Please show your work and use words to explain your steps where appropriate.

Problem 1 (10pts) On the defintion of a group.

(a) Suppose that G is a non-empty set equipped an operation. What 4 things do I need to check to see if G is a group? Give details.

1:

2:

3:

4:

What additional property needs to hold for G to be an **Abelian** group?

5:

(b) Let $G = \mathbb{Z}_{\geq 0}$ be the set of non-negative integers. It can be shown that $x \star y = \max\{x, y\}$ (example: $3 \star 1 = \max\{3, 1\} = 3$) is an associative, commutative (closed) binary operation on G with identity 0. However, G is not a group. Why? [Use a concrete counterexample.]

Problem 2 (10pts) Let G is a group. Prove $Z(G)$ (the center of G) is a subgroup of G.

Problem 3 (10pts) Let G be a group. **Prove:** for each $n \in \mathbb{N}$, if $a_1, \ldots, a_n \in G$ then $(a_1 a_2 \ldots a_n)^{-1} = a_n^{-1} \cdots a_2^{-1} a_1^{-1}$.

Problem 4 (10pts) Suppose $|g| = n$ where $n \in \mathbb{N}$. Prove $|g| = |xgx^{-1}|$ for each $x \in G$.

Problem 5 (10pts) Consider $\mathbb{Z}_{100}$. Find all the generators for $\langle 5 \rangle$. *I'll give you a hint: 5 is one of them.*

Problem 6 (5pts) Suppose there exist $x, y \in G$ with $x \neq y^2$ and $x^2 = 1$ and $y^4 = 1$. Is G cyclic? Argue for or against.

Problem 7 (5pts) Draw a Cayley table for $U(8)$.

Problem 8 (10pts) Recall $D_n = \{1, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\} = \langle x, y \mid x^n = 1, y^2 = 1, (xy)^2 = 1\rangle$. Use the relations for D_{10} to simplify $x^{-3}y^2x^{15}yx^4y^{-22}$

Problem 9 (15pts) Let $\alpha = (1234)$ and $\beta = (3476)$. Define $\tau = \alpha\beta$

(a.) Calculate the disjoint cycle decomposition of τ

(b.) Find the order of τ

(c.) Write τ as a product of transpositions.

(d.) Find the inverse of $\tau(89)$.

(e.) Calculate τ^{100}

Problem 10 (12pts) List the orders of elements in $\mathbb{Z}_{50}$. Then determine the number of elements of each order.

Order =						
Number of elements =						

Problem 11 (12pts) List the orders of elements in D_{50}. Then determine the number of elements of each order.

Order =						
Number of elements =						

Problem 12 (6pts) Prove that if $x \equiv y$ and $x' \equiv y'$ modulo n then $xy \equiv x'y'$ modulo n.

Problem 13 (15pts) Find $\langle A \rangle = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle$ in $\mathrm{GL}_2(\mathbb{Z}_6)$. What is the order of A? What is A^{-1}?

Problem 14 (5pts) Define the symmetry group of the circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ in terms of isometries (a sentence will do). Then geometrically explain why the symmetry group of the circle has elements of finite and infinite order.

Problem 15 choose your own adventure... of proof.

(a) Choose one of the following: (20pts)

I. Suppose that $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. Prove that G is abelian.

II. Suppose that $(ab)^2 = a^2b^2$ for all $a, b \in G$. Prove that G is abelian.

(b) Choose one of the following: (15pts)

I. Prove any subgroup of a cyclic group is cyclic.

II. Prove that $U(n) = \{x \in \mathbb{Z}_n \mid \gcd(n, k) = 1\}$ forms a group with respect to multiplication modulo n.