Please put your work on these sheets. If you need additional room to show your work then add paper as needed, but be sure to put your answer clearly near the problem statement. Box your answers. Make sure you name is on each page and the assignment is stapled. Thanks and enjoy.

Problem 1 [3pts] Consider the set of curves dsecribed by $y^2 = kx$. Find the orthogonal trajectories to the given set of curves.

We need to eliminate k; $k = y^2/x \Rightarrow 0 = \frac{2y}{x}\frac{dy}{dx} - \frac{y^2}{x^2}$

Thus $y^2 = kx$ is sol^{n} of $\frac{dy}{dx} = \left(\frac{y^2}{x^2}\right)\left(\frac{x}{2y}\right) = \frac{y}{2x}$

To find O.T. we solve $\frac{dy}{dx} = \frac{-1}{y/2x}$

$$y\,dy = -2x\,dx \Rightarrow \boxed{\frac{1}{2}y^2 = -x^2 + C}$$

Problem 2 [3pts] Find the velocity of a mass m which is launched vertically with velocity v_o from a planet with mass M and radius R. Recall that the gravitational force is given by:

$$F = -\frac{GmM}{(R+y)^2} = m\frac{dV}{dt} = m\frac{dy}{dt}\frac{dV}{dy} = mV\frac{dV}{dy}$$

if we assume the motion is directly vertical and y is the altitude of m. You may find the velocity as a function of y.

$$\int_{V_0}^{V_1} V\,dV = \int_0^{y_1} \frac{-GM\,dy}{(R+y)^2}$$

$$\frac{1}{2}(V^2 - V_0^2) = \left.\frac{GM}{R+y}\right|_0^{y_1} = GM\left(\frac{1}{R+y_1} - \frac{1}{R}\right)$$

$$\boxed{V = \pm\sqrt{V_0^2 + 2GM\left(\frac{1}{R+y} - \frac{1}{R}\right)}}$$

(note, no dependence on m)

Problem 3 [4pts] Suppose a rocket car has an initial speed of v_o as it hurtles across a speedway in a remote desert. Suppose the driver opens a parachute which developes a retarding force proportional to the cube of the velocity; $F_f = -kv^3$. Find the velocity as:

(a.) a function of time,

(b.) a function of position x taking x_o as the initial position

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a.) $$m\frac{dV}{dt} = -kV^3 \Rightarrow \frac{dV}{V^3} = \frac{-k}{m}dt$$

$$\Rightarrow \frac{-1}{2V^2} = \frac{-kt}{m} + C_1$$

$$\Rightarrow \frac{1}{V^2} = \frac{2kt}{m} + C_2$$

But, $\frac{1}{V_o^2} = 0 + C_2 \Rightarrow \frac{1}{V^2} = \frac{2kt}{m} + \frac{1}{V_o^2} \Rightarrow \boxed{V = \frac{1}{\sqrt{\frac{1}{V_o^2} + \frac{2kt}{m}}}}$

Can clean this up, $\boxed{V(t) = \frac{V_o}{\sqrt{1 + 2kV_o^2 t/m}}}$

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b.) $$mV\frac{dV}{dx} = -kV^3 \Rightarrow \frac{dV}{V^2} = -\frac{k}{m}dx \Rightarrow \frac{-1}{V} = \frac{-k}{m}x + C_1$$

But, $\frac{-1}{V_o} = \frac{-kx_o}{m} + C_1 \therefore \frac{-1}{V} = -\frac{kx}{m} + \frac{kx_o}{m} - \frac{1}{V_o}$

$$\boxed{V = \frac{1}{\frac{k}{m}(x - x_o) + \frac{1}{V_o}} = \frac{V_o}{1 + \frac{V_o(x-x_o)}{m}}}$$

Problem 4 [3pts] A tank initially contains 100 gallons of water with 10lb of lemon drink mix. Then at $t = 0$ fresh water is added to the tank at 3 gallons per minute and at the same time 3 gallons are drained per minute from the tank. Assume the tank is well-mixed during this process. Find the lb's of lemon drink mix as a function of time. If you like your drink with a concentration of 1lb per 20 gallons then at what time should you drink from the drain?

Let x = lbs of mix so $x(0) = 10$.

$$\frac{dx}{dt} = R_{in} - R_{out} = 0 - \left(\frac{3x}{100}\right)$$

$$\frac{dx}{x} = \frac{-3}{100}dt \longrightarrow \ln|x| = \frac{-3t}{100} + C_1$$

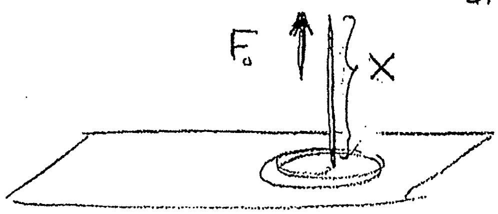
$$x(t) = C_2 e^{-3t/100}$$

Note, $x(0) = 10 = C_2 e^0 \therefore \boxed{x(t) = 10e^{-3t/100}}$

Concentration $\frac{1b}{20\text{gallons}} \Longrightarrow$ need $\frac{5lb's}{100\text{ gall}}$ solve $5 = 10e^{-3t/100}$

Hence, $\frac{1}{2} = e^{-3t/100} \Rightarrow \ln(1/2) = \frac{-3t}{100}$ $\boxed{t = \frac{100\ln(2)}{3}}$ (minutes)

Problem 5 [3pts] A chain is coiled on the ground. One end is then lifted with constant force. Find the velocity.



Let $\lambda = \frac{dm}{dx}$ = linear mass density.
If we assume λ constant then
$m(x) = \lambda x$ where x = length of chain lifted off ground. The chain lifted off ground has no momentum.

Thus $\underline{\frac{dP}{dt} = -mg + F_0}$,

$$\frac{dP}{dt} = \frac{d}{dt}(mv) = \frac{dm}{dt}v + m\frac{dv}{dt} = \lambda\frac{dx}{dt}v + m\frac{dx}{dt}\frac{dv}{dx} = \lambda v^2 + mv\frac{dv}{dx}$$

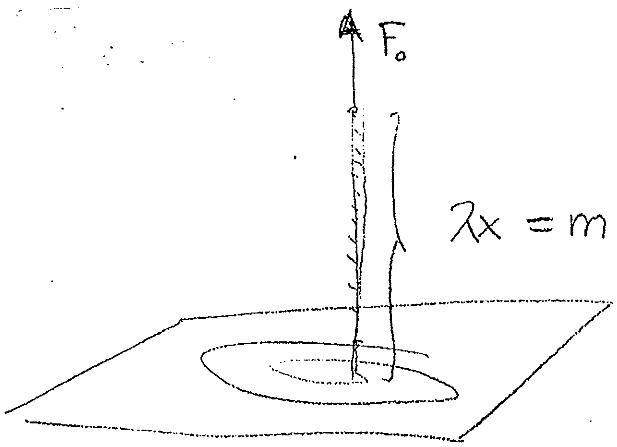
$$\lambda v^2 + mv\frac{dv}{dx} = -mg + F_0$$

where $m = \lambda x$

⋮ ← see next page for details

$$\boxed{v = \sqrt{\frac{F_0}{\lambda} - \frac{2}{3}gx}}$$

PROBLEM 5 continued



$$\frac{dP}{dt} = F_0 - mg$$

$$\frac{d}{dt}(mv) = F_0 - mg$$

$$\frac{dm}{dt}v + m\frac{dv}{dt} = F_0 - mg$$

$$\lambda\frac{dx}{dt}v + m\frac{dx}{dt}\frac{dv}{dx} = F_0 - mg$$

$$\lambda v^2 + \lambda x v\frac{dv}{dx} = F_0 - \lambda x g$$

$$v^2 + xv\frac{dv}{dx} = \frac{F_0}{\lambda} - gx$$

Let $w = v^2$ then $\frac{dw}{dx} = 2v\frac{dv}{dx}$ thus

$$w + \frac{x}{2}\frac{dw}{dx} = \frac{F_0}{\lambda} - gx$$

$$\frac{dw}{dx} + \left(\frac{2}{x}\right)w = \frac{2F_0}{\lambda x} - 2g \qquad I = e^{\int\frac{2dx}{x}} = e^{\ln x^2} = x^2$$

$$x^2\frac{dw}{dx} + 2xw = \frac{2F_0 x}{\lambda} - 2gx^2$$

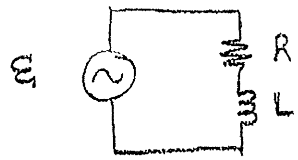
$$\frac{d}{dx}\left(x^2 w\right) = \frac{2F_0 x}{\lambda} - 2gx^2$$

$$x^2 w = \frac{F_0 x^2}{\lambda} - \frac{2}{3}gx^3$$

$$w = F_0/\lambda - \frac{2}{3}gx$$

$$\boxed{v = \sqrt{\frac{F_0}{\lambda} - \frac{2}{3}gx}}$$

Problem 6 [3pts] Suppose the RL-circuit has a voltage source which varies with time according to $\mathcal{E}(t) = V_o \cos(t)$. Find the current as a function of time and the initial current I_o. *hint: this is like an example in the notes, just replace the constant $\mathcal{E}$ with the sinuisoidal source $\mathcal{E}(t) = V_o \cos(t)$*



$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

$$\frac{dI}{dt} + \left(\frac{R}{L}\right) I = \frac{V_o}{L}\cos(t)$$

$$\underbrace{e^{\frac{R}{L}t}\frac{dI}{dt} + \frac{R}{L}e^{\frac{R}{L}t} I}_{} = \frac{V_o}{L} e^{\frac{R}{L}t}\cos t$$

$$\frac{d}{dt}\left[e^{\frac{R}{L}t} I\right] = \frac{V_o}{L} e^{\frac{R}{L}t}\cos t$$

$$e^{\frac{R}{L}t} I = \int \frac{V_o}{L} e^{\frac{R}{L}t}\cos t\, dt + C$$

$$I(t) = Ce^{-\frac{R}{L}t} + e^{-\frac{R}{L}t}\underbrace{\int \frac{V_o}{L} e^{\frac{R}{L}t} \cos t\, dt}_{\text{work out next-next-page}}$$

PROBLEM 6 (continued)

We can integrate $\int \frac{V_0}{L} e^{\frac{R}{L}t} \cos t \, dt$ by integrating by parts twice, I'll illustrate a different method for fun,

$$\cos t = \tfrac{1}{2}(e^{it} + e^{-it})$$ follows from $e^{it} = \cos t + i \sin t$

Note that, letting $\gamma = R/L$

$$e^{\gamma t} \cos t = e^{\gamma t} \tfrac{1}{2}(e^{it} + e^{-it})$$
$$= \tfrac{1}{2}(e^{(\gamma+i)t} + e^{(\gamma-i)t})$$

You can show $\frac{d}{dt}(e^{\lambda t}) = \lambda e^{\lambda t}$ for $\lambda \in \mathbb{C}$.

It follows $\int e^{\lambda t} dt = \frac{1}{\lambda} e^{\lambda t} + C$. Note,

$$\int e^{\gamma t} \cos t \, dt = \tfrac{1}{2} \int \left[e^{(\gamma+i)t} + e^{(\gamma-i)t}\right] dt$$
$$= \tfrac{1}{2}\left[\frac{1}{\gamma+i} e^{(\gamma+i)t} + \frac{1}{\gamma-i} e^{(\gamma-i)t}\right] + C$$
$$= \tfrac{1}{2}\left[\frac{\gamma-i}{\gamma^2+1} e^{(\gamma+i)t} + \frac{\gamma+i}{\gamma^2+1} e^{(\gamma-i)t}\right] + C$$
$$= \left(\frac{\gamma}{\gamma^2+1}\right)\left[\tfrac{1}{2}(e^{it} + e^{-it})\right] e^{\gamma t} + \left(\frac{1}{\gamma^2+1}\right)\left[\tfrac{1}{2i}(e^{it} - e^{-it})\right] e^{\gamma t} + C$$
$$= \frac{\gamma}{\gamma^2+1}(\cos t)\, e^{\gamma t} + \left(\frac{1}{\gamma^2+1}\right)(\sin t)\, e^{\gamma t} + C$$
$$= \frac{1}{(R/L)^2+1}\left[\left(\frac{R}{L}\right)\cos t + \sin t\right] e^{\frac{R}{L}t} + C$$

$$\boxed{I(t) = Ce^{-\frac{R}{L}t} + \left(\frac{1}{\left(\frac{R}{L}\right)^2+1}\right)\left[\left(\frac{R}{L}\right)\cos t + \sin t\right]}$$

Problem 7 [1pts] The Cartesian form of a complex number is $a+ib$. Find the Cartesian form of $\dfrac{2+i}{3-i}$.

$$\left(\frac{2+i}{3-i}\right)\left(\frac{3+i}{3+i}\right) = \frac{6+3i+2i+i^2}{9-i^2} = \frac{5+5i}{10} \quad \text{OR} \quad \boxed{\frac{1}{2}(1+i)}$$

$$\text{aka} \quad \frac{1}{2}+\frac{i}{2} \quad \text{or} \quad \frac{1}{2}+i\left(\frac{1}{2}\right) \quad \text{etc..}$$

Problem 8 [2pts] Differentiate $f(x) = e^{ax}(\cos(bx) + i\sin(bx))$ and show that $f'(x) = (a+ib)f(x)$.

$$\frac{df}{dx} = \frac{d}{dx}\left(e^{ax}\cos bx\right) + i\frac{d}{dx}\left(e^{ax}\sin bx\right) \quad : \text{by def}^{\underline{n}} \quad \frac{d(u+iv)}{dx} = \frac{du}{dx} + i\frac{dv}{dx}$$

$$= ae^{ax}\cos bx - be^{ax}\sin bx + i\left(ae^{ax}\sin bx + be^{ax}\cos bx\right)$$

$$= \left(a\cos bx + ia\sin bx\right)e^{ax} + \left(ib\cos bx - b\sin bx\right)e^{ax}$$

$$= e^{ax}\left[a(\cos bx + i\sin bx) + ib(\cos bx + i\sin bx)\right] \quad i^2 = -1$$

$$= (a+ib)e^{ax}(\cos bx + i\sin bx)$$

$$= (a+ib)f(x).$$

Problem 9 [2pts] If we denote $e^{(a+ib)x} = e^{ax}(\cos(bx) + i\sin(bx))$ then we have shown in the previous problem that $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$ for any $\lambda = a+ib \in \mathbb{C}$. In other words, the exponential function obeys a chain-rule in the complex case much the same as in the real case. And now the task: **Show** $y'' + 4y' + 5y = 0$ **has complex solution** $y = e^{(2+i)x}$.

$$y = e^{(2+i)x} \Rightarrow y' = (2+i)e^{(2+i)x}$$

$$\Rightarrow y'' = (2+i)^2 e^{(2+i)x}$$

$$\text{Thus} \quad y'' + 4y' + 5y = (2+i)^2 e^{(2+i)x} + 4(2+i)e^{(2+i)x} + 5e^{(2+i)x}$$

$$= \left[(2+i)^2 + 4(2+i) + 5\right]e^{(2+i)x}$$

$$= \left[4 + 4i - 1 + 8 + 4i + 5\right]e^{(2+i)x}$$

Actually: $y = e^{(2+i)x}$ is NOT a sol$^{\underline{n}}$! Why? As you can see $y'' + 4y' + 5y \neq 0$. I meant to give $e^{(-2+i)x}$.