Please show your work and use words to explain your steps where appropriate.

Problem 1 (10pts) Let D be an integral domain with $a, b, c \in D$. Prove that if $a \neq 0$ and $ab = ac$ then $b = c$.

Problem 2 (10pts) Let $\langle a, b\rangle = \{ra + sb \mid r, s \in R\}$ where R is a ring and $a, b \in R$. Prove $\langle a, b\rangle$ forms an ideal of R.

Problem 3 (10pts) Let $R = \langle 3, x\rangle$ where $x \in \mathbb{N}$. Prove that either $R = \langle 3\rangle$ or $R = \mathbb{Z}$.

Problem 4 (10pts) Let R be a ring and A an ideal of R. Prove that $(x + A)(y + A) = xy + A$ provides a well-defined operation on $R/A = \{x + A \mid x \in R\}$.

Problem 5 (10pts) Give an example of a field with 25 elements.

Problem 6 (10pts) Is $\mathbb{R}[x]/\langle x^2 - 3\rangle$ a field ? Explain.

Problem 7 (10pts) Show that $i + \sqrt{2}$ is algebraic over $\mathbb{R}$.

Problem 8 (15pts) Explain why the following polynomials are irreducible over $\mathbb{Q}$,

(a.) $x^5 + 10x^4 + 15x^2 + 20$

(b.) $x^3 + x + 1$

Problem 9 (10pts) Find the degree of $\alpha = \sqrt[5]{2}$ over $\mathbb{Q}$ and find a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

Problem 10 (20pts) Let R be a commutative ring with unity and let A be an ideal of R. Prove the quotient ring R/A is an integral domain if and only if A is a prime ideal.

Problem 11 (10pts) Let F be a field and suppose $p(x) \in F[x]$ is irreducible. Prove $\langle p(x) \rangle$ is a maximal ideal.

Problem 12 (15pts) Consider $J = \langle 1+2i \rangle$ in $\mathbb{Z}[i]$. How many elements are in $\mathbb{Z}[i]/J$? Make a multiplication table for the units in $\mathbb{Z}[i]/J$ and identify which group is isomorphic to the group of units in $\mathbb{Z}[i]/J$.

Problem 13 (20pts) Define $\psi : \mathbb{Q}[x] \to \mathbb{Q} \times \mathbb{Q}$ by $\psi(f(x)) = (f(1), f(-1))$. You are given ψ is a ring homomorphism.

(a.) Show that Ψ is a surjection.

(b.) Calculate $\text{Ker}(\Psi)$.

(c.) Is $\text{Ker}(\Psi)$ a prime ideal of $\mathbb{Q}[x]$? Explain.