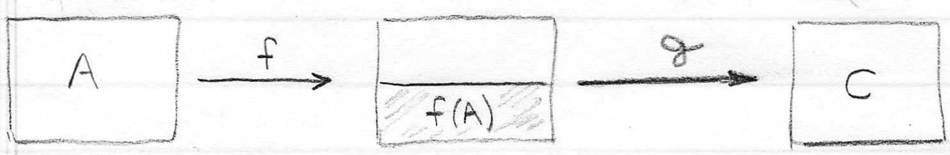
MATH 200, SOLUTIONS FOR §4.3 #8f, §5.1 # 17a,b, §6.2 #17

§4.3 #8f Give an example of $f: A \to B$ and $g: B \to C$ such that $g \circ f$ is one-one but g is not one-one.

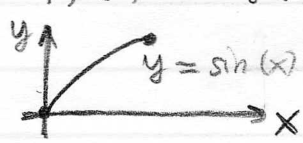
We need $(g \circ f)(x) = (g \circ f)(y) \Rightarrow x = y \ \forall x, y \in dom(f)$, however, $\exists a, b \in dom(g)$ such that $g(a) = g(b)$ but $a \neq b$. Perhaps a picture will help,

$$A \xrightarrow{f} B \xrightarrow{g} C$$



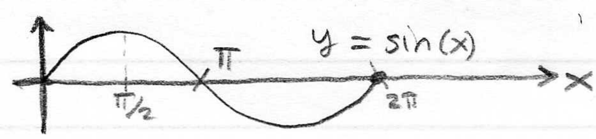
If we make f not onto B then we can exploit the difference between $f(A)$ and B to get what we want. Suppose $f: [0, \pi/2] \to [0, 2\pi]$ where $f(x) = x$ and $g: [0, 2\pi] \to [-1, 1]$ and $g(x) = \sin(x)$. This satisfies the needed criteria,

$$(g \circ f)(x) = g(x) = \sin(x) \quad \text{for} \quad 0 \leq x < \pi/2$$



($g \circ f$ is one-one)

$$g(x) = \sin(x) \quad \text{for} \quad 0 \leq x \leq 2\pi$$



(g is not one-one)

§5.1 # 17 Let A and B be finite sets with $A \approx B$ and $f: A \to B$

(a.) If f is one-one show f is onto B

Proof: Assume f is one-one, that is $\forall x, y \in A$ $f(x) = f(y) \Rightarrow x = y$. Observe that
$$f(A) = \{ f(a) \mid a \in A \}$$
has the same # of elements as A; $\overline{\overline{f(A)}} = \overline{\overline{A}}$. However, $A \approx B \Rightarrow \overline{\overline{A}} = \overline{\overline{B}} \Rightarrow \overline{\overline{f(A)}} = \overline{\overline{B}}$. The number of elements in $f(A)$ matches the # of elements in B, thus f is onto. //

(b.) If f is onto B, prove f is one-one

Proof: Assume f is onto B. Suppose (towards a $\rightarrow\leftarrow$) that $\exists x, y \in A$ such that $f(x) = f(y)$ and $x \neq y$. Notice $\overline{\overline{f(A)}} \leq \overline{\overline{A}} - 1$ since $f(x) = f(y)$. But, this is a contradiction as f onto $\Rightarrow f(A) = B \Rightarrow \overline{\overline{f(A)}} = \overline{\overline{B}}$ and $A \approx B \Rightarrow \overline{\overline{A}} = \overline{\overline{B}}$ so $\overline{\overline{B}} = \overline{\overline{f(A)}} \leq \overline{\overline{A}} - 1$

(contradicts $\overline{\overline{B}} = \overline{\overline{A}}$)

Thus, by proof by contradiction, f is one-one. //

Remark: these are counting arguments. Please note these fail for A, B infinite sets. When sets are finite we can list elements and make arguments on the basis of those finite lists. There are likely better proofs not using counting, since $A \approx B$ $\exists\, \varphi, \varphi^{-1}$ bijections.

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, "\varphi"'] & B \arrow[l, bend left, "\varphi^{-1}"] \end{tikzcd}$$

Note $f = f \circ \varphi^{-1} \circ \varphi$

I believe you could use this to make a non-counting proof. (it escapes me currently)

§6.2 #17 *If p is a prime integer, show that $(p-1)^{-1} = p-1$ in $(\mathbb{Z}_p - \{0\}, \cdot)$*

<u>Proof</u>: remember in $\mathbb{Z}_p$ we have $px \equiv 0$.

Consider then,

$$(p-1)(p-1) = p^2 - 2p + 1 \equiv 1$$

thus $(p-1)^{-1} = p-1$.

<u>Curious</u>: I didn't use $\gcd(p, x) = 1$ for $x < p$.

would this fail for $\mathbb{Z}_6$ or $\mathbb{Z}_4$ etc... ?

$\mathbb{Z}_6$ has $6-1 = 5$ and $5^2 = 25 = 1$ $\therefore 5^{-1} = 5$. (in $\mathbb{Z}_6$)

$\mathbb{Z}_8$ has $8-1 = 7$ and $7^2 = 49 = 1$ $\therefore 7^{-1} = 7$ (in $\mathbb{Z}_8$)

thus the assumption p be prime is not needed.

However, you can prove $(\mathbb{Z}_p - \{0\}, \cdot)$ is only a <u>group</u> if p is prime since otherwise the <u>factors</u> of $\mathbb{Z}_n$ turn out to be zero-divisors which have no multiplicative inverses (a big no-no for a multiplicative group!)