq n \text{ thus by induction hypothesis} \\
&= 5\cdot 2^n - 6\cdot 2^{n-1} \\
&= 5\cdot 2^n - \tfrac{1}{2}6\cdot 2^n \\
&= (5-3)2^n \\
&= 2\cdot 2^n = 2^{n+1} && \Rightarrow n+1 \in S.
\end{aligned}$$

and as n was arbitrary we find

$\forall n \in \mathbb{N}$, $k \in S$ for $k \leq n \Rightarrow n+1 \in S$.

Hence by PCI, $a^n = 2^n$ $\forall n \in \mathbb{N}$.

$\S 2.5 \# 7$ this is hard. Let $\sqrt{2} = \frac{m}{n} \rightarrow m^2 = 2n^2$

$$S = \{ n \in \mathbb{N} \mid \exists m \in \mathbb{N} \text{ with } m^2 = 2n^2 \}$$

We define this set in the hopes of showing it is empty. Assume $S \neq \emptyset$ then $S \subseteq \mathbb{N}$ hence by WOP $\exists n_o \in S$ which is the smallest element of S. ($n_o < n$ $\forall n \in S$ with $n \neq n_o$). Then your goal is to give a contradiction, you want to find a smaller element in S.

OBSERVATION: $\exists m_o$ with $m_o^2 = 2n_o^2$ notice this makes m_o^2 even hence m_o is even.

thus $\exists k_o \in \mathbb{Z}$ s.t. $m_o = 2k_o$:

$$m_o^2 = (2k_o)^2 = 2n_o^2 \Rightarrow 4k_o^2 = 2n_o^2$$
$$\Rightarrow 2k_o^2 = n_o^2$$
$$\Rightarrow k_o \in S.$$

BUT, notice $k_o^2 = \frac{n_o^2}{2} \Rightarrow k_o \leq \frac{n_o}{2} < n_o$.

thus k_o is smaller than the smallest thing in S. Thus $S \neq \emptyset$ is false, in fact $S = \emptyset$ which says $\nexists\, m, n \in \mathbb{N}$ such that $\sqrt{2} = \frac{m}{n}$. That is $\sqrt{2}$ is irrational.

§2.5 #9 I'll just suggest a set to think about. Let r be a particular rational #,

$$\underbrace{S = \{n \in \mathbb{N} \,/\, \exists m \in \mathbb{N} \text{ with } r = \tfrac{m}{n}\}}_{\text{subset of } \mathbb{N}.}$$

~~§2.3~~ §3.2 #9 Suppose S and R are $\sim$ relations on A. Prove $S \cap R$ is equivalence relation on A.

$S \subseteq A \times A$ and $R \subseteq A \times A$ is given.
Clearly $S \cap R \subseteq A \times A$ so it's a <u>relation</u>. Now

<u>Reflexive?</u> ~~Let $(a,b) \in S \cap R$~~
As we discussed today $\text{dom}(S) = \text{dom}(R) = A$ since it's a "relation on A".

① Let $a \in A$ then $(a,a) \in S$ and $(a,a) \in R$ since S and R are reflexive. Thus, $(a,a) \in S \cap R$. This shows $S \cap R$ is relexive

② Suppose $(a,b) \in S \cap R$ then $(a,b) \in S$ and $(a,b) \in R$ and by symmetric prop. of S and R respective $(b,a) \in S$ and $(b,a) \in R$ hence $(b,a) \in S \cap R$ which shows $(a,b) \in S \cap R \Rightarrow (b,a) \in S \cap R \;\; \forall (a,b) \in S \cap R$ hence $S \cap R$ is symmetric.

③ Suppose $(a,b) \in S \cap R$ and $(b,c) \in S \cap R$. Then $(a,b), (b,c) \in S$ and $(a,b), (b,c) \in R$. By transitivity for S $(a,c) \in S$ and trans. for R $(a,c) \in R$. Then $(a,c) \in S \cap R$. Consequently $S \cap R$ transitive.

<u>Remark</u>: I have the most trouble with reflexive for these sort of problems.