Please put your work on these sheets. If you need additional room to show your work then add paper as needed, but be sure to put your answer clearly near the problem statement. Box your answers. Make sure you name is on each page and the assignment is stapled. Thanks and enjoy. This is worth 20pts.

Problem 1 Solve,

4 pts

Problem 1 Solve,

(a.) $\frac{dy}{dx} = x^2e^{-4x} - 4y$. $\Rightarrow \frac{dy}{dx} + 4y = x^2e^{-4x}$: linear use $\int$-factor method.

$I = \exp\left(\int 4dx\right) = e^{4x}$, multiply by I,

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = x^2$$

$$\frac{d}{dx}\left(e^{4x}y\right) = x^2 \implies e^{4x}y = \frac{1}{3}x^3 + C \quad \therefore \boxed{y = \frac{1}{3}x^3e^{-4x} + Ce^{-4x}}$$

(b.) $\frac{dy}{dx} = \frac{y^2-1}{x^2-1}$ given $y(2) = 2$.

Note $\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$ for the integration below

multiply by 2 for convenience

$$\int\frac{2dy}{y^2-1} = \int\frac{2dx}{x^2-1} \implies \ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1| + C$$

$y(2) = 2 \Rightarrow -\ln 3 = -\ln(3) + C \quad \therefore \underline{C=0}$

$$\boxed{\ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right|}$$ or

$$\left|\frac{y-1}{y+1}\right| = \left|\frac{x-1}{x+1}\right|$$

etc...

(c.) $(e^x + y)dx + (2 + x + ye^y)dy = 0$ given $y(0) = 1$.

optional

$$\frac{\partial F}{\partial x} = e^x + y \implies F(x,y) = e^x + xy + C_1(y)$$

$$\frac{\partial F}{\partial y} = 2 + x + ye^y \implies F(x,y) = 2y + xy + ye^y - e^y + C_2(x)$$

Comparing we see $C_1(y) = ye^y - e^y + 2y$ & $C_2(x) = e^x$.

Thus, $F(x,y) = e^x + xy + ye^y - e^y + 2y = K$

But, $y(0) = 1$ hence

$1 + 0 + \cancel{1e} - \cancel{e} + 2(1) = K$

$$\boxed{e^x + xy + ye^y - e^y + 2y = 3}$$

(d.) $\cos(\theta)dr - (r\sin(\theta) - e^\theta)d\theta = 0$.

Let $F(r,\theta) = r\cos\theta + e^\theta$

and observe that

$$\frac{\partial F}{\partial r} = \cos\theta \quad \& \quad \frac{\partial F}{\partial \theta} = -r\sin\theta + e^\theta$$

Thus, $\boxed{r\cos\theta + e^\theta = K \text{ is the general, implicit sol}^{\underline{n}}}$

or $\underline{r = (K - e^\theta)\sec\theta}$ to be $\underline{\text{explicit}}$.

2pts

Problem 2 Find a continuous solution to the following IVP,

$$\frac{dy}{dx}+2y = f(x), \ y(0)=0, \text{ with } \quad f(x)=\begin{cases}1, & 0\le x\le 1\\ -1, & x>1\end{cases}$$

Integrating factor $I = \exp\left(\int 2dx\right) = e^{2x}$ gives

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}f(x) = \begin{cases} e^{2x} & 0\le x<1 \\ -e^{2x} & x>1\end{cases}$$

$$\frac{d}{dx}\left[e^{2x}y\right] = \begin{cases} e^{2x} & 0\le x<1 & \Rightarrow y = \frac{1}{2}+c_1e^{-2x} \text{ for } 0\le x<1 \\ -e^{2x} & x>1 & \Rightarrow y = \frac{-1}{2}+c_2e^{-2x} \text{ for } x>1\end{cases}$$

Note $y(0)=0 \Rightarrow 0 = \frac{1}{2}+c_1 \therefore \boxed{y(x) = \frac{1}{2}(1-e^{-2x}) \text{ for, } 0\le x<1}$

As $x\to 1$ we find $y(1) = \frac{1}{2}(1-e^{-2})$ thus we need

$$\frac{1}{2}(1-e^{-2}) = \frac{-1}{2}+c_2e^{-2} \Rightarrow \left(1-\frac{1}{2}e^{-2}\right)e^{2} = c_2 \quad \boxed{y = \frac{-1}{2}+\left(e^2-\frac{1}{2}\right)e^{-2x} \text{ for } x>1}$$

2pts **Problem 3** Show the following differential equation is not exact:

Sorry I forget, but $A=0, B=0$ shows that $\partial_x N \neq \partial_y M$.

$$(5x^2y+6x^3y^2+4xy^2)dx + (2x^3+3x^4y+3x^2y)dy = 0.$$

Solve this ODE by the integrating factor method. Hint: $I = x^Ay^B$ will work here.

$$0 = \underbrace{\left(5x^{A+2}y^{B+1}+6x^{A+3}y^{B+2}+4x^{A+1}y^{B+2}\right)}_{IM}dx + \underbrace{\left(2x^{A+3}y^{B}+3x^{A+4}y^{B+1}+3x^{A+2}y^{B+1}\right)}_{IN}dy$$

$$\frac{\partial}{\partial y}(IM) = \underline{5(B+1)x^{A+2}y^{B}} + \underline{6(B+2)x^{A+3}y^{B+1}} + \underline{4(B+2)x^{A+1}y^{B+1}}$$

$$\frac{\partial}{\partial x}(IN) = \underline{2(A+3)x^{A+2}y^{B}} + \underline{3(A+4)x^{A+3}y^{B+1}} + \underline{3(A+2)x^{A+1}y^{B+1}}$$

Need $\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial x}(IN)$ to make $IM\,dx + IN\,dy$ exact.

$$\left.\begin{aligned} 5(B+1) &= 2(A+3) \\ 6(B+2) &= 3(A+4) \\ 4(B+2) &= 3(A+2)\end{aligned}\right\} \longrightarrow B+2 = \underbrace{\frac{1}{2}(A+4) = \frac{3}{4}(A+2)}$$

$$2A+8 = 3A+6 \Rightarrow A=2 \Rightarrow B=1$$

Let $I = x^2y$ and note

$$(5x^4y^2+6x^5y^3+4x^3y^3)dx + (2x^5y+3x^6y^2+3x^4y^2)dy = 0$$

is exact with sol'n $\boxed{x^5y^2+x^6y^3+x^4y^3 = C}$

2pts

Problem 4 Use the substitution $y = vx^2$ to solve

$$\frac{dy}{dx} = \frac{2y}{x} + \cos(y/x^2).$$

$$\frac{y}{x} = vx \quad \& \quad \frac{y}{x^2} = v$$

$$y = vx^2 \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x^2 + 2xv$$

Thus $x^2\frac{dv}{dx} + \cancel{2xv} = \cancel{2vx} + \cos(v)$

Sep. variables,

$$\frac{dv}{\cos(v)} = \frac{dx}{x^2}$$

$$\int \sec(v)\,dv = \int \frac{dx}{x^2}$$

$$\ln|\sec v + \tan v| = \frac{-1}{x} + C$$

$$\boxed{\ln\left|\sec(y/x^2) + \tan(y/x^2)\right| = \frac{-1}{x} + C}$$

8pts **Problem 5** Solve the following by one of the substitution methods we discussed:

typo.

(a.) $\frac{dy}{dx} - y = e^{2x}y^3$ BERNOULLI !

$$y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x} \qquad \text{Let } z = y^{-2} \Rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$-\frac{1}{2}\frac{dz}{dx} - z = e^{2x}$$

$$\frac{dz}{dx} + 2z = -2e^{2x} \Rightarrow e^{2x}\frac{dz}{dx} + 2e^{2x}z = -2e^{4x}$$

$$\frac{d}{dx}\left[e^{2x}z\right] = -2e^{4x}$$

Hence, $e^{2x}z = -\frac{1}{2}e^{4x} + C \Rightarrow \boxed{\frac{1}{y^2} = -\frac{1}{2}e^{2x} + Ce^{-2x}}$

(b.) $\frac{dy}{dx} = (x+y+2)^2$.

Let $v = x+y+2 \Rightarrow \frac{dv}{dx} = 1 + \frac{dy}{dx}$ thus

$$\frac{dv}{dx} - 1 = v^2 \Rightarrow \frac{dv}{dx} = v^2+1 \Rightarrow \int\frac{dv}{v^2+1} = \int dx$$

Hence, $\tan^{-1}(v) = x + C$

$$v = \tan(x+C)$$

$$x+y+2 = \tan(x+C) \quad \therefore \quad \boxed{y = -2 - x + \tan(x+C)}$$

(c.) $m\dot{v} = mg - kv^2$ where m, g, k are constants. Also, find terminal velocity.

Recall $\dot{v} = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$ thus $mv\frac{dv}{dx} = mg - kv^2$

Hence $\frac{dv}{dx} = \frac{g}{v} - \frac{kv}{m} \Rightarrow \frac{dv}{\frac{g}{v} - \frac{kv}{m}} = dx$

Remark: Can also solve in t, v ... partial fractions

Thus, $\int \frac{v\,dv}{\underbrace{g - \frac{k}{m}v^2}} = \int dx \Rightarrow \int\frac{-m\,du}{2k\,u} = \frac{-m}{2k}\ln|u| = x + C_1$

$u = g - \frac{kv^2}{m} \longrightarrow du = \frac{-2kv\,dv}{m}$

$$\ln\left|g - \frac{kv^2}{m}\right| = \frac{-2kx}{m} + C_2$$

$$g - \frac{kv^2}{m} = C_3 e^{-\frac{2kx}{m}}$$

$$g - C_3e^{-\frac{2kx}{m}} = \frac{kv^2}{m}$$

$$\boxed{\pm\sqrt{\frac{mg}{k} - C_4e^{-\frac{2kx}{m}}} = v(x)}$$

As $x \to \infty$ we find

$v_\infty = \pm\sqrt{\frac{mg}{k}}$ (set $m\dot{v} = 0$ find same!)

(d.) $(x^2 - 2y^2)dx + 2xydy = 0$. Hint: divide by x^2, tilt head, think.

$$\left(1 - \frac{2y^2}{x^2}\right)dx + \frac{2y}{x}dy = 0$$

Homogeneous ODE. Subst.

$v = y/x$ so $xv = y$

$$v\,dx + x\,dv = dy$$

Problem 5d continuing

$$(1-2v^2)\,dx + 2v\,dy = 0$$

$$(1-2v^2)\,dx + 2v(v\,dx + x\,dv) = 0$$

$$(1-\cancel{2v^2}+\cancel{2v^2})\,dx + 2xv\,dv = 0$$

$$dx = -2xv\,dv$$

$$\int -\frac{dx}{x} = \int 2v\,dv$$

$$-\ln|x| = v^2 + C$$

$$-\ln|x| = \frac{y^2}{x^2} + C$$

$$\boxed{y^2 = -Cx^2 - x^2\ln|x|}$$

2pts **Problem 6** Suppose f is a function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that either f is the function which is identically zero on $\mathbb{R}$ or f is an exponential function.

$$\frac{\partial}{\partial y}\left[f(x+y)\right] = \frac{\partial}{\partial y}\left[f(x)f(y)\right]$$

$$f'(x+y) = f(x)f'(y)$$

$\underline{y=0}$ $\quad f'(x) = f(x)$ $\quad$ & $\quad \underline{x=0}$ $\quad \underbrace{f'(y) = f(0)f'(y)}$

If $f'(y) \neq 0$ then gives us $f(0) = 1$.

$$\frac{df}{dx} = f$$

$$\frac{df}{f} = dx \Rightarrow \ln|f| = x + C_1$$

$$f(x) = \pm e^{x+C_1} = ke^x \qquad f(0) = ke^0 = 1$$

$$\therefore \boxed{f(x) = e^x}$$

(or $y = 0$ is also a sol^{n})

1pt

Problem 7 Let $\star$ be the DEqn $y^2\sin(x)dx + yf(x)dy = 0$. Find all functions f such that $\star$ is an exact DEqn.

We need $\frac{\partial}{\partial y}\left(y^2 \sin(x)\right) = \frac{\partial}{\partial x}\left[y f(x)\right]$

$$\underline{2y\sin(x)} = y\frac{\partial f}{\partial x} = \underline{y\frac{df}{dx}}$$

$$\Rightarrow \frac{df}{dx} = 2\sin(x)$$

$$\Rightarrow \boxed{f(x) = -2\cos(x) + C}$$

3 pts

Problem 8 The DEqn $y = xy' + f(y')$ is called **Clairaut's equation** .

1. Show that the lines $y = cx + f(c)$ are solutions of **Clairaut's equation**
2. Suppose $f(y') = \frac{1}{2}(y')^2$ and show $y = xy' + f(y')$ has solution $y = -\frac{1}{2}x^2$.
3. Plot the so-called **singular** solution $y = -\frac{1}{2}x^2$ and plot the linear solutions at $(-2,-2)$, $(-1,-1/2)$, $(0,0)$, $(1,-1/2)$ and $(2,-2)$. Clearly Clairaut's equation does not have a unique solution at each point. Does this contradict the uniqueness theorem we discussed in lecture? (why not!)

1.) Suppose $y = cx + f(c)$ defines a line. Differentiate implicitly, for fixed c, and derive $\frac{dy}{dx} = c$ then $f(\frac{dy}{dx}) = f(c)$ thus

$$xy' + f(y') = xc + f(c)$$

But, $y = xc + f(c)$ thus $y = xy' + f(y')$. We've shown $y = cx + f(c)$ solves Clairaut's Eq^{n} for arbitrary c.

2.) Given $f(y') = \frac{1}{2}(y')^2$ we face Clairaut's Eq^{n}

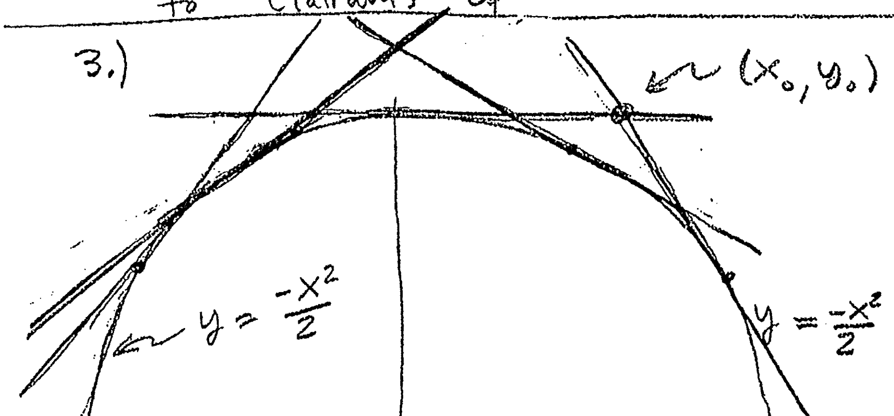
$$y = xy' + \tfrac{1}{2}(y')^2$$

Let $y = -x^2/2$ then $y' = -x$ and we calculate,

$$xy' + \tfrac{1}{2}(y')^2 = x(-x) + \tfrac{1}{2}(-x)^2 = -x^2 + x^2/2 = -x^2/2$$

Thus $xy' + \frac{1}{2}(y')^2 = y$ and we've shown $y = -x^2/2$ is a sol^{n} to Clairaut's Eq^{n} with the choice $f(y') = \frac{1}{2}(y')^2$

3.)



$y = cx + \frac{1}{2}c^2$

$(0,0)$: $y = 0$

$(-1, 1/2)$: $y =$

$(1, 1/2)$:

$(2,-2)$:

$(-2,-2)$:

We need $\frac{dy}{dx} = F(x,y)$ to apply the uniqueness Th^{m}.

Observe $(y')^2 + 2xy' = 2y \Rightarrow (y'+x)^2 = 2y + x^2$

Hence, $\boxed{y' = -x^2 \pm \sqrt{2y + x^2}}$

$\frac{\partial F}{\partial y} = \frac{\pm 1}{\sqrt{2y+x^2}}$ is divergent along $y = \frac{-1}{2}x^2$, certainly uniqueness Th^{m} does not apply. The $\pm$ suggests problem at (x_0, y_0)