MATH 334: MISSION 6

Please put your work on these sheets. If you need additional room to show your work then add paper as needed, but be sure to put your answer clearly near the problem statement. Box your answers. Make sure you name is on each page and the assignment is stapled. Thanks and enjoy.

Topics: singular points, method of Frobenius

3pt **Problem 1** (Ritger & Rose section 7-2 problem 2)

Find all singularities of the following DEq^{ns}.

a.) $y'' + xy' + 3y = 0$ has analytic coefficients over $\mathbb{R}$ $\Rightarrow$ no singularities.

b.) $(x^2 - 3x + 2)y'' + \sqrt{x}\, y' + x^2 y = 0$

$$y'' + \frac{\sqrt{x}}{(x-1)(x-2)} y' + \frac{x^2}{(x-1)(x-2)} y = 0$$

$\boxed{\text{singular pts } 1, 2 \text{ and } 0}$ $\Leftarrow$ ($\sqrt{x}$ not analytic at $x = 0$)

c.) $\underbrace{(1-x^2)}\, y'' - 2xy' + n(n+1)y = 0$

zero when $x = \pm 1$ $\Rightarrow$ $y'' - \left(\frac{2x}{1-x^2}\right) y' + \frac{n(n+1)}{1-x^2} y = 0$

has nonanalytic coeff. at $\boxed{x = \pm 1 \text{ singular pts.}}$

d.) $(x^2 - x)y'' + x^2 y' - 3xy = 0$

$$\Rightarrow \quad y'' + \left(\frac{x^2}{x(x-1)}\right) y' - \left(\frac{3x}{x(x-1)}\right) y = 0$$

These coeffhicients are singular at $\boxed{x = 0 \text{ and } 1}$

($x = 0$ is removeable, it corresponds to a hole in the graph)

e.) $(e^x - 1)y'' + xy = 0$

$\Rightarrow$ $y'' + \frac{x}{e^x - 1} y = 0$ $\Rightarrow$ $\boxed{x = 0 \text{ is singular}}$

f.) $x\underbrace{(x^2 + 2x + 2)}_{(x+1)^2 + 1}y'' + (x^2+1)y' + 3y = 0$

$\Rightarrow$ $\boxed{x = 0 \text{ and } x = -1 \pm i \text{ singular}}$

5pt **Problem 2** Find the complete Frobenius solution of

$$x^2y'' + x(x - \tfrac{1}{2})y' + \tfrac{1}{2}y = 0.$$

(it turns out this one has real exponents)

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2}$

$$\sum_{n=0}^{\infty} (n+r-1)(n+r)a_n x^{n+r} + \left(x^2 - \tfrac{1}{2}x\right)\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} \frac{a_n}{2} x^{n+r} = 0$$

$$\Rightarrow 0 = \sum_{n=0}^{\infty} a_n(n+r-1)(n+r)x^{n+r} + \underbrace{\sum_{n=0}^{\infty} (n+r)a_n x^{n+r+1}}_{n+r+1 = j+r} - \sum_{n=0}^{\infty} \left(\frac{n+r}{2}\right)a_n x^{n+r} + \sum_{n=0}^{\infty} \frac{a_n}{2} x^{n+r}$$

$$\Rightarrow \left((r-1)ra_0 - \tfrac{1}{2}ra_0 + \tfrac{1}{2}a_0 \right)x^r + \sum_{j=1}^{\infty} \Big(a_j(j+r-1)(j+r) + (j+r-1)a_{j-1} - \tfrac{1}{2}(j+r)a_j + \tfrac{1}{2}a_j \Big) x^{j+r} = 0$$

$$\Rightarrow \left[r^2 - \tfrac{3}{2}r + \tfrac{1}{2}\right]a_0 x^r + \sum_{j=1}^{\infty} \Big(\underbrace{\left[(j+r-1)(j+r) - \tfrac{1}{2}(j+r) + \tfrac{1}{2}\right]}_{} a_j + (j+r-1)a_{j-1} \Big) x^{j+r}$$

$$0 = \left[(j+r-\tfrac{3}{2})(j+r) + \tfrac{1}{2}\right]a_j + (j+r-1)a_{j-1}$$

$$\Rightarrow a_j = \left(\frac{-(j+r-1)}{(j+r-\frac{3}{2})(j+r) + \frac{1}{2}} \right) a_{j-1} \quad (\star)$$

The indicial eq$^{\underline{n}}$ $r^2 - \frac{3}{2}r + \frac{1}{2} = 0 \Rightarrow \underline{r = 1/2 \text{ or } r = 1}$.

Eq$^{\underline{n}}$ $(\star)$ for $r = 1/2$ yields $a_j = \frac{-1}{j} a_{j-1}$. To see this note that:

$$(j-1)(j+\tfrac{1}{2}) + \tfrac{1}{2} = j^2 + \tfrac{1}{2}j - j - \tfrac{1}{2} + \tfrac{1}{2}$$

$$= j^2 - \tfrac{1}{2}j$$

$$= j\underbrace{(j - 1/2)}_{}$$

cancels with

$$j + r - 1 = j + \tfrac{1}{2} - 1 = j - 1/2.$$

PROBLEM 2 continued

☆ yields, $a_j = \frac{-1}{j} a_{j-1}$. Consider,

$$a_1 = \frac{-1}{1} a_0$$

$$a_2 = \frac{-1}{2} a_1 = \frac{(-1)^2}{2!} a_0$$

$$a_3 = \frac{-1}{3} a_2 = \frac{(-1)^3}{3!} a_0 \quad \cdots \quad \underline{a_j = \frac{(-1)^j}{j!} a_0}.$$

Hence $\underline{y_1 = a_0 \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^{j+\frac{1}{2}}}$.

Likewise $r = 1$ gives (☆)

$$a_j = \frac{-j\, a_{j-1}}{(j+1-\frac{3}{2})(j+1)+\frac{1}{2}} = \frac{-j\, a_{j-1}}{(j-\frac{1}{2})(j+1)+\frac{1}{2}}$$

$$= \frac{-j\, a_{j-1}}{j^2 - \frac{1}{2} j + j - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}}$$

$$= \frac{-\cancel{j}\, a_{j-1}}{\cancel{j}(j+\frac{1}{2})} = \underline{\frac{-a_{j-1}}{j+\frac{1}{2}}}.$$

Consider,

$$a_1 = \frac{(-1)^1}{3/2} a_0, \quad a_2 = \frac{-a_1}{5/2} = \frac{(-1)^2 a_0}{\frac{3}{2}\cdot\frac{5}{2}}, \quad a_3 = \frac{(-1)^3 a_0}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}$$

$$\Rightarrow a_j = \frac{(-1)^j 2^j a_0}{3\cdot5\cdot7\cdots(2j+1)} \quad \therefore \underline{y_2 = a_0 \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{3\cdot5\cdot7\cdots(2j+1)} x^{j+1}}.$$

In summary, for $x>0$

$$\boxed{y = c_1 \sqrt{x} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^j + c_2\, x \sum_{j=0}^{\infty} \left[(-1)^j \frac{2^j}{3\cdot5\cdots(2j+1)}\right] x^j}$$

For $x<0$ replace $\sqrt{x}$ with $|x|^{1/2}$ and x with $|x|$.

4pts **Problem 3** Find the Frobenius solution near $x = 0$ for $x > 0$ up to terms of order x^2 for

$$x^2y'' + \sin(x)y' - \cos(x)y = 0.$$

Note, $\sin x \approx x - \frac{1}{6}x^3$ and $\cos(x) = 1 - \frac{1}{2}x^2$ for $x \approx 0$. Hence we face, up to 2^{nd} order,

$$x^2y'' + xy' - (1 - \tfrac{1}{2}x^2)y = 0$$

Suppose
$$y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \cdots$$
$$y' = a_0rx^{r-1} + a_1(r+1)x^r + a_2(r+2)x^{r+1}$$
$$y'' = a_0r(r-1)x^{r-2} + a_1r(r+1)x^{r-1} + a_2(r+1)(r+2)x^r$$

Consequently,

$$a_0r(r-1)x^r + a_1r(r+1)x^{r+1} + a_2(r+1)(r+2)x^{r+2} + \cdots$$
$$+ a_0rx^r + a_1(r+1)x^{r+1} + a_2(r+2)x^{r+2} + \cdots$$
$$- a_0x^r - a_1x^{r+1} - a_2x^{r+2} + \cdots + \tfrac{1}{2}a_0x^{r+2} + \cdots = 0$$

Hence,

x^r: $a_0(r(r-1) + r - 1) = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow \underline{r = \pm 1}$.

x^{r+1}: $a_1r(r+1) + a_1(r+1) - a_1 = 0 \Rightarrow \underline{a_1 = 0}$.

x^{r+2}: $a_2(r+1)(r+2) + a_2(r+2) - a_2 + \frac{1}{2}a_0 = 0$

Let $r = -1$ thus $\underbrace{a_2[0 + -1 - 1] + \tfrac{1}{2}a_0 = 0}$

$\hookrightarrow -2a_2 = \frac{-1}{2}a_0 \Rightarrow \underline{a_2 = \frac{1}{4}a_0}$.

Thus, $\boxed{y_1 = a_0\left(x^{-1} + \frac{1}{4}x + \cdots\right)}$ (the next nontrivial term is at $n = 3$ since $a_3 = 0$)

Remark: $r_2 - r_1 = 2 \in \mathbb{Z}$ means finding y_2 has some complications. I did not teach that aspect of Frobenius this semester so if you did at least as much as I showed here you earned full credit. See pg. 199 part B. for what to do to complete the sol$^{\text{n}}$.

3pts **Problem 4**: (this is Problem 41 of section 8.6 of the 5-th Ed. of Nagle, Saff and Snider) Solve $x^3y'' - x^2y' - y = 0$ for $x >> 0$ by making the subsitution $z = 1/x$ and solving the resulting differential equation in z about the regular singular point $z = 0$. Find the first four nonzero terms in the series expansion about infinity.

Let $z = \frac{1}{x}$ then $\frac{dy}{dz} = \frac{dx}{dz}\frac{dy}{dx} = \frac{-1}{z^2}\frac{dy}{dx} \Rightarrow \underline{\frac{dy}{dx} = -z^2\frac{dy}{dz}}$ ☆

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left[\frac{-1}{z^2}\frac{dy}{dx}\right] = \frac{2}{z^3}\frac{dy}{dx} - \frac{1}{z^2}\frac{d}{dz}\left[\frac{dy}{dx}\right] = \frac{2}{z^3}\frac{dy}{dx} + \frac{1}{z^4}\frac{d^2y}{dx^2}$$

Note, $\frac{d^2y}{dz^2} = \frac{2}{z^3}\left(-z^2\frac{dy}{dz}\right) + \frac{1}{z^4}\frac{d^2y}{dx^2} \Rightarrow \underline{\frac{d^2y}{dx^2} = z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz}}$ ☆☆

Using ☆ & ☆☆ we obtain,

$$x^3\left(z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz}\right) - x^2\left(-z^2\frac{dy}{dz}\right) - y = 0$$

$$\Rightarrow z\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + \frac{dy}{dz} - y = 0$$

$$\Rightarrow z\frac{d^2y}{dz^2} + 3\frac{dy}{dz} - y = 0$$

Let $y = a_0z^r + a_1z^{r+1} + a_2z^{r+2} + \cdots$

$y' = a_0rz^{r-1} + a_1(r+1)z^r + a_2(r+2)z^{r+1} + \cdots$

$y'' = a_0r(r-1)z^{r-2} + a_1r(r+1)z^{r-1} + a_2(r+1)(r+2)z^r + \cdots$

Hence,

$$a_0r(r-1)z^{r-1} + a_1r(r+1)z^r + a_2(r+1)(r+2)z^{r+1} + \cdots$$
$$+ 3a_0rz^{r-1} + 3a_1(r+1)z^r + 3a_2(r+2)z^{r+1} + \cdots$$
$$- a_0z^r - a_1z^{r+1} - \cdots = 0$$

We find,

$\underline{z^{r-1})}$ $a_0[r(r-1) + 3r] = 0 \Rightarrow r(r+2) = 0$

$\underline{z^r\,]}$ $\underbrace{a_1r(r+1) + 3a_1(r+1) - a_0 = 0}$

$(r+1)(r+3)a_1 = a_0 \hookrightarrow a_1 = \frac{a_0}{(r+1)(r+3)}$

$\underline{z^{r+1})}$ $\underbrace{a_2(r+1)(r+2) + 3a_2(r+2) - a_1 = 0}$

$(r+2)(r+4)a_2 = a_1 \hookrightarrow a_2 = \frac{a_1}{(r+2)(r+4)}$

PROBLEM 4 continued

We found,

$$a_1 = \frac{a_0}{(r+1)(r+3)} \quad \text{and} \quad a_2 = \frac{a_1}{(r+2)(r+4)}$$

$\star$

$$\Rightarrow \quad a_2 = \frac{a_0}{(r+1)(r+2)(r+3)(r+4)}$$

$\star\star$

The indicial eq^{n} was found from coeff. of z^{r-1} : $r(r+2) = 0 \hookrightarrow r = 0$ or $r = -2$

Context: $z = \frac{1}{x}$ and $y = a_0 z^r + a_1 z^{r+1} + \cdots$

or $y = a_0 \left(\frac{1}{x}\right)^r + a_1 \left(\frac{1}{x}\right)^{r+1} + \cdots$ if $r = -2$ then

the sol^{n} has form $y = \underbrace{a_0 x^2 + a_1 x^1}_{} + a_2 + a_3/x + \cdots$

blow-up at ∞ and the $\star$ & $\star\star$ force coupling of a_0, a_1 to a_j for $j \geq 2$.

The sol^{n} at ∞ must stem from $r = 0$.

$$a_1 = \frac{a_0}{3}, \quad a_2 = \frac{a_0}{2 \cdot 3 \cdot 4} = \frac{a_0}{24}$$

Thus

$$\boxed{y = a_0 \left(1 + \frac{1}{3x} + \frac{1}{24x^2} + \frac{1}{360x^3} + \cdots \right)}$$

I leave the 4th term to the reader ☺.

However, the pattern is clear

$$a_3 = \frac{a_2}{(r+3)(r+5)} = \frac{a_0}{(r+1)(r+2)(r+3)^2(r+4)(r+5)} = \frac{a_0}{2 \cdot 9 \cdot 4 \cdot 5}$$

$$r = 0 \Rightarrow \frac{a_0}{360}.$$

(this is the answer in Nagle, Saff & Snider) in any event, these should be a welcome relief from the trouble of Frobenius

2pts Problem 5 Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Calculate A^2.

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1+8+21 & 2+10+24 & 3+12+27 \\ 4+20+42 & 8+25+48 & 12+30+45 \\ 7+32+63 & 14+40+72 & 21+48+81 \end{bmatrix}$$

$$\therefore \boxed{A^2 = \begin{bmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \\ 102 & 126 & 150 \end{bmatrix}}$$

2pts Problem 6 Let A be as in the previous problem. Suppose $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$.

(a.) calculate Av_1 $\quad AV_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 14 \\ 23 \end{bmatrix}}$.

(b.) calculate Av_2

$$AV_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = -\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ 14 \\ 20 \end{bmatrix}}$$

(c.) calculate $A[v_1|v_2]$ (here $[v_1|v_2]$ is the 3×2 matrix made from gluing (aka concatenating) the column vectors v_1 and v_2)

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{3\times 3}\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}}_{3\times 2} = \boxed{\underbrace{\begin{bmatrix} 5 & 8 \\ 14 & 14 \\ 23 & 20 \end{bmatrix}}_{3\times 2}}.$$

(d.) Does $A[v_1|v_2] = [Av_1|Av_2]$?

YES!

Problem 7 (2pts) A square matrix X is invertible iff there exists Y such that $XY = YX = I$ where I is the identity matrix. Moreover, linear algebra reveals that X is invertible iff $det(X) \neq 0$. For a 2×2 matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define $det(X) = ad - bc$. Suppose X is invertible and show $X^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. This formula is worth memorizing for future use in two-dimensional problems. Please understand, all I'm asking here is for you to multiply X and my propsed formula for X^{-1} to obtain $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\left[\begin{array}{c|c} ad-bc & db-bd \\ \hline -ca+ac & -bc+ad \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $X^{-1}X = I$, I was also happy if you showed $XX^{-1} = I$.

Problem 8 (2pts) Differentiation of matrices of functions is not hard. Let $X(t) = \begin{bmatrix} 2e^t & t \\ 1/t & e^{-t} \end{bmatrix}$.

$det(X(t)) = 1$. So $X^{-1} = \begin{bmatrix} e^{-t} & -t \\ -1/t & 2e^t \end{bmatrix}$

Calculate:

(a.) calculate $\frac{dX}{dt} = \frac{d}{dt}\begin{bmatrix} 2e^t & t \\ 1/t & e^{-t} \end{bmatrix} = \underline{\left[\begin{array}{c|c} 2e^t & 1 \\ \hline -1/t^2 & -e^{-t} \end{array}\right]}$.

(b.) calculate $\frac{dX^{-1}}{dt} = \frac{d}{dt}\begin{bmatrix} e^{-t} & -t \\ -1/t & 2e^t \end{bmatrix} = \underline{\left[\begin{array}{c|c} -e^{-t} & -1 \\ \hline 1/t^2 & 2e^t \end{array}\right]}$.

(c.) simplify $\frac{dX}{dt}X^{-1} + X\frac{dX^{-1}}{dt} = \frac{d}{dt}(XX^{-1}) = \frac{d}{dt}(I) = 0$. - (easy way) -

- (hard way) -

$$\frac{dX}{dt}X^{-1} + X\frac{dX^{-1}}{dt} = \begin{bmatrix} 2e^t & 1 \\ -1/t^2 & -e^{-t} \end{bmatrix}\begin{bmatrix} e^{-t} & -t \\ -1/t & 2e^t \end{bmatrix} + \begin{bmatrix} 2e^t & t \\ 1/t & e^{-t} \end{bmatrix}\begin{bmatrix} -e^{-t} & -1 \\ 1/t^2 & 2e^t \end{bmatrix}$$

$$= \left[\begin{array}{c|c} 2 - 1/t & -2te^t + 2e^t \\ \hline \frac{-e^{-t}}{t^2} + \frac{e^{-t}}{t} & \frac{1}{t} - 2 \end{array}\right] + \left[\begin{array}{c|c} -2 + 1/t & -2e^t + 2te^t \\ \hline \frac{-e^{-t}}{t} + \frac{e^{-t}}{t^2} & \frac{-1}{t} + 2 \end{array}\right]$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(d.) explain the previous part by differentiating $X(t)X^{-1}(t) = I$. Note: the product rule for matrix products is simply $\frac{d}{dt}(AB) = \frac{dA}{dt}B + A\frac{dB}{dt}$.

$$\frac{d}{dt}(\underbrace{XX^{-1}}_{I}) = \frac{dX}{dt}X^{-1} + X\frac{dX^{-1}}{dt} = \frac{d}{dt}(I) = 0.$$

Problem 9 (2pts) Reformulate the system of differential equations $y'' + z = 0$ and $z'' - y = 0$ as a system of four first order linear differential equations via the substitution $y = x_1, y' = x_2, z = x_3$ and $z' = x_4$. Write the system of first order differential equations in matrix form.

$$x_1' = y' = x_2$$

$$y'' = -z \Rightarrow x_2' = y'' = -z = -x_3$$

$$z'' = y \Rightarrow x_3' = z' = x_4$$

$$x_4' = z'' = y = x_1$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}$$

We can solve this via various methods.
(this was not req^{d} here, but,

$$y'' = -z \quad \text{and} \quad z'' = y$$

$$\Rightarrow z'''' = -z$$

$$\Rightarrow (D^4 + 1)[z] = 0$$

$$(D^2 - i)(D^2 + i)[z] = 0$$

$$(D - \sqrt{i})(D + \sqrt{i})(D - \sqrt{-i})(D + \sqrt{-i})[z] = 0$$

Anyway... by some nontrivial algebra

$$\lambda^4 + 1 = (\lambda^2 + \sqrt{2}\lambda + 1)(\lambda^2 - \sqrt{2}\lambda + 1)$$

$$= \left[\left(\lambda + \tfrac{1}{\sqrt{2}}\right)^2 + \tfrac{1}{2}\right]\left[\left(\lambda - \tfrac{1}{\sqrt{2}}\right)^2 + \tfrac{1}{2}\right]$$

$$\Rightarrow \underline{z(t) = A_1 \exp(-t/\sqrt{2}) \sin(t/\sqrt{2} + \phi_1) + A_2 \exp(t/\sqrt{2}) \sin(t/\sqrt{2} + \phi_2)}.$$

then $y = z''$, but, I spare us the details.