Please show your work and use words to explain your steps where appropriate.

Problem 1 (15pts)

(a) Given: $H = \{1, x^2\}$ is a normal subgroup of $D_4 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1\}$.

The order of D_4/H is ______________________________.

The identity of D_4/H is _______________. $(xyH)^{-1} =$ ______________________________.

The order of xyH in D_4/H is _______________. The size of the set xyH is _______________.

Scratch work:

(b) $D_4/H \approx U(n)$ for $n = 5$ or $n = 8$? (choose either 5 or 8 and explain your choice)

Problem 2 (5pts) Define what is meant by writing $G = G_1 \oplus G_2 \oplus G_3$ where G is an additive group. What conditions are needed on the subgroups G_1, G_2, G_3 (if any).

Problem 3 (10pts) Consider $G = D_6 \times \mathbb{Z}_8$

(a.) The number of subgroups in G of order 15 is ______________________________.

(b.) The number of subgroups in G of order 8 is ______________________________.

Scratch work:

Problem 4 (10pts) How many automorphisms of $\mathbb{Z}_{100}$ have order 2 ?

Problem 5 (10pts)

(a) List all of the non-isomorphic abelian groups of order $54 = 3^3 \cdot 2$. Circle any that are cyclic.

(b) Are the groups $\mathbb{Z}_3 \times \mathbb{Z}_{50}$ and $\mathbb{Z}_6 \times \mathbb{Z}_{25}$ isomorphic? Explain your answer.

Problem 6 (15pts) Let $\phi : G \to H$ be a group homomorphism. Prove $\mathrm{Ker}(\phi)$ is a normal subgroup of G.

Problem 7 (10pts) Show that $\mathrm{SL}(n, \mathbb{R})$ is a normal subgroup of $\mathrm{GL}(n, \mathbb{R})$. Recall $\mathrm{SL}(n, \mathbb{R})$ is the set of $n \times n$ matrices over $\mathbb{R}$ for which the determinant is one.

Problem 8 (10pts) Let $f : \mathbb{Z}_7 \to \mathbb{R}$ be defined by $f([x]_7) = x$. Is f a homomorphism? Explain why f fails or succeeds at being a homomorphism of these additive groups.

Problem 9 (10pts) Let $H \leq G$. Prove $aH = bH$ if and only if $ab^{-1} \in H$.

Problem 10 (10pts) Prove $(G \times H)/(\{e\} \times H) \approx G$.

Problem 11 (10pts) (pick one of the following)

(a.) Recall $\phi_a(g) = aga^{-1}$ defines the inner automorphism induced by a. Suppose x and y induce the same inner automorphism. Show $x^{-1}y \in \mathrm{Z}(G)$.

(b.) Suppose $H, K \trianglelefteq G$. Prove $H \cap K \trianglelefteq G$.

Problem 12 (15pts) (pick one of the following)

(a.) Suppose $H \leq G$ where G is a finite group. Prove $|H| \mid |G|$. That is, prove Lagrange's Theorem.

(b.) Suppose $H, K \trianglelefteq G$ and $H \cap K = \{e\}$. For all $x, x' \in H$ and $y, y' \in K$, you are given (i.) $xy = x'y'$ implies $x = x'$ and $y = y'$ and (ii.) $xy = yx$. Prove that $H \times K \approx H \oplus K$.

Problem 13 (10pts) Suppose $\star : G \times S \to S$ is a group action. Prove that $G_x \leq G$ for each $x \in S$.

Problem 14 (20pts) Suppose $\star : G \times S \to S$ and $\diamond : H \times T \to T$ are group actions of G on S and H on T. Define $\bullet : (G \times H) \times (S \times T) \to S \times T$ by

$$(g, h) \bullet (x, y) = (g \star x, h \diamond y)$$

For each $(g, h) \in G \times H$ and $(x, y) \in S \times T$.

(a.) Prove that $\bullet$ is a group action.

(b.) If $G = S_4$ acts on $S = S_4$ by conjugation and $H = \langle (123)(457) \rangle \leq S_{10}$ acts on $T = \mathbb{N}_{10}$ by $\sigma \diamond x = \sigma(x)$ then find the orbit of $((123), 4)$

Problem 15 (20pts) Let $R \in \mathrm{SO}(3, \mathbb{R})$ act on $\mathbb{R}^3$ via matrix multiplication. Show $G_p \approx \mathrm{SO}(2, \mathbb{R})$ for $p \neq 0$. Here $\mathrm{SO}(n, \mathbb{R})$ denotes the group of orthogonal $n \times n$ matrices with determinant one.