Please show your work and use words to explain your steps where appropriate. At least 300pts to earn.

Problem 1 (10pts) Let $F = \mathbb{Q}(\sqrt[3]{2})$. Calculate $[F : \mathbb{Q}]$ and find a basis for F over $\mathbb{Q}$.

Problem 2 (10pts) Explain why the following polynomials are irreducible over $\mathbb{Q}$,

(a.) $x^5 + 10x^4 + 15x^2 + 20$

(b.) $x^3 + x + 1$

Problem 3 (10pts) Suppose R is a ring with $x^2 = x$ for each $x \in R$. Prove R is commutative.

Problem 4 (10pts) Explain why S_5 is not isomorphic to D_{60} [Note: $5! = 120$]

Problem 5 (10pts) List all of the non-isomorphic abelian groups of order $36 = 2^2 3^2$. Circle any that are cyclic.

Problem 6 (20pts) Calculate and answer in $\mathbb{Z}_{12}$:

(a) $(3) = \langle 3 \rangle = \{$____________________________$\}$ and $\mathbb{Z}_{12}/(3) = \{$____________________________$\}$.

(b) Create addition and multiplication tables for $\mathbb{Z}_{12}/(3)$.

(c) Is $\mathbb{Z}_{12}/(3)$ a cyclic group under addition? Why or why not?

(d) Is $\mathbb{Z}_{12}/(3)$ an integral domain or field? Why or why not?

(e) Is (3) a prime or maximal ideal in $\mathbb{Z}_{12}$? Why or why not?

Problem 7 (15pts) Decide if x given below is a unit or a zero-divisor in $\mathbb{Z}_{300}$.
If it is a zero-divisor then prove it. If x is a unit then calculate a^{-1}.

(a.) $x = 40$

(b.) $x = 37$

Problem 8 (10pts) Consider $\mathbb{Z}_{88}$ as an additive group. Find the order of $\langle 11 \rangle$ and find all generators for this subgroup.

Problem 9 (10pts) Is $U(150) \approx \mathbb{Z}_2 \times \mathbb{Z}_{20}$? (here $\approx$ denotes isomorphism of groups) Argue for or against.

Problem 10 (10pts) Determine how many automorphisms of $\mathbb{Z}_{75}$ have order 20.

Problem 11 (15pts) Let $D_n = \{1, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y \mid x^n = 1, y^2 = 1, (xy)(xy) = 1\}$.

(a.) Prove that $x^k y = yx^{-k}$ for $k = 1, \ldots, n-1$

(b.) Show that $|x^k y| = 2$ for $k = 1, \ldots, n-1$.

Problem 12 (10pts) Let $H = \{(1), (12)\}$. Explain why H is a subgroup of S_3, then show it is **not** a **normal** subgroup of S_3.

Problem 13 (15pts) Show that $\varphi : \mathbb{Z}_9 \to \mathbb{Z}_6$ defined by $\varphi([x]_9) = [2x]_6$ where the notation $[x]_k = x + k\mathbb{Z}$ as usual.

(a.) prove φ is a **well-defined** function

(b.) prove φ is group **homomorphism**.

(c.) show φ is **not** a **ring** homomorphism.

Problem 14 (15pts) Let $\psi : G \to H$ be a homomorphism of groups. Prove ψ is injective if and only if $\text{Ker}(\psi) = \{e\}$.

Problem 15(15pts) Let $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$. Prove $\mathrm{SL}(n, \mathbb{R})$ is a normal subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Problem 16 (10pts) Let $\star : G \times S \to S$ is a group action. Prove $G_x \leq G$ for each $x \in S$. Recall $G_x = \{g \in G \mid g \star x = x\}$.

Problem 17 (10pts) State the orbit stabilizer theorem. Then count the number of symmetries for a cube via an appropriate application of the orbit stabilizer theorem.

Problem 18 (15pts) Consider $J = \langle 2i \rangle$ in $\mathbb{Z}[i]$. How many elements are in $\mathbb{Z}[i]/J$? Identify the ring to which $\mathbb{Z}[i]/J$ is isomorphic. Also, find and state the explicit isomorphism (making Cayley Table(s) might be helpful)

Problem 19 (15pts) Let R be a commutative ring with identity. Prove a is a unit if and only if $\langle a \rangle = R$.

Problem 20 (15pts) Let F be a field. Prove $F[x]$ is a principal ideal domain.

Problem 21 (15pts) Let $\phi : R \to S$ be a surjective homomorphism of rings and R has unity 1. Prove $\phi(1)$ is the unity of S.

Problem 22 (15pts) Let $A = \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \in \mathbb{Z}_6^{2\times 2}$. Determine $x \in \mathbb{Z}_6$ provide an invertible matrix A and calculate A^{-1} for each such x. (list all the answers explicitly)

Problem 23 (20pts)

(a) Let R be a commutative ring and I an ideal of R. Prove that ${}^{R}\!/_{I}$ is commutative.

(b) Although quotients of commutative rings are commutative and quotients of rings with 1 are rings with 1. It is not the case that quotients of integral domain are integral domains. Give an example which illustrates this fact.

(c) Let R be an integral domain and S a subring of R which contains the multiplicative identity of R. Show S must be an integral domain (i.e. a sub-domain of R).

(d) Part (c) says that subrings (containing 1) of integral domains are themselves integral domains. The analogous statement does *not* hold for fields. Give an example of a subring (containing 1) of a field which is itself *not* a field.

Problem 24 (50pts) Let $\sigma = (134)$ and consider $\langle \sigma \rangle$ in S_4. A real matrix representation of G is a subgroup G' of $\mathrm{Gl}(n, \mathbb{R})$ which is homomorphic to G. If $\phi : G \to \mathrm{Gl}(n\mathbb{R})$ is injective and $\phi(G) = G'$ then G' is known as a *faithful* representation of G. Let $G = \langle \sigma \rangle$ and find a faithful representation of G inside $\mathrm{GL}(4, \mathbb{R})$.