Lecture 17: Properties of Determinants

We discussed the expansion by minors in our previous Lecture and we found a couple basic tricks. Let's review those here,

1.) $$\det \underbrace{\begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}}_{A \text{ either upper-}\Delta\text{, lower-}\Delta \text{ or both.}} = a_1 a_2 \cdots a_n$$

2.) If $\{v_1, v_2, \dots, v_n\}$ is linearly dependent then $\det[v_1 | v_2 | \cdots | v_n] = 0$.

These make for easy calculation!

$\boxed{E1}$ $$\det(I_n) = \det\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} = 1 \cdot 1 \cdots 1 = \boxed{1}.$$

$\boxed{E2}$ $$\det\begin{pmatrix} 6 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 0 & 11 \end{pmatrix} = 6(9)(11) = \boxed{594}.$$

$\boxed{E3}$ $$\det\begin{bmatrix} 1 & 3 & e & 0 & 2 \\ 1 & 4 & 2 & 3 & 2 \\ 1 & 5 & 7 & 3 & 2 \\ 1 & 7 & 9 & 4 & 2 \\ 1 & \pi & 0 & 5 & 2 \end{bmatrix} = \boxed{0.}$$

linearly dependent!

The proof of many of the following claims follow from some simple calculations from the definition

$$\det(A) = \sum_{i_1,\dots,i_n=1}^{n} \epsilon_{i_1 i_2 \cdots i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$$

Thm/ Let $A \in \mathbb{R}^{n\times n}$ then

(1.) $\det(A^T) = \det(A)$

(2.) if $\exists j$ such that $row_j(A) = 0$ then $\det(A) = 0$

(3.) if $\exists j$ such that $col_j(A) = 0$ then $\det(A) = 0$

(4.) $\det(A_1|\cdots|aA_k + bB_k|\cdots|A_n) = a\det(A_1|\cdots|A_k|\cdots|A_n) + b\det(A_1|\cdots|B_k|\cdots|A_n)$

that is, we have linearity in each column of A.

(5.) the determinant is linear in each row as well

(6.) $\det(kA) = k^n \det(A)$

(7.) If $B = \{A : r_k \longleftrightarrow r_j\}$ then $\det(B) = -\det(A)$.

(8.) If $B = \{A : r_k + ar_j \longrightarrow r_k\}$ then $\det(B) = \det(A)$.

(9.) If $row_i(A) = k\, row_j(A)$ for $i \neq j$ then $\det(A) = 0$.

(10.) If there is any linear dep. amongst rows of A then $\det(A) = 0$.

E4

$$\det\begin{bmatrix} 0 & 3 & 4 & 5 & 6 \\ 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 8 & 9 \end{bmatrix} = \det\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix} = \underbrace{2\cdot 3\cdot 4\cdot 8\cdot 7}_{\boxed{1344}}$$

two flips $(-1)(-1) = 1$.

If a matrix is block-diagonal then there is a very nice formula for the determinant,

$$\boxed{\text{Th}^{\text{m}}/ \ \det(A_1 \oplus A_2 \oplus \cdots \oplus A_k) = \det(A_1)\det(A_2)\cdots\det(A_k)}$$

In fact, while it is not easy to prove, it can be shown by clever elementary matrix arguments or via the algebra of the wedge product that:

$$\boxed{\text{Th}^{\text{m}}/ \ \det(AB) = \det(A)\det(B) \ \text{ for } A, B \in \mathbb{R}^{n \times n}}$$

E5

$$\det\left[\begin{array}{cc|ccc} 4 & 5 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ \hline 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right] = \det\begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} \det\begin{bmatrix} 4 & 5 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix} = (-2)(20) = \boxed{-40}$$

E6 Suppose $\det(A) = 3$ and $\det(B) = 5$ where A is a 2×2 matrix and B is a 3×3 matrix then calculate $\det((2A)\oplus(7B))$.

$$\begin{aligned} \det((2A)\oplus(7B)) &= \det(2A)\det(7B) \\ &= (2^2\det(A))(7^3\det(B)) \\ &= 2^2 \cdot 3 \cdot 7^3 \cdot 5 \\ &= \boxed{20{,}580} \end{aligned}$$

E7 Suppose A is invertible, show $\det(A) \neq 0$

$AA^{-1} = I$ if we know A is invertible,

thus $\det(AA^{-1}) = \det(I) \Rightarrow \det(A)\det(A^{-1}) = 1$

$\Rightarrow \underline{\det(A) \neq 0}$.

We ought to state the last example as a Theorem since this is an important idea,

Thm/ Given $A \in \mathbb{R}^{n\times n}$, if A is invertible with inverse A^{-1} then $\det(A) \neq 0$ and $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

<u>Proof</u>: Given $A \in \mathbb{R}^{n\times n}$ and the existence of $A^{-1} \in \mathbb{R}^{n\times n}$ for which $AA^{-1} = I$ we find that

$$\det(AA^{-1}) = \det(I) = 1$$

$$\Rightarrow \det(A)\det(A^{-1}) = 1 \quad \therefore \det(A^{-1}) = \frac{1}{\det(A)}$$

and clearly $\det(A) \neq 0$. //

You see the identity $\det(AB) = \det(A)\det(B)$ is central to our understanding of many of the deeper applications of the determinant.

Thm/ If $T: V \to V$ is a linear transformation and β is a basis for V and $\bar{\beta}$ is any other basis for V then $\det[T]_{\beta\beta} = \det[T]_{\bar{\beta}\bar{\beta}}$

<u>Proof</u>: we showed $[T]_{\bar{\beta}\bar{\beta}} = P^{-1}[T]_{\beta\beta}P$ for some invertible matrix P thus,

$$\begin{aligned} \det[T]_{\bar{\beta}\bar{\beta}} &= \det(P^{-1}[T]_{\beta\beta}P) \\ &= \cancel{\det(P^{-1})}\det[T]_{\beta\beta}\cancel{\det(P)} \quad \text{by Th}^{m}\text{ above.} \\ &= \det[T]_{\beta\beta} \end{aligned}$$

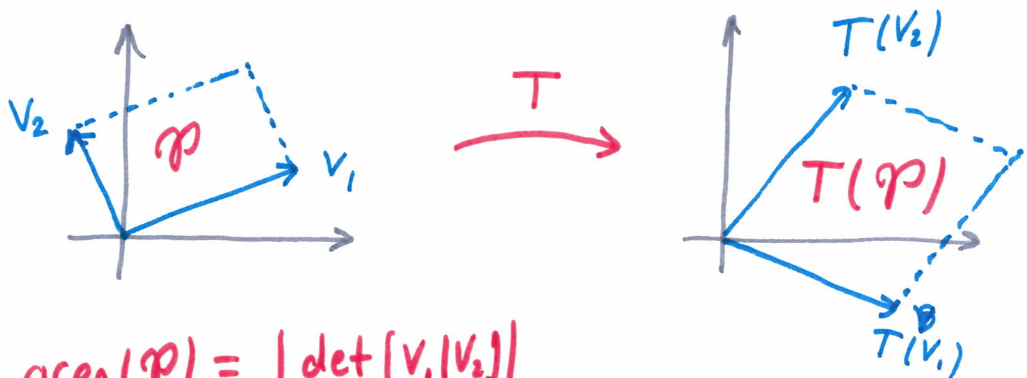
Defn/ If $T: V \to V$ linear then $\det(T) = \det([T]_{\beta\beta})$ where β is any basis for V

The definition just given for det(T) makes sense as any two similar matrices have the same determinant. Once again, if $B = P^{-1}AP$ then, I'll calculate it another way,

$$\begin{aligned}\det(B) &= \det(P^{-1}AP)\\ &= \det(P^{-1})\det(AP)\\ &= \det(AP)\det(P^{-1})\\ &= \det(APP^{-1})\\ &= \det(AI)\\ &= \det(A).\end{aligned}$$

#'s commute.

Let's return to the problem of geometry,



$$\text{area}(\mathcal{P}) = |\det[V_1|V_2]|$$

$$\begin{aligned}\text{area}(T(\mathcal{P})) &= |\det[T(V_1)|T(V_2)]|\\ &= |\det[[T]V_1 \,|\, [T]V_2]|\\ &= |\det([T][V_1|V_2])|\\ &= |\det T|\,|\det[V_1|V_2]|\\ &= |\det T|\,\text{area}(\mathcal{P})\end{aligned}$$

$$\boxed{\text{Th}^{m}/\ \ \text{area}(T(\mathcal{P})) = |\det(T)|\,\text{area}(\mathcal{P})}$$

We can prove a similar Th$^{\underline{m}}$ for $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ since $\text{Vol}(\mathcal{P}(v_1, v_2, v_3)) = |\det[v_1|v_2|v_3]|$ and

$$[T(v_1)|T(v_2)|T(v_3)] = [\,[T]v_1\,|\,[T]v_2\,|\,[T]v_3\,]$$
$$= [T][v_1|v_2|v_3]$$

Therefore, using $\det([T][v_1|v_2|v_3]) = \det[T]\det([v_1|v_2|v_3])$

note, $\det(T) = \det([T])$.

$$\boxed{\text{Th}^{\underline{m}}/\ \text{Vol}(T(\mathcal{P})) = |\det T|\,\text{Vol}(\mathcal{P})}$$

Remark:
If $\det T > 0$ then T maintains the orientation of a set of vectors, if v_1, v_2, v_3 are positively oriented then $T(v_1), T(v_2), T(v_3)$ are also positively oriented (a.k.a. right-handed)