LECTURE 16 : DETERMINANTS

Given a square matrix A we can calculate a # known as the <u>determinant</u> of A, we denote this by $\det(A)$. I'll begin by giving formulas for 2×2 or 3×3 cases,

$$\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A_{11}A_{22} - A_{12}A_{21}$$

$$\det\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = A_{11}\det\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - A_{12}\det\begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} + A_{13}\det\begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

$$= A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}$$

For the 2×2 case, $\epsilon_{12} = 1$, $\epsilon_{21} = -1$ and $\epsilon_{11} = \epsilon_{22} = 0$ allows us to express the formula via

$$\boxed{\det(A) = \sum_{i,j=1}^{2} \epsilon_{ij} A_{1i} A_{2j}}$$

Likewise if we set $\epsilon_{123} = 1 = \epsilon_{231} = \epsilon_{312}$ and $\epsilon_{321} = -1 = \epsilon_{213} = \epsilon_{132}$ whereas $\epsilon_{ijk} = 0$ for any choice of i,j,k where there is a repeated index

$$\boxed{\det(A) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} A_{1i} A_{2j} A_{3k}}$$

Then, the general definition is given below using $\epsilon_{123\cdots n} = 1$ and $\epsilon_{i_1 i_2 \cdots i_n}$ is antisymmetric in every pair of indices.

$$\xrightarrow[(n\times n\ A)]{\text{Def}^{\text{n}}} \boxed{\det(A) = \sum_{i_1, i_2, \dots, i_n = 1}^{n} \epsilon_{i_1 i_2 \cdots i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}}$$

In the case of ϵ_{ijk} we can connect to the cross product of vectors in $\mathbb{R}^3$,

Defⁿ/ $\vec{V} \times \vec{W} = \sum_{i,j,k} \epsilon_{ijk} V_i W_j \hat{x}_k$

where $\hat{x}_1 = \langle 1,0,0 \rangle$, $\hat{x}_2 = \langle 0,1,0 \rangle$, $\hat{x}_3 = \langle 0,0,1 \rangle$

Both the cross product and the determinant of a 3x3 are built with the same pattern. We should do some examples,

E1

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{bmatrix} = 1 \cdot \det\begin{pmatrix} 6 & 8 \\ 7 & 9 \end{pmatrix} - 2\det\begin{pmatrix} 4 & 8 \\ 5 & 9 \end{pmatrix} + 3\det\begin{pmatrix} 4 & 6 \\ 5 & 7 \end{pmatrix}$$

$$= (54-56) - 2(36-40) + 3(28-30)$$

$$= -2 - 2(-4) + 3(-2)$$

$$= \boxed{0}.$$

For a 4x4 matrix we expand across top row $+ - + -$

E2

$$\det\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 7 \\ 1 & 4 & 1 & 1 \\ 5 & 5 & 6 & 8 \end{bmatrix} = -2\det\begin{bmatrix} 0 & 0 & 7 \\ 1 & 1 & 1 \\ 5 & 6 & 8 \end{bmatrix}$$

$$= (-2)(7)\det\begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$$

$$= -14\,(6-5)$$

$$= \boxed{-14}.$$

In fact, we can expand across any row, this is known as Laplace's expansion by minors.

The expansion by minors requires us to systematically calculate the determinants of submatrices. Following Jacob's suggestion we introduce a notation

Def^n/ Let $A \in \mathbb{R}^{n \times n}$ then define $\cancel{A}_{ij}$ to be the $(n-1) \times (n-1)$ matrix formed by deleting the i^{th} row and j^{th} column.

In this notation the expansion across top row is

$$\det(A) = A_{11}\det(\cancel{A}_{11}) - A_{12}\det(\cancel{A}_{12}) + A_{13}\det(\cancel{A}_{13}) + \cdots + (-1)^{1+n}A_{1n}\det(\cancel{A}_{1n})$$

Then it turns out we can calculate by expanding across the i^{th} row as follows:

Th^m/ $\det(A) = (-1)^{i+1}A_{i1}\det(\cancel{A}_{i1}) + (-1)^{i+2}A_{i2}\det(\cancel{A}_{i2}) + \cdots + (-1)^{i+n}A_{in}\det(\cancel{A}_{in})$

which we could write nicely by

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\cancel{A}_{ij})$$

E3

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 4 & 5 & 6 & 7 \\ 8 & 0 & 9 & 0 \end{bmatrix} = (-1)^{2+3} 2 \det\begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 7 \\ 8 & 0 & 0 \end{bmatrix}$$

$$= (-2)(-1)^{3+1} 8 \det\begin{bmatrix} 1 & 1 \\ 5 & 7 \end{bmatrix}$$

$$= -16(7-5)$$

$$= \boxed{-32}.$$

Another important fact about det is that $\det(A) = \det(A^T)$ hence whatever we know for rows equally well applies to columns.

$\boxed{E4}$

$$\det\begin{bmatrix} 0 & 3 & 0 & 9 & 1 \\ 0 & 4 & 0 & 8 & 3 \\ 0 & 5 & 0 & 7 & 6 \\ 1 & 6 & 11 & 5 & 4 \\ 1 & 7 & 0 & 2 & 0 \end{bmatrix} = (-1)^{4+3}\, 11 \det\begin{bmatrix} 0 & 3 & 9 & 1 \\ 0 & 4 & 8 & 3 \\ 0 & 5 & 7 & 6 \\ 1 & 7 & 2 & 0 \end{bmatrix}$$

$$= -11(-1)^{4+1}(1)\det\begin{bmatrix} 3 & 9 & 1 \\ 4 & 8 & 3 \\ 5 & 7 & 6 \end{bmatrix}$$

$$= (11)\left[3(48-21) - 9(24-15) + 1(28-40)\right]$$

$$= 11\left(3(27) - 9(9) - 12\right)$$

$$= \boxed{-132}.$$

We repeatedly expand down the 1st column in the next example.

$\boxed{E5}$

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 1\cdot\det\begin{bmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix}$$

$$= 1\cdot 5\cdot\det\begin{bmatrix} 8 & 9 \\ 0 & 10 \end{bmatrix}$$

$$= 1\cdot 5\cdot 8\cdot 10$$

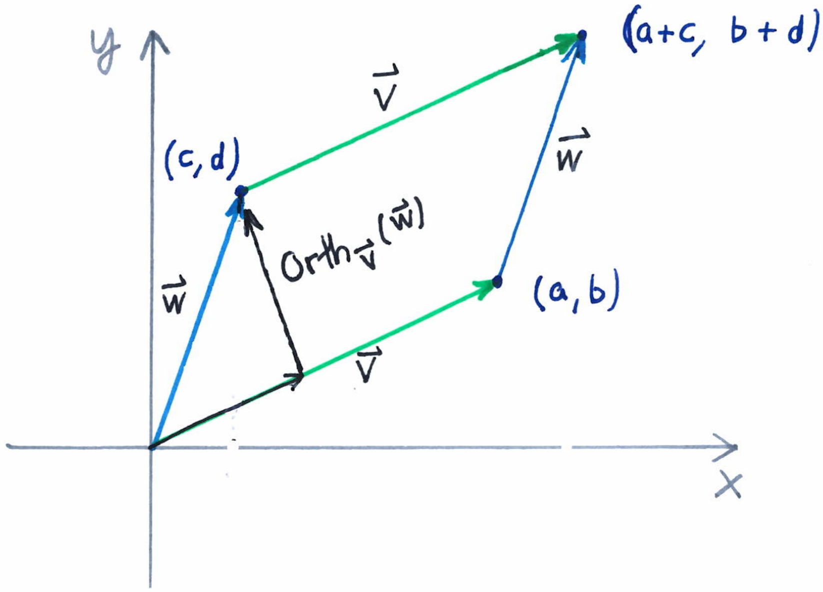
$$= \boxed{400}.$$

The pattern of the above example reveals the Th^{m} below,

Th^{m}/ If A is upper or lower triangular then the $\det(A)$ is simply the product of the diagonal entries

GEOMETRY OF 2×2 or 3×3 DETERMINANTS

Suppose vector $\vec{v} = \langle a, b \rangle$ and vector $\vec{w} = \langle c, d \rangle$



$$\begin{aligned}
\text{Orth}_{\vec{v}}(\vec{w}) &= \vec{w} - \text{Proj}_{\vec{v}}(\vec{w}) \\
&= \langle c, d \rangle - \left(\frac{\langle c, d \rangle \cdot \langle a, b \rangle}{\langle a, b \rangle \cdot \langle a, b \rangle} \right) \langle a, b \rangle \\
&= \langle c, d \rangle - \left(\frac{ac + bd}{a^2 + b^2} \right) \langle a, b \rangle \\
&= \left\langle c - \frac{a(ac+bd)}{a^2+b^2}, \ d - \frac{b(ac+bd)}{a^2+b^2} \right\rangle \\
&= \left\langle \frac{c(a^2+b^2) - a(ac+bd)}{a^2+b^2}, \ \frac{d(a^2+b^2) - b(ac+bd)}{a^2+b^2} \right\rangle
\end{aligned}$$

The height of the parallelogram is $\|\text{Orth}_{\vec{v}}(\vec{w})\|$,

$$H = \frac{1}{a^2+b^2} \sqrt{(cb^2 - abd)^2 + (da^2 - bac)^2}$$

The base of the parallelogram is $\|\vec{v}\| = \sqrt{a^2+b^2} = B$

$$\begin{aligned}
\text{area} = BH &= \frac{\sqrt{a^2+b^2}}{a^2+b^2} \sqrt{b^2(bc-ad)^2 + a^2(ad-bc)^2} \\
&= \frac{1}{\sqrt{a^2+b^2}} \sqrt{(a^2+b^2)(ad-bc)^2} \\
&= \sqrt{(ad-bc)^2} \\
&= |ad - bc| .
\end{aligned}$$

Geometry of Determinants Continued

I derived via vector algebraic techniques that the parallelogram with side vectors $\langle a,b\rangle$, $\langle c,d\rangle$ has area $|ad-bc|$. Notice

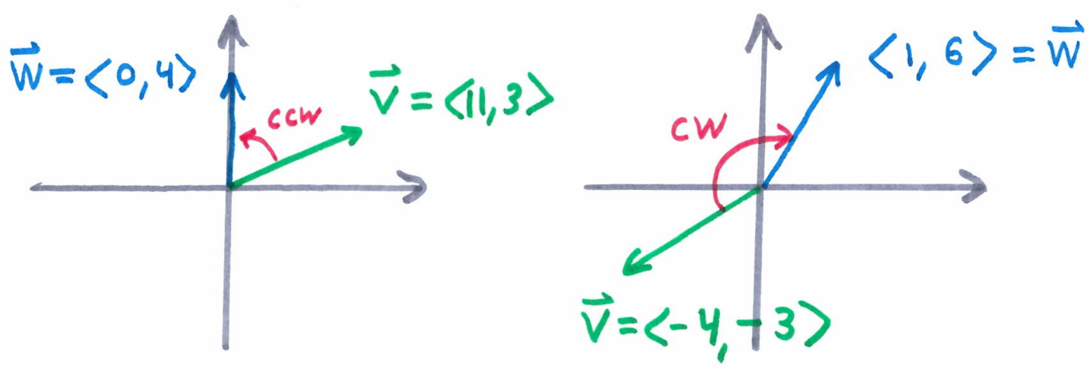
$$\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad-bc = \det[\vec{v}\,|\,\vec{w}]$$

$$\text{area}(\mathcal{P}(\vec{v},\vec{w})) = \pm\det(\vec{v}\,|\,\vec{w})$$

where $+$ if a ccw rotation of $\vec{v}$ generates $\vec{w}$
and $-$ if a cw rotation of $\vec{v}$ generates $\vec{w}$

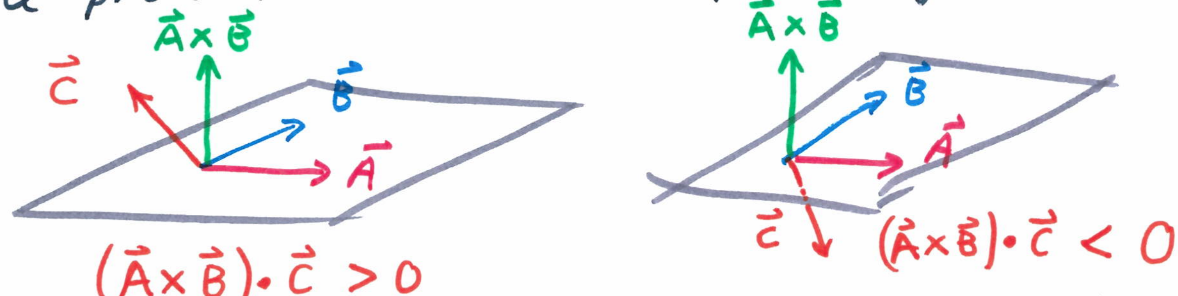
Here I intend a rotation by less than $180°$ ☺,

E6



$\det[\vec{v}|\vec{w}] = \det\begin{bmatrix} 11 & 0 \\ 3 & 4 \end{bmatrix} = 44$ $\qquad$ $\det[\vec{v}|\vec{w}] = \det\begin{bmatrix} -4 & 1 \\ -3 & 6 \end{bmatrix} = -21$

Finally, recall from Calculus III that the volume of a parallel-piped with side vectors $\vec{A}, \vec{B}, \vec{C}$ is given by $|(\vec{A}\times\vec{B})\cdot\vec{C}|$ where the sign of the triple product is determined by the right-hand-rule



since $\det(\vec{A}|\vec{B}|\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$ we can understand $\pm$ for det in this fashion.

Let us derive $\det(\vec{A}|\vec{B}|\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$

$$
\begin{aligned}
(\vec{A}\times\vec{B})\cdot\vec{C} &= \sum_{k=1}^{3} (\vec{A}\times\vec{B})_k C_k \\
&= \sum_{k=1}^{3}\left(\sum_{i,j=1}^{3} \epsilon_{ijk} A_i B_j\right) C_k \\
&= \sum_{i,j,k=1}^{3} \epsilon_{ijk} M_{1i} M_{2j} M_{3k}, \quad M = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \\
&= \det(M) = \det(M^T) \\
&= \det\left[\vec{A}|\vec{B}|\vec{C}\right].
\end{aligned}
$$

In short, the **determinant of a matrix** $A = [\vec{V}_1|\vec{V}_2|\vec{V}_3|\cdots|\vec{V}_n]$ gives the signed-hypervolume of the n-dim'l boxlike solid with edge vectors $\vec{V}_1, \vec{V}_2, \dots, \vec{V}_n$

$$\mathrm{Vol}(\mathcal{P}(\vec{V}_1,\dots,\vec{V}_n)) = \left|\det[\vec{V}_1|\vec{V}_2|\cdots|\vec{V}_n]\right|$$

Geometrically, we obtain the following:

> Th^{m}/ If $\{\vec{V}_1, \vec{V}_2, \dots, \vec{V}_n\}$ is linearly dependent then $\det[\vec{V}_1|\vec{V}_2|\cdots|\vec{V}_n] = 0$

(this can be proven via the multilinearity and complete antisymmetry of the det, but I leave that for another course.)