MATH 334 : MISSION 4 SOLUTION

§4.3 # 41, 43
§4.5 # 43, 45 ← bonus.
§ 4.6 # 1, 3, 5, 14, 18
§ 4.7 # 17
§ 4.8 # 3, 5, 14
§ 4.9 # 7
§ 6.3 # 13, 15, 19, 23, 25, 29

§4.3 #41 $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-6y=0$. Solve it, $x>0$

$$y=x^R \hookrightarrow R(R-1)+2R-6=0$$
$$R^2+R-6=0$$
$$(R+3)(R-2)=0 \quad \therefore \quad R=-3,2$$
$$\boxed{y=c_1x^{-3}+c_2x^2=\frac{c_1}{x^3}+c_2x^2}$$

§4.3 #43

$$x^2y''+9xy'+17y=0$$
$$R(R-1)+9R+17=0$$
$$R^2+8R+17=0$$
$$(R+4)^2+1=0 \quad \therefore \quad R=-4\pm i.$$

$$x^{-4+i}=x^{-4}x^{i}$$
$$=x^{-4}(\cos(\ln x)+i\sin(\ln x))$$

$$\boxed{y=c_1x^{-4}\cos(\ln(x))+c_2x^{-4}\sin(\ln(x))}$$

Remark: I'm ignoring text instruction here ☺ of course #38 also a good way to do these.

$\boxed{\S 4.5 \# 43}$ spring/mass ext. force $g(t) = 5\sin t$

$m = 1, \quad k = 3, \quad b = 4$

$y(0) = 1/2, \quad y'(0) = 0$ find $y(t)$ {aha: find the equation of motion}

$$\underbrace{y'' + 4y' + 3y}_{} = 5\sin t$$

$$\lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3) = 0 \therefore y_h = c_1 e^{-t} + c_2 e^{-3t}$$

$$y_p = A\sin t + B\cos t$$

$$y_p'' + 4y_p' + 3y_p = -A\sin t - B\cos t + 4(A\cos t - B\sin t) + 3(A\sin t + B\cos t)$$

$$= 5\sin t$$

oh, so,

$$\sin t\,[-A + 3A - 4B] + \cos t\,[-B + 4A + 3B] = 5\sin t$$

$$2A - 4B = 5$$
$$4A + 2B = 0 \Rightarrow \underline{B = -2A}.$$

$$2A - 4(-2A) = 5$$
$$10A = 5 \therefore \underline{A = 1/2}.$$

Hence, $B = 2A = -1$.

$$y(t) = c_1 e^{-t} + c_2 e^{-3t} + \frac{1}{2}\sin t - \cos t$$

$$y(0) = c_1 + c_2 - 1 = 1/2$$
$$y'(0) = -c_1 - 3c_2 + \frac{1}{2} = 0$$

$$\rightarrow -2c_2 - \frac{1}{2} = \frac{1}{2}$$
$$-2c_2 = 1 \therefore \underline{c_2 = \frac{-1}{2}}.$$
$$c_1 = \frac{3}{2} - c_2 = \frac{4}{2} = 2.$$

§4.5 #43 continued

we find $\boxed{y = 2e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}\sin t - \cos t}$

§4.5 #45 $y(t) = 0$ for $t \leq -L/2V$

$$my'' + ky = \begin{cases} \cos\left(\frac{\pi V t}{L}\right) & \text{for } |t| < L/2V \\ 0 & \text{for } L/2V \leq t \end{cases}$$

(a.) $m = k = 1$, $L = \pi$ for convenience
solve and show $y(t) = A\sin t$ where $A = A(V)$

$$y'' + y = \cos\left(\frac{\pi V t}{L}\right) = \cos(Vt) \qquad (L = \pi)$$

$$y_p = A\cos(Vt) + B\sin(Vt) \quad \text{if } V \neq 1.$$

$$y_p'' = -V^2 y_p$$

$$y_p'' + y_p = (1 - V^2)y_p = \cos(Vt)$$

$$(1 - V^2)A\cos(Vt) + B(1 - V^2)\sin(Vt) = \cos(Vt)$$

$$A(1 - V^2) = 1 \quad \& \quad B(1 - V^2) = 0$$

Thus, $y = c_1\cos t + c_2\sin t + \frac{1}{1 - V^2}\cos(Vt)$ $\quad L = \pi$

$$0 = y(-L/2V) = c_1\cos\left(\frac{-L}{2V}\right) + c_2\sin\left(\frac{-L}{2V}\right) + \cancel{\frac{1}{1-V^2}\cos\left(-V\left(\frac{-L}{2V}\right)\right)}$$

$$0 = y'(-L/2V) = -c_1\sin\left(\frac{-L}{2V}\right) + c_2\cos\left(\frac{-L}{2V}\right) - \frac{V}{1 - V^2}\sin\left(\frac{\pi}{2}\right) \quad 0$$

§4.5 #45 continued

$$\underline{c_1 = \tan(L/2v)\, c_2}.$$

$$c_2 \cos\left(\frac{L}{2v}\right) + c_1 \sin\left(\frac{L}{2v}\right) = \frac{v}{1-v^2}$$

$$c_2 \cos\left(\frac{L}{2v}\right) + c_2 \tan\left(\frac{L}{2v}\right)\sin\left(\frac{L}{2v}\right) = \frac{v}{1-v^2}$$

$$c_2\left(\cos^2(L/2v) + \sin^2(L/2v)\right) = \frac{v}{1-v^2}\cos\left(\frac{L}{2v}\right)$$

$$\underline{c_2 = \frac{v}{1-v^2}\cos\left(\frac{L}{2v}\right)}.$$

$$\boxed{y = \frac{v}{1-v^2}\cos\left(\frac{L}{2v}\right)\left[\tan(L/2v)\cos t + \sin t\right] + \frac{\cos(vt)}{1-v^2}}$$

for $|t| < L/2v = \pi/2v$

$$y = \frac{v}{1-v^2}\left[\sin\left(\frac{\pi}{2v}\right)\cos t + \cos\left(\frac{\pi}{2v}\right)\sin t\right] + \frac{\cos(vt)}{1-v^2}$$

$$y = \frac{v}{1-v^2}\sin\left(t + \frac{\pi}{2v}\right) + \frac{\cos(vt)}{1-v^2}$$

$$y\left(\frac{L}{2v}\right) = y\left(\frac{\pi}{2v}\right) = \frac{v}{1-v^2}\sin\left(\frac{\pi}{v}\right) + \cancel{\frac{\cos(\pi/2)}{1-v^2}} \xrightarrow{} 0$$

$$y(t) = \bar{c}_1\cos t + \bar{c}_2 \sin t \quad \text{for } t \geq L/2v = \pi/2v$$

§4.5 ## 45 continued

$$y\left(\frac{\pi}{2v}\right) = \bar{C}_1 \cos\left(\frac{\pi}{2v}\right) + \bar{C}_2 \sin\left(\frac{\pi}{2v}\right) = \frac{v}{1-v^2}\sin\left(\frac{\pi}{v}\right) \quad (1)$$

$$y'\left(\frac{L}{2v}\right) = \frac{v}{1-v^2}\cos\left(\frac{\pi}{v}\right) - \frac{v}{1-v^2}\sin\left(\frac{\pi}{2}\right)$$

$$y'(L/2v) = \frac{v}{1-v^2}\left(\cos\frac{\pi}{v} - 1\right)$$

$$\cancel{y'\left(\frac{\pi}{2v}\right) = -\frac{\bar{C}_1\pi}{2v}\sin\left(\frac{\pi}{}\right.}$$

$$y'(t) = -\bar{C}_1 \sin t + \bar{C}_2 \cos t$$

$$y'(L/2v) = \frac{v}{1-v^2}\left(\cos\frac{\pi}{v} - 1\right) = -\bar{C}_1 \sin\left(\frac{\pi}{2v}\right) + \bar{C}_2 \cos\left(\frac{\pi}{2v}\right) \quad (2)$$

$$\sin\frac{\pi}{v} = 2\sin\left(\frac{\pi}{2v}\right)\cos\left(\frac{\pi}{2v}\right) \longrightarrow \bar{C}_1 = \sin\left(\frac{\pi}{2v}\right)\cdot\left(\frac{v}{1-v^2}\right)$$

$$\longrightarrow \bar{C}_2 = \cos\left(\frac{\pi}{2v}\right)\cdot\left(\frac{v}{1-v^2}\right)$$

$$\Big[\underbrace{\cos^2\left(\frac{\pi}{2v}\right) - \sin^2\left(\frac{\pi}{2v}\right)}_{2\cos^2\left(\frac{\pi}{2v}\right) - 1}\Big]\left(\frac{v}{1-v^2}\right) = \frac{v}{1-v^2}\left[\cos\frac{\pi}{v} - 1\right]$$

§4.6 #1 $\quad \underbrace{y'' + 4y} = \tan 2t$

$$\lambda^2 + 4 = 0 \quad \therefore$$

$$y_1 = \cos 2t, \quad y_2 = \sin 2t$$

$$y_1 y_2' - y_2 y_1' = 2\cos^2(2t) + 2\sin^2(2t) = 2.$$

$$V_1 = \int \frac{-\sin(2t)\tan(2t)}{2}\,dt = \frac{-1}{2}\int \frac{\sin^2(2t)}{\cos(2t)}\,dt$$

$$V_1 = \frac{-1}{2}\int \left(\sec(2t) - \cos(2t)\right)dt$$

$$\underline{V_1 = \frac{-1}{2}\left(\frac{1}{2}\ln|\sec 2t + \tan 2t| - \frac{1}{2}\sin(2t)\right)}.$$

$$V_2 = \int \frac{\cos 2t \tan 2t}{2}\,dt = \int \frac{1}{2}\sin(2t)\,dt = \frac{-1}{4}\cos(2t)$$

cancel in sol$^{\underline{n}}$

$$\boxed{y = c_1\cos(2t) + c_2\sin(2t) - \frac{1}{4}\cos(2t)\ln|\sec 2t + \tan 2t|}$$

§4.6#3 $2x'' - 2x' - 4x = 2e^{3t}$

$$0 = 2\lambda^2 - 2\lambda - 4 = (2\lambda - 4)(\lambda + 1) \Rightarrow \lambda_1 = 2,\ \lambda_2 = -1$$

$$\left.\begin{array}{l} x_1 = e^{2t} \\ x_2 = e^{-t} \end{array}\right\} \quad x_1 x_2' - x_2 x_1' = e^{2t}(-e^{-t}) - e^{-t}(2e^{2t}) = \underline{-3e^{t} = W}.$$

We suppose $y_p = x_1 v_1 + x_2 v_2$ and this leads us to calculate

$$v_1 = \int \frac{-g x_2}{aW}\,dt = \frac{1}{2}\int \frac{2e^{3t}e^{-t}}{3e^{t}}\,dt = \frac{1}{3}\int e^{t}\,dt = \underline{\frac{1}{3}e^{t} = v_1}$$

$$v_2 = \int \frac{g x_1}{aW}\,dt = \frac{1}{2}\int \frac{(2e^{3t})(e^{2t})}{-3e^{t}}\,dt = -\frac{1}{3}\int e^{4t}\,dt = \underline{\frac{-1}{12}e^{4t} = v_2}$$

$$\therefore\ x(t) = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{3}e^{2t}e^{t} - \frac{1}{12}e^{-t}e^{4t}$$

$$\boxed{x(t) = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{4}e^{3t}}$$

Remark: I followed the text's instructions here. For the record, undet. coeff is way easier.

$$x_p = Ae^{3t}$$
$$x_p' = 3Ae^{3t} = 3x_p$$
$$x_p'' = 9Ae^{3t} = 9x_p$$

$$2x_p'' - 2x_p' - 4x_p = (18 - 6 - 4)x_p = 2e^{3t}$$

$$\therefore\ x_p = \frac{2}{8}e^{3t} = \frac{1}{4}e^{3t}.$$

§4.6 #5) solve via variation of parameters.

$$y'' - 2y' + y = \frac{1}{t}e^t$$

$$\lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0 \Rightarrow y_1 = e^t, \; y_2 = te^t$$

$$W = y_1 y_2' - y_2 y_1' = e^t(1+t)e^t - te^te^t = \underline{e^{2t} = W}.$$

$$V_1 = \int \frac{-\frac{1}{t}e^t te^t}{e^{2t}}\, dt = -\int dt = \underline{-t = V_1}.$$

$$V_2 = \int \frac{\frac{1}{t}e^t(e^t)}{e^{2t}}\, dt = \int \frac{dt}{t} = \underline{\ln|t| = V_2}.$$

$$\therefore \quad y = c_1 e^t + c_2 te^t \overbrace{- te^t + te^t \ln|t|}^{y_1V_1 + y_2V_2}$$

(can absorb y_1V_1 into c_2 term here)

or $$\boxed{y = c_1e^t + c_2te^t + te^t \ln|t|}$$

§4.6 #14) $y'' + y = \sec^3\theta$ ← scary.

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_1 = \cos\theta, \; y_2 = \sin\theta \;\therefore\; \underline{W = y_1y_2' - y_2y_1' = \cos^2\theta + \sin^2\theta = 1}.$$

$$V_1 = \int \frac{-\sec^3\theta \sin x}{1}\, dx = -\int \frac{\sin x\, dx}{\cos^3\theta} = \int \frac{du}{u^3} = \frac{-1}{2u^2} = \underline{\frac{-1}{2}\sec^2\theta = V_1}.$$

$$V_2 = \int \sec^3\theta \cos x\, dx = \int \sec^2\theta\, d\theta = \underline{\tan\theta = V_2}$$

So, actually not so scary ☺,

$$y = c_1\cos\theta + c_2\sin\theta - \frac{1}{2}\cos\theta\sec^2\theta + \tan\theta\sin\theta$$

$$\boxed{y = c_1\cos\theta + c_2\sin\theta - \frac{1}{2}\sec\theta + \sin\theta\tan\theta}$$

§4.6 #18 / Solve via variation of parameters,

$$y'' - 6y' + 9y = \frac{1}{t^3}e^{3t}$$

$$\lambda^2 - 6\lambda + 9 = (\lambda-3)^2 \quad \therefore \quad y_1 = e^{3t}, \; y_2 = te^{3t}$$

$$W = y_1 y_2' - y_2 y_1' = e^{3t}(1+3t)e^{3t} - te^{3t}(3e^{3t}) = \underline{e^{6t} = W}.$$

$$V_1 = \int \frac{\frac{-1}{t^3}e^{3t}e^{3t}}{e^{6t}}\,dt = \int \frac{-dt}{t^3} = \underline{\frac{1}{2t^2} = V_1}.$$

$$V_2 = \int \frac{\frac{1}{t^3}e^{3t}te^{3t}}{e^{6t}}\,dt = \int \frac{dt}{t^2} = \underline{\frac{-1}{t} = V_2}$$

$$\therefore \quad y = c_1 e^{3t} + c_2 te^{3t} + \frac{1}{2t^2}e^{3t} - \frac{1}{t}te^{3t}$$

$$\boxed{y = c_1 e^{3t} + c_2 te^{3t} + \frac{1}{2t^2}e^{3t}}$$

§4.7 #17 ARMAGEDDON!

$$m\frac{dv}{dt} = -\frac{GmM}{r^2}$$

(find time for earth to fall into sun, ignoring orbital angular momentum...)

$$\frac{d}{dt}\Big(\underbrace{\frac{1}{2}mv^2 - \frac{GmM}{r}}\Big) = 0$$

$E(r,v) =$ constant so $\frac{E}{m}$ also constant.

$$\frac{E_0}{m} = \frac{1}{2}v^2 - \frac{GM}{r} \quad , \quad \frac{E_0}{m} = \frac{1}{2}(0)^2 - \frac{GM}{a}$$

$$v^2 = \frac{2E_0}{m} + \frac{2GM}{r}$$

because r is decreasing.

$$v = -\sqrt{\frac{2E_0}{m} + \frac{2GM}{r}} = -\sqrt{\frac{-2GM}{a} + \frac{2GM}{r}}$$

$$v = -\underbrace{\sqrt{2GM}}_{\alpha}\sqrt{\frac{1}{r} - \frac{1}{a}} = -\alpha\sqrt{\frac{1}{r} - \frac{1}{a}}$$

$$\frac{dr}{dt} = -\alpha\sqrt{\frac{1}{r} - \frac{1}{a}}$$

$$\underbrace{\int_a^0 \frac{dr}{\sqrt{\frac{1}{r} - \frac{1}{a}}}}_{\text{How ?}} = -\int_0^T \alpha\, dt = -\alpha T$$

$$\int \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{a}}} = \int \frac{dx}{\sqrt{\frac{a-x}{xa}}} = \int \frac{\sqrt{xa}\, dx}{\sqrt{a-x}}$$

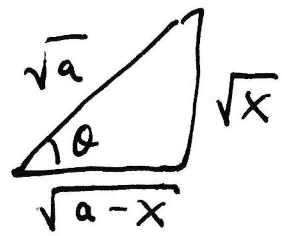
Let $\sqrt{x} = \sqrt{a}\sin\theta$ then $x = a\sin^2\theta$

and $a - x = a - a\sin^2\theta = a(1-\sin^2\theta) = a\cos^2\theta$.

and $\sqrt{xa} = \sqrt{a}\sqrt{x} = \sqrt{a}\sqrt{a}\sin\theta = a\sin\theta$

$dx = d(a\sin^2\theta) = 2a\sin\theta\cos\theta\, d\theta$. Hence,

$$\int \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{a}}} = \int \frac{(a\sin\theta)(2a\sin\theta\cos\theta)\,d\theta}{\sqrt{a\cos^2\theta}}$$



$$= \frac{a^2}{\sqrt{a}} \int \sin^2\theta\, d\theta$$

$$= a\sqrt{a} \int \frac{1}{2}(1 - \cos(2\theta))\,d\theta$$

$$= \frac{a\sqrt{a}}{2}\left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right) + C$$

$$= \frac{a\sqrt{a}}{2}\left(\frac{1}{2}\sin^{-1}\left(\sqrt{\frac{x}{a}}\right) - \frac{1}{2}\sin\theta\cos\theta\right) + C$$

$$= \underbrace{\frac{a\sqrt{a}}{4}\sin^{-1}\left(\sqrt{\frac{x}{a}}\right) - \frac{a\sqrt{a}}{4}\sqrt{\frac{x}{a}}\,\frac{\sqrt{a-x}}{\sqrt{a}} + C}_{F(x)}.$$

$$\int_0^a \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{a}}} = \lim_{\varepsilon \to 0^+}\left(F(a/2) - F(\varepsilon)\right) + \lim_{b \to a^-}\left(F(b) - F(a/2)\right)$$

$$= \lim_{b \to a^-}\left(\frac{a\sqrt{a}}{4}\underbrace{\sin^{-1}\sqrt{\frac{b}{a}}}_{\to \pi/2} - \underbrace{\sqrt{a}\sqrt{b}\sqrt{a-b}}_{\to 0}\right) - \lim_{\varepsilon \to 0^+} F(\varepsilon)$$

$$\lim_{\varepsilon \to 0^+} \left(F(\varepsilon) \right) = \lim_{\varepsilon \to 0^+} \left(\frac{a\sqrt{a}}{4} \underbrace{\sin^{-1}\left(\sqrt{\frac{\varepsilon}{a}}\right)}_{\sin^{-1}(0)=0} - \sqrt{a}\,\underbrace{\sqrt{\varepsilon}}_{0}\,\overbrace{\sqrt{a-\varepsilon}}^{\sqrt{a}} \right)$$

$$= -\sqrt{a}\,(0)\,\sqrt{a}$$

$$= 0.$$

$$\therefore \int_0^a \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{a}}} = \frac{a\sqrt{a}}{4} \cdot \frac{\pi}{2} = \frac{a\sqrt{a}\,\pi}{8}$$

Thus,

$$\int_a^0 \frac{dr}{\sqrt{\frac{1}{r} - \frac{1}{a}}} = -\alpha T$$

$$\Rightarrow \quad \frac{-a\sqrt{a}\,\pi}{8} = -\alpha T \quad \therefore \quad T = \frac{a\sqrt{a}\,\pi}{8\alpha}$$

But, $\alpha = \sqrt{2GM}$ hence

$$T = \frac{a\sqrt{a}\,\pi}{8\sqrt{2GM}} \quad \text{vs.} \quad T_{yr} = 2\pi\sqrt{\frac{a^3}{GM}}$$

$$\frac{T}{T_{yr}} = \frac{\cancel{a\sqrt{a}}\,\pi}{8\sqrt{2\cancel{GM}}} \cdot \frac{\cancel{\sqrt{GM}}}{2\pi\cancel{\sqrt{a^3}}} = \frac{1}{16\sqrt{2}} \Rightarrow \boxed{\frac{T}{T_{yr}} = \frac{1}{16\sqrt{2}}}$$

–(I disagree with text's key)–

§4.8 #3) $y'' + by' + 16y = 0$
$y(0) = 1$, $y'(0) = 0$. Find $y(t)$ for
$b = 0, \quad , \quad 6, 8, \; 10$ and sketch graphs

See back of text for nice pictures

$b=0$) $y'' + 16y = 0 \Rightarrow \lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i \Rightarrow y = c_1 \cos 4t + c_2 \sin 4t$

$y'(0) = 4c_2 = 0$ & $y(0) = c_1 = 1 \quad \therefore \quad \boxed{y_0 = \cos 4t}$

//

$b=6$) $y'' + 6y' + 16y = 0$

$\lambda^2 + 6\lambda + 16 = (\lambda+3)^2 + 7 = 0$

$\lambda = -3 \pm i\sqrt{7}$

$y(t) = c_1 e^{-3t} \cos(\sqrt{7}t) + c_2 e^{-3t} \sin(\sqrt{7}t)$

$y(0) = c_1 e^0 \cos(0) = 1 \quad \therefore \quad \underline{c_1 = 1}.$

$y'(0) = -3c_1 e^0 \cos(0) - 3c_2 e^0 \sin(0) + \sqrt{7}\, c_2 e^0 \cos(0)$

$0 = -3 + \sqrt{7}\, c_2 \quad \therefore \quad \underline{c_2 = 3/\sqrt{7}}.$

Remark: the point of this problem was to see the effect of increasing damping as we go from SHM to over damped

$$y = e^{-3t}\left(\cos(\sqrt{7}t) + \frac{3}{\sqrt{7}} \sin(\sqrt{7}t)\right)$$

$$\boxed{y_6 = \frac{4e^{-3t}}{\sqrt{7}} \sin(\sqrt{7}t + \phi)}, \quad \phi = \tan^{-1}\left(\frac{\sqrt{7}}{3}\right) \approx 0.723.$$

$b=8$) $y'' + 8y' + 16y = 0$

$\lambda^2 + 8\lambda + 16 = (\lambda+4)^2 = 0 \quad \therefore \quad y = c_1 e^{-4t} + c_2 t e^{-4t}$

$y(0) = c_1 = 1$, $y'(0) = c_1(-4) + c_2(1 - \cancel{4}) = 0 \quad \therefore \quad \underline{c_2 = 4}.$

$$\boxed{y_8 = e^{-4t}(1 + 4t)}$$

$b=10$) $y'' + 10y' + 16y = 0$

$\underbrace{\lambda^2 + 10\lambda + 16}_{(\lambda+2)(\lambda+8) = 0} = (\lambda+5)^2 - 9$

$\therefore \quad y = c_1 e^{-2t} + c_2 e^{-8t}$

$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= -2c_1 - 8c_2 = 0 \end{aligned}\right\} *$

Solving $*$ gives $c_1 = \frac{4}{3}$, $c_2 = \frac{-1}{3} \quad \therefore \quad \boxed{y_{10} = \frac{4}{3} e^{-2t} - \frac{1}{3} e^{-8t}}$

§4.8 #5

$$y'' + 10y' + ky = 0 \qquad \begin{array}{l} k=20 \rightarrow \text{over damped} \\ k=25 \rightarrow \text{critically damped} \\ k=30 \rightarrow \text{under damped} \end{array}$$

$y(0) = 1$
$y'(0) = 0$

Intuitively, as k gets further from 10 we get closer & closer to $y'' + ky \approx 0$ (SHM) the progression of motion is opposite of #3.

-(the sol$^{=}$'s are in text, I make this comment to explain why I assigned this)-

§4.8#14 for an underdamped system, verify as $b \to 0$ the damping factor $\to A$ and the quasifreq. approaches the nat. freq $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Fixing m & t we observe as $b \to 0$

$$A\exp\left(\frac{-bt}{2m}\right) \longrightarrow A\exp(0) = \boxed{A}.$$

Also, the quasifrequency, for fixed k, m as $b \to 0$ we have:

$$\frac{\sqrt{4mk - b^2}}{4m\pi} = \left(\sqrt{\frac{4mk - b^2}{m^2}}\right)\frac{1}{4\pi}$$

$\frac{1}{2} = \frac{1}{\sqrt{4}}$ bring the 2 inside the $\sqrt{}$.

$$= \frac{1}{2\pi}\sqrt{\frac{4mk - b^2}{4m^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{k}{m} - \underbrace{\frac{b^2}{4m^2}}_{\to 0}}$$

$$= \boxed{\frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

§4.9 #7 the sol$^{\underline{n}}$ is in back of text. I'll just say a word or two.

1.) the derivation of y_p is almost independent of the under/over damping condition. The main point is that no overlap is possible as the forcing term is a pure sinusoidal fnct. and our homogeneous sol$^{\underline{n}}$ cannot have that form in the presence of any damping.

2.) the homogeneous part of the sol$^{\underline{n}}$ is naturally different in the under/over/critical cases.

§6.3 #13 e^{-7x} is annihilated by $\boxed{A = D+7}$

§6.3 #15 $e^{2x} - 6e^{x}$ is annihilated by $A = \underbrace{(D-2)}_{\text{kills } e^{2x}}\underbrace{(D-1)}_{\text{kills } e^{x}}$

§6.3 #19 $\underbrace{xe^{-2x}}_{\lambda = -2 \text{ twice}} + \underbrace{xe^{-5x}\sin(3x)}_{\lambda = -5 \pm 3i \text{ twice}}$ $\Rightarrow$ $A = (D+2)^2\left((D+5)^2 + 9\right)^2$ ← also good.

$$\boxed{A = (D+2)^2(D^2 + 10D + 34)^2}$$

§6.3 #23 $y'' - 5y' + 6y = e^{3x} - x^2$

$D^3(D-3)(D^2 - 5D + 6)[y] = (D-3)D^3[e^{3x} - x^2] = 0$

$D^3(D-3)^2(D-2)[y] = 0$

$y = c_1 + c_2x + c_3x^2 + c_4e^{3x} + c_5xe^{3x} + c_6e^{2x}$

homogeneous sol$^{\underline{n}}$, every else has to be y_p

$$\Rightarrow \boxed{y_p = A + Bx + Cx^2 + Dxe^{3x}}$$

§6.3 #25

$$y'' - 6y' + 9y = \sin(2x) + x$$

$$(D^2 - 6D + 9)[y] = \underbrace{\sin(2x) + x}_{\text{annihilated via } (D^2+4)(D^2)}$$

Hence,

$$D^2(D^2+4)(D-3)^2[y] = 0$$

$$y = \underbrace{c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x}_{y_p} + \underbrace{c_5 e^{3x} + c_6 x e^{3x}}_{y_h}$$

$$\boxed{y_p = A + Bx + C\cos(2x) + D\sin(2x)}$$ (no overlap here)

§6.3 #29

$$z''' - 2z'' + z' = x - e^x$$

$$(D^3 - 2D^2 + D)[z] = x - e^x \leftarrow \text{annihilated by } D^2(D-1)$$

$$D^2(D-1)(D^3 - 2D^2 + D)[z] = 0$$

$$D^2(D-1)D(D^2 - 2D + 1)[z] = 0$$

$$D^3(D-1)^3[z] = 0$$

$$z = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 x e^x + c_6 x^2 e^x$$

z_h

$$\therefore \boxed{z_p = Ax + Bx^2 + Cx^2 e^x}$$