SELECT MATH 200 HOMEWORK SOLUTIONS CONCERNING $\mathbb{Z}_n$

$\boxed{\S 3.2 \#6}$ Calculate the equivalence classes for the relation of

(a.) congruence modulo 5 says $x, y \in \mathbb{Z}$ are congruent iff $\exists k \in \mathbb{Z}$ such that $x - y = 5k$. That is,

$$x \equiv_5 y \quad \text{iff} \quad \exists k \in \mathbb{Z} \text{ s.t. } x = y + 5k$$

Consequently the <u>equivalence classes</u> of $\equiv_5$ are,

$$\begin{aligned}
\overline{x} &= \{ y \in \mathbb{Z} \mid x \equiv_5 y \} \\
&= \{ x + 5k \mid k \in \mathbb{Z} \} \\
&= x + 5\mathbb{Z}.
\end{aligned}$$

In particular,

$$\begin{aligned}
\overline{0} &= 0 + 5\mathbb{Z} = \{ \dots, -10, -5, 0, 5, 10, \dots \} \\
\overline{1} &= 1 + 5\mathbb{Z} = \{ \dots, -4, 1, 6, 11, \dots \} \\
\overline{2} &= 2 + 5\mathbb{Z} = \{ \dots, -3, 2, 7, 12, \dots \} \\
\overline{3} &= 3 + 5\mathbb{Z} = \{ \dots, -2, 3, 8, 13, \dots \} \\
\overline{4} &= 4 + 5\mathbb{Z} = \{ \dots, -1, 4, 9, 14, \dots \}
\end{aligned}$$

<u>Remark</u>: $\equiv_5$ is an equivalence relation on $\mathbb{Z}$ and we can see that the eq. classes <u>partition</u> $\mathbb{Z}$ as expected; $\mathbb{Z} = \overline{0} \cup \overline{1} \cup \overline{2} \cup \overline{3} \cup \overline{4}$ and $\overline{x} \cap \overline{y} = \phi$ for $x \not\equiv_5 y$.

(b.) $\equiv_8$ has equivalence classes $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}$ each of which has the form $\overline{x} = x + 8\mathbb{Z}$.

(c.) $\equiv_1$ has a rather boring equivalence class, it's $\overline{0} = \mathbb{Z}$.

(d.) $\equiv_7$ has eq. classes $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}$ each of which has the form $\overline{x} = x + 7\mathbb{Z}$.

$\boxed{\S 3.2 \# 7}$ Given that $\equiv_m$ is an equivalence relation on $\mathbb{Z}$, prove that $\forall x, y \in \mathbb{Z}$:

(b.) $\boxed{x \in \overline{x}}$ we are given $\equiv_m$ is reflexive thus $x \equiv_m x \quad \forall x \in \mathbb{Z}$. Thus $x \in \overline{x}$.

(g.) $\boxed{\text{if } \overline{x} \neq \overline{y}, \text{ then } \overline{x} \cap \overline{y} = \phi}$

The contrapositive of the above is that if $\overline{x} \cap \overline{y} \neq \phi$ then $\overline{x} = \overline{y}$.
Assume $\overline{x} \cap \overline{y} \neq \phi$, thus $\exists z \in \overline{x} \cap \overline{y}$.
The point of intersection has $z \in \overline{x}$ and $z \in \overline{y}$ hence $z \equiv_m x$ and $z \equiv_m y$. By symmetric property $x \equiv_m z$. Then by transitive property $x \equiv_m z$ and $z \equiv_m y \Rightarrow x \equiv_m y$.
Therefore, $a \in \overline{x} \Leftrightarrow a \equiv_m x \Leftrightarrow a \equiv_m y \Leftrightarrow a \in \overline{y}$.
We find $\overline{x} = \overline{y}$ as they share the same elements.

Remark: we should be able to prove these w/o assuming $\equiv_m$ is an equivalence relation. The proofs are about the same,

(b.) Observe $x - x = 0 = 0 \cdot m$ thus $x \equiv_m x$ hence $x \in \overline{x}$.

(d.) if $x \equiv_m y$ then $y - x = mk$ for some $k \in \mathbb{Z}$.
Thus $y = x + mk$. We seek to prove $\overline{x} = \overline{y}$.
Let $a \in \overline{y} \Leftrightarrow a = y + ml$, for some $l \in \mathbb{Z}$

$\Leftrightarrow a = x + mk + ml$, using $y = x + mk$

$\Leftrightarrow a = x + m(k + l)$

$\Leftrightarrow a \in \overline{x}$ $\therefore$ as a was arbitrary we find $\overline{x} = \overline{y}$.

$\boxed{\S 4.1 \# 8a}$ Denote $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ and $\mathbb{Z}_6 = \{[0], [1], \dots\}$ is the relation $f : \mathbb{Z}_3 \to \mathbb{Z}_6$ defined by $f(\bar{x}) = [x]$ a function?

Notation can be decieving. Notice that for any $k \in \mathbb{Z}$ we have $\bar{x} = \overline{x+3k}$ and $[x] = [x+6k]$. Let $k \in \mathbb{Z}$,

$$[x] = f(\bar{x}) = f(\overline{x+3k})$$
$$= [x+3k]$$

If f is a function we need $[x] = [x+3k]$ for any $k \in \mathbb{Z}$. That is false since $[0] \neq [0+3]$ (counter-example) This relation is not single valued, $f(\bar{0}) = [0]$ and $f(\bar{0}) = f(\bar{3}) = [3] \neq [0]$.

$\boxed{\S 4.1 \# 8b}$ $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ where $f(\bar{x}) = [x+1]$ Same question as #8a

Here, for any $k \in \mathbb{Z}$, $\bar{x} = \overline{x+6k}$ and $[x] = [x+6k]$. Consider then, for $k \in \mathbb{Z}$,

$$f(\overline{x+6k}) = [x+6k+1] = [x+1] = f(\bar{x})$$

Thus f is well-defined. (it is single-valued since $\bar{x} \longmapsto f(\bar{x})$ and $f(\bar{x})$ is just one thing)

$\boxed{\S 4.1 \# 8c}$ $f : \mathbb{Z}_3 \to \mathbb{Z}_6$ with $f(\bar{x}) = [2x]$, again same?

For $k \in \mathbb{Z}$ we have $\overline{x+3k} = \bar{x}$ and $[x] = [x+6k]$. Let $k \in \mathbb{Z}$ and consider, for any $x \in \mathbb{Z}$,

$$f(\overline{x+3k}) = [2(x+3k)]$$
$$= [2x+6k]$$
$$= [2x]$$

Thus $f(\overline{x+3k}) = f(\bar{x})$, f is defined independent of the representative, it's single-valued, it's a function.

$\boxed{\S 4.1 \# 8d}$ $f : \mathbb{Z}_4 \to \mathbb{Z}_6$ with $f(\overline{x}) = [2x+1]$ is this a function?

Let $k \in \mathbb{Z}$ and consider $\overline{x} = \overline{x+4k}$ and $[x] = [x+6k]$,

$$\begin{aligned} f(\overline{x+4k}) &= [2(x+4k)+1] \\ &= [2x+1+8k] \\ &= [2x+1+2k] \end{aligned}$$

Notice $[2x+1+2k] \neq [2x+1]$ in general.

$f(\overline{0}) = [1]$ yet $f(\overline{0}) = f(\overline{4}) = [8+1] = [9] = [3]$

thus f is not a function.

$\boxed{\S 4.1 \# 8e}$ $f : \mathbb{Z}_3 \to \mathbb{Z}_4$ with $f(\overline{x}) = [x]$, is f a function?

Observe that $f(\overline{0}) = [0]$ and $f(\overline{3}) = [3]$ yet $[0] \neq [3]$ hence f is <u>not</u> a function.

$\boxed{\S 4.1 \# 8f}$ $f : \mathbb{Z}_4 \to \mathbb{Z}_2$ with $f(\overline{x}) = [3x]$, is f a function?

Let $k, x \in \mathbb{Z}$ and consider,

$$\begin{aligned} f(\overline{x+4k}) &= [3(x+4k)] \\ &= [3x + 12k] \\ &= [3x + 2(6k)] \\ &= [3x] = f(\overline{x}). \end{aligned}$$

Thus f is well-defined, it is a function.

<u>Remark</u>: in $\mathbb{Z}_n$ the formula for a function can appear to be single-valued, but we must examine the ambiguity that arises from the non-unique choice of representative ($\overline{x} = \overline{x+n}$ etc...)

$\boxed{\S 4.3\ \#14a}$ Let $f: \mathbb{Z}_4 \longrightarrow \mathbb{Z}_8$ be given by $f(\bar{x}) = [2x]$
Prove f is 1-1 but not onto

Notice $f(\mathbb{Z}_4) = \{[0], [2], [4], [6]\}$ thus f is not onto.

Suppose $f(\bar{x}) = f(\bar{y}) \Rightarrow [2x] = [2y]$
$\Rightarrow 2x \equiv_8 2y$
$\Rightarrow x \equiv_8 y$
$\Rightarrow x = y + 8k$ for some $k \in \mathbb{Z}$.
$\Rightarrow x = y + 4(2k)$ for $2k \in \mathbb{Z}$.
$\Rightarrow x \equiv_4 y$
$\Rightarrow \bar{x} = \bar{y}$, $\therefore$ <u>f is 1-1</u>

$\boxed{\S 4.3\ \#14b}$ Let $f: \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ where $f(\bar{x}) = [3x]$.
Prove f is onto but not 1-1

Notice $f(\bar{0}) = [3 \cdot 0] = [0]$ and $f(\bar{1}) = [3] = [1]$, thus f is surjective. Consider that $f(\bar{2}) = [3(2)] = [0]$ hence $f(\bar{0}) = f(\bar{2})$ and $\bar{0} \neq \bar{2}$ $\therefore$ <u>f is not injective</u>.

$\boxed{\S 4.3\ \#14c}$ Let $f: \mathbb{Z}_6 \longrightarrow \mathbb{Z}_6$ with $f(\bar{x}) = \overline{x+1}$.
Prove f is a bijection

Suppose $f(\bar{x}) = f(\bar{y})$ then $\overline{x+1} = \overline{y+1}$ hence
$x+1 \equiv_6 y+1 \Rightarrow x+1 = y+1+6k$ for some $k \in \mathbb{Z}$.
Subtracting 1 from both sides, $x = y + 6k$ for $k \in \mathbb{Z}$
hence $x \equiv_6 y \Rightarrow \bar{x} = \bar{y}$. $\therefore$ <u>f is 1-1</u>.
Let $\bar{z} \in \mathbb{Z}_6$ observe that $\overline{z-1} \in \mathrm{dom}(f)$ and
$f(\overline{z-1}) = \overline{z-1+1} = \bar{z}$ $\therefore$ <u>f is onto</u>.
Consequently f is a <u>bijection</u> since it's <u>1-1 and onto</u>