MATH 334 : MISSION 3 SOLUTION

§4.2 # 9, 11, 15, 29 (p. 167)
§4.3 # 9, 11, 12, 15, 25 (p. 177)
§4.5 # 3, 7, 11, 14, 17, 25 (p. 192)
§6.1 # 9, 13, 23 (p. 324–325)
§6.2 # 1, 9, 13, 14, 16 (p. 331)

§4.2 #9

$$y'' - y' - 11y = 0$$

$$\lambda^2 - \lambda - 11 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+44}}{2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{45} = \frac{1 \pm 3\sqrt{5}}{2}$$

$$\boxed{y = c_1 \exp\left(\left(\frac{1+3\sqrt{5}}{2}\right)x\right) + c_2 \exp\left(\left(\frac{1-3\sqrt{5}}{2}\right)x\right)}$$

§4.2 #11

$$4w'' + 20w' + 25w = 0$$

$$4\lambda^2 + 20\lambda + 25 = 0$$

$$(2\lambda + 5)(2\lambda + 5) = 0 \Rightarrow \lambda_1 = \lambda_2 = \frac{-5}{2}$$

$$\boxed{y = c_1 e^{\frac{-5x}{2}} + c_2 x e^{\frac{-5x}{2}}}$$

(can set $x = t$ etc...)

§4.2 #15

$$y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = -3$$

$$\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1$$

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

$$y'(x) = -c_1 e^{-x} + c_2(1-x)e^{-x}$$

$$\left.\begin{aligned} y(0) &= 1 = c_1 + c_2 \\ y'(0) &= -3 = -c_1 + c_2 \end{aligned}\right\} \quad \begin{aligned} 2c_2 &= -2 \\ 2c_1 &= 4 \end{aligned}$$

$\therefore$ $c_1 = 2$ & $c_2 = -1$

$$\boxed{y = 2e^{-x} - xe^{-x}}$$

§4.2 # 29 $y_1 = te^{2t}$ and $y_2 = e^{2t}$

determine if y_1 & y_2 are lin. dep on $(0,1)$.

Consider $\dfrac{y_2}{y_1} = \dfrac{e^{2t}}{te^{2t}} = \dfrac{1}{t}$ $\Rightarrow$ y_2 not constant multiple of y_1.

Thus $\{y_1, y_2\}$ are linearly independant.

§4.3 # 9 $y'' - 8y' + 7y = 0$

$$\lambda^2 - 8\lambda + 7 = (\lambda - 1)(\lambda - 7) = 0$$

$$\therefore \lambda_1 = 1, \lambda_2 = 7 \Rightarrow \boxed{y = c_1 e^{t} + c_2 e^{7t}}$$

§4.3 # 11 $z'' + 10z' + 25z = 0$

$$\lambda^2 + 10\lambda + 25 = 0$$

$$(\lambda + 5)^2 = 0 \Rightarrow \boxed{z = c_1 e^{-5x} + c_2 x e^{-5x}}$$

Remark: with notation y', y'' etc.. we could use x or t or whatever for the indep. variable.

§4.3 # 12 $u'' + 7u = 0$

$$\lambda^2 + 7 = 0 \Rightarrow \lambda^2 = -7 \Rightarrow \lambda = \pm i\sqrt{7}$$

$$\therefore \boxed{y = c_1 \cos(t\sqrt{7}) + c_2 \sin(t\sqrt{7})}$$

§4.3 # 15 $y'' + 10y' + 41y = 0$

$$0 = \lambda^2 + 10\lambda + 41 = (\lambda + 5)^2 + 16 \Rightarrow \underline{\lambda = -5 \pm 4i}.$$

$$\boxed{y = c_1 e^{-5x}\cos(4x) + c_2 e^{-5x}\sin(4x)}$$

§4.3 #25 $y'' - 2y' + 2y = 0,\quad y(\pi) = e^{\pi},\ y'(\pi) = 0$

$$\lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = 0$$

$$\lambda = 1 \pm i$$

$$y = c_1 e^x \cos(x) + c_2 e^x \sin(x)$$

$$y(\pi) = c_1 e^{\pi} \underbrace{\cos\pi}_{-1} + c_2(0) = e^{\pi} \Rightarrow \underline{c_1 = -1}.$$

$$y'(x) = c_1(\cos x - \sin x)e^x + c_2(\sin x + \cos x)e^x$$

$$y'(\pi) = -c_1 e^{\pi} + c_2 \cos\pi\, e^{\pi}$$

$$y'(\pi) = -c_1 e^{\pi} - c_2 e^{\pi} = 0 \Rightarrow c_1 + c_2 = 0$$

or $c_2 = -c_1 = 1.$

$$\therefore \quad y = -e^x \cos(x) + e^x \sin x$$

$$\boxed{y = e^x(\sin x - \cos x)}$$

§4.5 #3 $y'' - y = t$ and $y_p(t) = -t$ ← given

find the general sol^{n}

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \therefore y_h = c_1 e^{t} + c_2 e^{-t}$$

Thus, $\boxed{y = \underbrace{c_1 e^{t} + c_2 e^{-t}}_{\text{homog. sol}^{n}} \underbrace{- t}_{\text{particular sol}^{n}}}$

$\boxed{\S 4.5\#11}$ $y'' - 6y' - 4y = 4\sin 3t - t^2 e^{3t} + \frac{1}{t}$

can we solve via undet. coeff?

No. The $\frac{1}{t}$ term cannot be covered by undet. coefficients ($\frac{1}{t}$ not annihilated by A in $\mathbb{R}[\frac{d}{dt}]$ ...)

$\boxed{\S 4.5\#14}$ $y'' - 2y' + 3y = \cosh t$

YES. Undetermined coeff. is possible here, $\cosh t = \frac{1}{2}(e^t + e^{-t})$ so $y_p = Ae^t + Be^{-t}$ will do nicely.

$\boxed{\S 4.5\#17}$ Find general sol$^{\underline{n}}$ to $y'' - y = 1 - 11t$

Let $y_p = A + Bt \Rightarrow y_p'' = 0$ and

$y_p'' - y_p = 1 - 11t \quad \therefore \quad -A - Bt = 1 - 11t$ $\quad A = -1$, $B = 11$

Also, $\lambda^2 - 1 = 0 \quad \therefore \quad \lambda = \pm 1$ so $y_h = c_1 e^t + c_2 e^{-t}$

and we find $\boxed{y = c_1 e^t + c_2 e^{-t} + 11t - 1}$

$\boxed{\S 4.5\#25}$ $z'' + z = 2e^{-x}$ with $z(0) = 0$, $z'(0) = 0$

$z_p = Ae^{-x} \Rightarrow z_p'' = Ae^{-x}$ so $z_p'' + z_p = 2e^{-x}$

yields $2Ae^{-x} = 2e^{-x} \quad \therefore \quad A = 1$ & $\underline{z_p = e^{-x}}$.

Since $\lambda^2 + 1 = 0 \Rightarrow z_h = c_1 \cos x + c_2 \sin x$

and $z = c_1 \cos x + c_2 \sin x + e^{-x}$

$z(0) = c_1 + 1 = 0$ & $z'(0) = c_2 - 1 = 0$ $\quad \therefore \boxed{z = \sin x - \cos x + e^{-x}}$

$c_1 = -1$ $\qquad c_2 = 1$

§6.1 #9 Determine LI of $\{\sin^2 x, \cos^2 x, 1\}$ on $\mathbb{R}$.

well, $1 = \cos^2 x + \sin^2 x \quad \forall x \in \mathbb{R}$
so these are linearly dependent on $(-\infty, \infty)$.

§6.1 # 13 Consider $\{x, x^2, x^3, x^4\}$ on $(-\infty, \infty)$

If $c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 = 0$ for $x \in \mathbb{R}$
then diff. w.r.t. x to get $\underline{c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 = 0}$ 1.
set $x = 0$ in (1.) to obtain $c_1 = 0$. Next
$\frac{d}{dx}$ (1.) gives $\underline{2c_2 + 6c_3 x + 12 c_4 x^2 = 0}$ (2.) Then
set $x=0$ in (2.) to get $2c_2 = 0 \therefore \underline{c_2 = 0}$.
$\frac{d}{dx}$ (2.) yields $\underline{6c_3 + 24 c_4 x = 0}$ (3.). Set $x=0$
in (3.) to obtain $6c_3 = 0 \therefore \underline{c_3 = 0}$. Finally,
$\frac{d}{dx}$ (3.) gives $24 c_4 = 0 \Rightarrow \underline{c_4 = 0}$. Hence
$c_1 = c_2 = c_3 = c_4 = 0$ which demonstrates
the LI of $\{x, x^2, x^3, x^4\}$.

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Remark: you would have been within
your rights to say $c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 = 0$
implies $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$.
I'm merely showing how to derive this
via calculus. (to the grader, they don't have to do what I do..)

$\boxed{§6.1 \# 23}$

$$L[y] = y''' + y' + xy \qquad y_1 = \sin x \quad y_2 = x$$

Calculate $L[y_1]$ and $L[y_2]$

then solve

(a.) $L[y] = 2x\sin x - x^2 - 1$

(6.) $L[y] = 4x^2 + 4 - 6x\sin x$

$$L[y_1] = (\sin(x))''' + (\sin x)' + x\sin x$$
$$= -\cos x + \cos x + x\sin x$$
$$= \underline{x\sin x}.$$

$$L[y_2] = x''' + x' + x(x) = \underline{1 + x^2}.$$

(a.) $L[2y_1 - y_2] = 2L[y_1] - L[y_2] = 2x\sin x - x^2 - 1$

$\therefore \boxed{y = 2\sin x - x}$

(6.) $L[4y_2 - 6y_1] = L[4x - 6\sin(x)] = 4x^2 + 4 - 6x\sin x$

so $\boxed{y = 4x - 6\sin(x)}$ is the sol$^{\underline{n}}$ we want.

$\boxed{§6.2 \# 1}$ $y''' + 2y'' - 8y' = 0$

$$\lambda^3 + 2\lambda^2 - 8\lambda = \lambda(\lambda^2 + 2\lambda - 8)$$
$$= \lambda(\lambda+4)(\lambda-2) \rightarrow \lambda_1 = 0, \ \lambda_2 = -4, \ \lambda_3 = 2$$

$$\therefore \boxed{y = c_1 + c_2 e^{-4t} + c_2 e^{2t}}$$

§6.2 #9 $u''' - 9u'' + 27u' - 27u = 0$

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = 0$$

$$(\lambda - 3)^3 = 0$$

$$\boxed{u = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}}$$

§6.2 #13 $y^{(4)} + 4y'' + 4y = 0$

$$\lambda^4 + 4\lambda^2 + 4 = 0$$

$$(\lambda^2 + 2)^2 = 0 \quad \therefore \quad \lambda = \pm i\sqrt{2} \ \underline{\text{twice}}$$

$$\boxed{y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + c_3 x \cos(x\sqrt{2}) + c_4 x \sin(x\sqrt{2})}$$

§6.2 #14 $y^{(4)} + 2y''' + 10y'' + 18y' + 9y = 0$

$$\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = 0 \quad (*)$$

<u>Hint</u>: $y = \sin 3x$ is sol^{n} $\Rightarrow$ $(\lambda^2 + 9)$ factor in

$$\begin{array}{r|l}
 & \lambda^2 + 2\lambda + 1 \\
\lambda^2 + 9 & \lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 \\
 & \lambda^4 \qquad\qquad + 9\lambda^2 \\
\hline
 & \qquad 2\lambda^3 + \lambda^2 + 18\lambda + 9 \\
 & \qquad 2\lambda^3 + \qquad 18\lambda \\
\hline
 & \qquad\qquad \lambda^2 + 9 \\
 & \qquad\qquad \lambda^2 + 9 \\
\hline
 & \qquad\qquad\qquad 0.
\end{array}$$

Hence, $(*) \rightarrow (\lambda^2 + 9)(\lambda^2 + 2\lambda + 1) = 0$

continued

§6.2 #14 continued

$$(\lambda^2+9)(\lambda^2+2\lambda+1) = (\lambda^2+9)(\lambda+1)^2 = 0 \quad \begin{array}{l} \nearrow \lambda = \pm 3i \\ \searrow \lambda = -1 \text{ twice} \end{array}$$

$$\therefore \boxed{y = c_1 \cos 3x + c_2 \sin 3x + c_3 e^{-x} + c_4 x e^{-x}}$$

§6.2 #16

$$(D+1)^2 (D-6)^3 (D+5)(D^2+1)(D^2+4)[y] = 0$$

$$\underbrace{\lambda_1 = -1}_{\text{twice}}, \quad \underbrace{\lambda_2 = 6}_{\text{thrice}} \quad \lambda_3 = -5, \ \lambda_4 = \pm i, \ \lambda_5 = \pm 2i$$

Oh, fine $\lambda_1 = \lambda_2 = \lambda_3 = -1$, $\lambda_4 = \lambda_5 = 6$, $\lambda_6 = -5$
$\lambda_{7,8} = \pm i$ and $\lambda_{9,10} = \pm 2i$. Label them however you like. We all should agree,

$$\boxed{\begin{aligned} y = {} & c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{6x} + c_4 x e^{6x} + c_5 x^2 e^{6x} \\ & + c_6 e^{-5x} + c_7 \cos x + c_8 \sin x \\ & + c_9 \cos 2x + c_{10} \sin 2x \end{aligned}}$$

Remark: you can impress your friends with this if they've not had DEqn's. It looks way harder than it is ☺.