MATH 421 TEST 2

Please show your work and use words to explain your steps where appropriate.

Problem 1 (15pts)

(a) Given: $H = \{1, x^2\}$ is a normal subgroup of $D_4 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1\}$.

The order of D_4/H is $8/2 = 4.$

The identity of D_4/H is H. $(xyH)^{-1} = (xy)^{-1}H = xyH$.

The order of xyH in D_4/H is 2. The size of the set xyH is 2.

coincidendental

Scratch work:

$$(xyH)(xyH) = (xy)(xy)H = (xy)^2H = H$$

$$D_4/H = \{H, xH, yH, xyH\}$$

$\{1, x^2\}$ $\{x, x^3\}$ $\{y, yx^2\} = \{y, x^2y\}$ $\{xy, xyx^2\} = \{xy, xx^{-2}y\} = \{xy, x^3y\}$

(b) $D_4/H \approx U(n)$ for $n = 5$ or $n = 8$? (choose either 5 or 8 and explain your choice)

Notice xH, yH, xyH all have order 2

thus $D_4/H \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \approx U(8)$.

Problem 2 (5pts) Define what is meant by writing $G = G_1 \oplus G_2 \oplus G_3$ where G is an additive group. What conditions are needed on the subgroups G_1, G_2, G_3 (if any).

$G_1 \cap G_2 = \{0\}$ and $(G_1 \oplus G_2) \cap G_3 = \{0\}$ make G_1, G_2, G_3 independent subgroups. We also need $G = G_1 + G_2 + G_3$. Normality of subgroups is req^d, but since G abelian all subgroups are normal so no need to make extra requirement.

Problem 3 (10pts) Consider $G = D_6 \times \mathbb{Z}_8$

(a.) The number of subgroups in G of order 15 is ZERO.

(b.) The number of subgroups in G of order 8 is $32/4 = 8$ subgroups of order 8 (each has 4 order 8 elements)

Scratch work:

(a.) elements $(a, b) \in D_6 \times \mathbb{Z}_8$ have order 15 if $\text{lcm}(|a|, |b|) = 15$. But, $5 \nmid 12$ and $5 \nmid 8$ so obtaining the 5 in $15 = 3(5)$ is impossible. In short $\nexists\ |(a, b)| = 15$ for any $(a, b) \Rightarrow$ No subgps. of order 15.

In D_6 we have $x^3, y, xy, x^2y, x^3y, x^4y, x^5y$ of order 2, x, x^5 order 6, x^2, x^4 order 3.

In $\mathbb{Z}_8$ we have $|2| = 4$, $|4| = 2$, $|1| = |3| = |5| = |7| = 8$, $|6| = 4$

To obtain $\text{lcm}(|a|, |b|) = 8$ we need $|a| = 2$, $|b| = 8$ or $|a| = 1$, $|b| = 8$.

$|a| = 2$ has 7 choices, $|b| = 8$ gives 4 choices $\Rightarrow$ 28 from $|a| = 2$.

$|a| = 1$ has 1 choice, $|b| = 8$ gives 4 choices $\Rightarrow$ 4 choices from $|a| = 1$ $\therefore$ 32 to be explicit, $(0,10)$, $(1,0)$, $(1,10)$ are the order two element in $\mathbb{Z}_2 \times \mathbb{Z}_{20}$

Problem 4 (10pts) How many automorphisms of $\mathbb{Z}_{100}$ have order 2 ?

Recall, $\mathrm{Aut}(\mathbb{Z}_{100}) \approx U(100) = U(4\cdot 25) \approx U(4) \times U(25)$

Hence $\mathrm{Aut}(\mathbb{Z}_{100}) \approx \mathbb{Z}_2 \times \mathbb{Z}_{20}$ as $U(4) = \{1,3\} \approx \mathbb{Z}_2$

$U(25) \approx \mathbb{Z}_{5^2-5} = \mathbb{Z}_{20}$.

To count $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_{20}$ with order 2 we consider $\underbrace{|a| = 1,\ |b| = 2}_{1 \text{ choice}}$ and $\underbrace{|a| = 2,\ |b| = 1 \text{ or } |b| = 2}_{2 \text{ choices}}$.

Thus, as isomorphism preserves order, $\exists$ 3 automorphisms of order 2 in $\mathrm{Aut}(\mathbb{Z}_{100})$.

Problem 5 (10pts)

(a) List all of the non-isomorphic abelian groups of order $54 = 3^3 \cdot 2$. Circle any that are cyclic.

$\mathbb{Z}_{27} \times \mathbb{Z}_2$ ← cyclic as $\gcd(27,2) = 1$.

$\mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_2$

$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$

not cyclic.

(b) Are the groups $\mathbb{Z}_3 \times \mathbb{Z}_{50}$ and $\mathbb{Z}_6 \times \mathbb{Z}_{25}$ isomorphic? Explain your answer.

YES! Observe, $\mathbb{Z}_3 \times \mathbb{Z}_{50} \approx \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$

$\approx \mathbb{Z}_6 \times \mathbb{Z}_{25}$.

Problem 6 (15pts) Let $\phi : G \to H$ be a group homomorphism. Prove $\mathrm{Ker}(\phi)$ is a normal subgroup of G.

Note, $\phi(e) = e$ thus $e \in \mathrm{Ker}\,\phi = \{g \in G \mid \phi(g) = e\} \neq \emptyset$. Suppose $a, b \in \mathrm{Ker}\,\phi$ then $\phi(a) = e$ and $\phi(b) = e$ thus,

$$\phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = \phi(a)(\phi(b))^{-1} = e(e)^{-1} = e.$$

hence $ab^{-1} \in \mathrm{Ker}\,\phi$ and we find $\mathrm{Ker}\,\phi \leq G$ by one-step-subgroup test. Next let $g \in G$ and suppose $x \in g\,\mathrm{Ker}\,\phi\, g^{-1}$ hence $x = gyg^{-1}$ where $\phi(y) = e$. Observe,

$$\begin{aligned} \phi(x) &= \phi(gyg^{-1}) \\ &= \phi(g)\phi(y)\phi(g^{-1}) && : \text{as } \phi \text{ homomorphism.} \\ &= \phi(g)\phi(g^{-1}) && : \text{as } \phi(y) = e \\ &= \phi(gg^{-1}) && : \phi \text{ homomorphism} \\ &= \phi(e) \\ &= e. \end{aligned}$$

Thus $x \in \mathrm{Ker}\,\phi$ and $g\,\mathrm{Ker}\,\phi\, g^{-1} \subseteq \mathrm{Ker}\,\phi$ $\forall g \in G$ $\therefore$ $\mathrm{Ker}\,\phi \trianglelefteq G$. //

Problem 7 (10pts) Show that $\mathrm{SL}(n, \mathbb{R})$ is a normal subgroup of $\mathrm{GL}(n, \mathbb{R})$. Recall $\mathrm{SL}(n, \mathbb{R})$ is the set of $n \times n$ matrices over $\mathbb{R}$ for which the determinant is one.

Let $\phi : GL(n, \mathbb{R}) \to \mathbb{R}^{\times}$ be defined by $\phi(A) = \det(A)$ $\forall A \in GL(n, \mathbb{R})$.
Note $A \in GL(n, \mathbb{R}) \Rightarrow \det(A) \neq 0$ $\therefore$ ϕ is into. Moreover,
$\phi(AB) = \det AB = \det A \det B = \phi(A)\phi(B)$ $\therefore$ ϕ homomorphism.
Finally, $\ker \phi = \{ A \in GL(n, \mathbb{R}) \mid \det(A) = 1 \} = SL(n, \mathbb{R})$
hence $SL(n, \mathbb{R}) \trianglelefteq GL(n, \mathbb{R})$ by <u>PROBLEM 6</u> or Notes etc..

Problem 8 (10pts) Let $f : \mathbb{Z}_7 \to \mathbb{R}$ be defined by $f([x]_7) = x$. Is f a homomorphism? Explain why f fails or succeeds at being a homomorphism of these additive groups.

Observe $[6]_7 = [13]_7$ yet
$f([6]_7) = 6$ and $f([6]_7) = f([13]_7) = 13$
hence f is not single-value $\Rightarrow$ f <u>not</u> <u>a</u> <u>function</u>
$\therefore$ <u>f not a homomorphism</u>.

Problem 9 (10pts) Let $H \leq G$. Prove $aH = bH$ if and only if $ab^{-1} \in H$. typo! should be $a^{-1}b \in H$.

$\Rightarrow$ Suppose $aH = bH$ where $a, b \in G$ and $H \leq G$. Since $a = ae \in aH$
we find $a \in bH$ $\therefore$ $\exists h \in H$ such that $a = bh \Rightarrow b^{-1}a \in H$
hence $(b^{-1}a)^{-1} = a^{-1}(b^{-1})^{-1} = a^{-1}b \in H$.

$\Leftarrow$ Suppose $a^{-1}b \in H$. Thus $a^{-1}b = \bar{h} \in H$. Observe, $b = a\bar{h}$ and,

$$\begin{aligned} aH &= \{ah \mid h \in H\} = \{a\bar{h}\bar{h}^{-1}h \mid h \in H\} \\ &= \{bk \mid k \in H\} \\ &= bH. \end{aligned}$$

$k = \bar{h}^{-1}h$ is arbitrary as h is arbitrary.

(see notes for a different and perhaps superior argument)

Problem 10 (10pts) Prove $(G \times H)/(\{e\} \times H) \approx G$.

Let $\phi : G \times H \to G$ be defined by $\phi(x, y) = x$ $\forall (x,y) \in G \times H$.
Observe, $\phi((x, y)(a, b)) = \phi((xa, yb)) = xa = \phi((x,y))\phi((a,b))$
hence ϕ a homomorphism. Furthermore,

$$\ker \phi = \{ (x, y) \in G \times H \mid x = e \} = \{e\} \times H$$

Thus, by 1st isomorphism Thm, $\underline{\dfrac{G \times H}{\{e\} \times H} \approx G}$.

(noting $g \in G$ has $\phi((g, e)) = g$
$\therefore$ $\phi(G \times H) = G$,)

Problem 11 (10pts) (pick one of the following)

(a.) Recall $\phi_a(g) = aga^{-1}$ defines the inner automorphism induced by a. Suppose x and y induce the same inner automorphism. Show $x^{-1}y \in Z(G)$.

(b.) Suppose $H, K \trianglelefteq G$. Prove $H \cap K \trianglelefteq G$.

(a.) Suppose $\phi_x = \phi_y \Rightarrow \phi_x(g) = \phi_y(g)\ \forall g \in G \Rightarrow xgx^{-1} = ygy^{-1}\ \forall g \in G$.
Thus, $x^{-1}xgx^{-1}y = x^{-1}ygy^{-1}y$ and we find $g(x^{-1}y) = (x^{-1}y)g\ \forall g \in G$.
That is, we find $x^{-1}y \in Z(G)$.

(b.) § $H, K \trianglelefteq G$. Let $x \in g(H\cap K)g^{-1}$ then $x = gyg^{-1}$ where $y \in H\cap K$ thus $y \in H$ and $y \in K$. As $H \trianglelefteq G$ we know $gHg^{-1} \subseteq H$ hence $x = gyg^{-1}$ where $y \in H \Rightarrow x \in H$. Likewise, as $K \trianglelefteq G$ we know $gKg^{-1} \subseteq K$ hence $x = gyg^{-1}$ where $y \in K$ yields $x \in K$. In total, $x \in H \cap K$ $\therefore$ $g(H\cap K)g^{-1} \subseteq H \cap K$ and it follows $H \cap K \trianglelefteq G$.

–(However, it remains to show $H\cap K \leq G$, I leave that to you use the usual subgroup tests, it's easy)–

Problem 12 (15pts) (pick one of the following)

(a.) Suppose $H \leq G$ where G is a finite group. Prove $|H| \mid |G|$. That is, prove Lagrange's Theorem.

(b.) Suppose $H, K \trianglelefteq G$ and $H \cap K = \{e\}$. For all $x, x' \in H$ and $y, y' \in K$, you are given (i.) $xy = x'y'$ implies $x = x'$ and $y = y'$ and (ii.) $xy = yx$. Prove that $H \times K \approx H \oplus K$.

(a.) Since cosets aH partition G we parse G into distinct cosets $a_1H, a_2H, \dots, a_kH$. However, $|a_iH| = |a_jH| = |H|$ for each $i = 1, 2, \dots, k$ hence $G = a_1H \cup a_2H \cup \dots \cup a_kH$ has

$$|G| = |a_1H| + |a_2H| + \dots + |a_kH| = \underbrace{|H| + |H| + \dots + |H|}_{k\text{-fold}} = k|H|$$

Thus $|H| \mid |G|$. (in fact $[G:H] = k = \dfrac{|G|}{|H|}$ is clear aswell).

(b.) I gave you the Lemma that drives the isomorphism $\phi : H\oplus K \longrightarrow H\times K$ by $x = ab \in H \oplus K$ where $a \in H$, $b \in K$ maps to (a,b) ; $\phi(ab) = (a,b)$. Note ϕ well-defined by (i.)
~~And~~, Also,

$$\begin{aligned}
\phi(xy) &= \phi((ab)(a'b')) &&: \text{where } x = ab,\ y = a'b' \text{ and } a, a' \in H,\ b, b' \in K. \\
&= \phi((aa')(bb')) &&: \text{by (ii.)} \\
&= (aa', bb') &&: \text{def}^{\underline{n}} \text{ of } \phi \\
&= (a,b)(a',b') &&: \text{def}^{\underline{n}} \text{ of } H\times K \text{ product.} \\
&= \phi(ab)\,\phi(a'b') = \phi(x)\phi(y)
\end{aligned}$$

hence ϕ homomorphism.
Moreover, if $(a,b) \in H\times K$ then $\phi(ab) = (a,b)$ $\therefore$ $\phi(H\oplus K) = H\times K$
$\ker\phi = \{ab \mid (a,b) = (e,e)\} = \{e\}$
$\therefore$ ϕ is isomorphism of $H\oplus K \approx H\times K$.

$*$ since I have trouble producing $\star$.

Problem 13 (10pts) Suppose $\star : G \times S \to S$ is a group action. Prove that $G_x \leq G$ for each $x \in S$.

By def$^{\underline{n}}$ of group action we have $e * x = x$ for each $x \in S$ thus $e \in G_x \neq \emptyset$. Suppose $a, b \in G_x$ then $a * x = x$ and $b * x = x$ thus $a^{-1} * (a * x) = a^{-1} * x \Rightarrow (a^{-1}a) * x = a^{-1} * x$ by Axiom 2 of group actions thus $e * x = x = a^{-1} * x \Rightarrow a^{-1} \in G$. Also, $(ab) * x = a * (b * x) = a * x = x$ using $a, b \in G_x$ and again Axiom 2 of group actions. Consequently, $ab \in G_x$ thus by two-step-subgroup test $G_x \leq G$.

Problem 14 (20pts) Suppose $\star : G \times S \to S$ and $\diamond : H \times T \to T$ are group actions of G on S and H on T. Define $\bullet : (G \times H) \times (S \times T) \to S \times T$ by

$$(g, h) \bullet (x, y) = (g \star x, h \diamond y)$$

For each $(g, h) \in G \times H$ and $(x, y) \in S \times T$.

(a.) Prove that $\bullet$ is a group action.

(b.) If $G = S_4$ acts on $S = S_4$ by conjugation and $H = \langle (123)(457) \rangle \leq S_{10}$ acts on $T = \mathbb{N}_{10}$ by $\sigma \diamond x = \sigma(x)$ then find the orbit of $((123), 4)$

(a.) $(e_G, e_H) = e_{G \times H}$ and we have $e_G * g = g \ \forall g \in G$ and $e_H \diamond h = h \ \forall h \in H$ by Axiom 1 of group action.

Consider, $(e_G, e_H) \bullet (x, y) = (e_G * x, e_H \diamond y) = (x, y) \ \forall (x,y) \in$

thus $\bullet$ has $e_{G \times H} \bullet z = z \ \forall z \in G \times H$. Next, consider,

$$((g_1, h_1)(g_2, h_2)) \bullet (x, y) = (g_1 g_2, h_1 h_2) \bullet (x, y) \quad : \text{def}^{\underline{n}} \text{ of product in } G \times H$$

$$= ((g_1 g_2) * x, (h_1 h_2) \diamond y) \quad : \text{def}^{\underline{n}} \text{ of } \bullet$$

$$= (g_1 * (g_2 * x), h_1 \diamond (h_2 \diamond y)) \quad : \text{Axiom 2 for } * \text{ and } \diamond.$$

$$= (g_1, h_1) \bullet (g_2 * x, h_2 \diamond y) \quad : \text{def}^{\underline{n}} \text{ of } \bullet$$

$$= (g_1, h_1) \bullet ((g_2, h_2) \bullet (x, y)) \quad : \text{def}^{\underline{n}} \text{ of } \bullet$$

Thus $\bullet$ is a group action.

(b.)

$$\begin{aligned}
\mathcal{O}(((123), 4)) &= \{ (\alpha, \sigma) \bullet ((123), 4) \mid \alpha \in S_4, \ \sigma = (123)^k(457)^k, k \in \mathbb{Z} \} \\
&= \{ (\alpha (123) \alpha^{-1}, \sigma(4)) \mid \alpha \in S_4, \ \sigma = (123)^k(457)^k, k \in \mathbb{Z} \} \\
&= \{ (\alpha(123)\alpha^{-1}, 4), (\alpha(123)\alpha^{-1}, 5), (\alpha(123)\alpha^{-1}, 7) \mid \alpha \in S_4 \} \\
&= \boxed{\{(123), (132), (124), (142), (134), (143), (234), (243)\} \times \{4, 5, 7\}}
\end{aligned}$$

-(24 things in here)-

Problem 15 (20pts) Let $R \in \mathrm{SO}(3,\mathbb{R})$ act on $\mathbb{R}^3$ via matrix multiplication. Show $G_p \approx \mathrm{SO}(2,\mathbb{R})$ for $p \neq 0$. Here $\mathrm{SO}(n,\mathbb{R})$ denotes the group of orthogonal $n \times n$ matrices with determinant one.