SOLUTION TO LECTURE 10 PROBLEMS 37 → 40

P37 Let $\phi : G_1 \longrightarrow G_2$ be homomorphism. Prove $\ker \phi \leq G_1$ and $Im(\phi) \leq G_2$

① observe $\phi(e_1) = e_2$ hence $e_1 \in \ker\phi = \{x \in G_1 \mid \phi(x) = e_2\}$.
Thus $\ker\phi \neq \emptyset$. Let $a, b \in \ker\phi$ then as ϕ is homomorphism

$$\phi(ab^{-1}) = \phi(a)\phi(b)^{-1} = e_2 e_2^{-1} = e_2$$

thus $ab^{-1} \in \ker\phi$ $\therefore$ $\ker\phi \leq G_1$ by one-step subgroup test.

② Again, $\phi(e_1) = e_2$ thus $e_2 \in Im\phi = \{\phi(x) \mid x \in G_1\}$.
Thus $Im\phi \neq \emptyset$. Let $a, b \in Im\phi$ then $\exists\, x, y \in G_1$ for which $a = \phi(x)$ & $b = \phi(y)$.
Note that $ab^{-1} = \underbrace{\phi(x)(\phi(y))^{-1} = \phi(x)\phi(y^{-1}) = \phi(xy^{-1})}_{\text{as } \phi \text{ is homomorphism}}$
and $xy^{-1} \in G_1$ thus $ab^{-1} \in Im\phi$ and by one-step subgroup test
$Im\,\phi \leq G_2$

<u>Remark</u>: PROP. 2.1-8 produces the above results as special cases

$$\phi^{-1}\{e_2\} = \ker\phi \quad \& \quad \phi(G_1) = Im\,\phi$$

the point of this problem was to work through the proof in these important and common cases.

P38 Gallian #5 on pg. 130

Show $U(8)$ is isomorphic to $U(12)$

$U(8)$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$U(12)$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

these Cayley Tables demonstrate the isomorphism
$\psi(1) = 1$, $\psi(3) = 5$, $\psi(5) = 7$, $\psi(7) = 11$
is an isomorphism from $U(8)$ to $U(12)$.

Remark: much later, $U(8) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$
and $U(12) \approx U(3) \times U(4) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$.

FROM THE FUTURE!

P39 Gallian #7 on pg. 130

Prove S_4 is not isomorphic to D_{12}

Well $|S_4| = 4! = 24$ and $|D_{12}| = 24$ so that's no help! We'll need something about elements. In D_{12} we have
$y, xy, x^2y, x^3y, x^4y, x^5y, \dots, x^{11}y$ and x^6 of order 2.
That is, in D_{12} there are 13 order 2 elements.
However, in S_4 there are just (I'll be lazy and list them, $(12), (13), (14), (23), (24), (34)$ that's 6 ~~element~~ order 2 elements. If $\phi : D_{12} \to S_4$ ~~is~~ was an isomorphism then the image of $y, xy, \dots, x^{11}y, x^6$ have order 2 (and are distinct) in S_4 which is impossible!
$\therefore S_4 \not\approx D_{12}$.

P40 # 24, pg. 130 - 131 Gallian

$G = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \}$

$H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \middle| \ a, b \in \mathbb{Q} \right\}$

① Show $(G, +) \approx (H, +)$.
② Prove G, H closed under multiplication
③ Does ψ from 1 also preserve the multiplication?

① Let $\psi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$

$$\begin{aligned} \psi((a+b\sqrt{2}) + (x + y\sqrt{2})) &= \psi((a+x) + (b+y)\sqrt{2}) \\ &= \begin{bmatrix} a+x & 2(b+y) \\ b+y & a+x \end{bmatrix} \\ &= \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \begin{bmatrix} x & 2y \\ y & x \end{bmatrix} \\ &= \psi(a+b\sqrt{2}) + \psi(x+y\sqrt{2}) \end{aligned}$$

Also, $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \in H$ has $\psi(a+b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ for $a + b\sqrt{2} \in G$ thus ψ is surjective. Moreover, $\operatorname{Ker} \psi = \left\{ a + b\sqrt{2} \ \middle| \ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} = \{0\}$ hence ψ is 1-1. In summary, ψ is bijective homomorphism, an isomorphism of $(G,+) \approx (H,+)$.

P40 continued

$a+b\sqrt{2},\ x+y\sqrt{2} \in G \quad (*)$

$$(a+b\sqrt{2})(x+y\sqrt{2}) = ax + 2by + (ay+bx)\sqrt{2}$$

Hence, as $ax+2by$, $ay+bx \in \mathbb{Q}$ since we assumed $a, b, x, y \in \mathbb{Q}$ implicitly at $(*)$ we find $(a+b\sqrt{2})(x+y\sqrt{2}) \in G$. Thus G is closed under multiplication.

Remark: you could go further and prove $G^{\times} = G - \{0\}$ has inverses & identity 1.

Likewise, for $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}, \begin{bmatrix} x & 2y \\ y & x \end{bmatrix} \in H$ calculate

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\begin{bmatrix} x & 2y \\ y & x \end{bmatrix} = \underbrace{\begin{bmatrix} ax+2by & 2ay+2bx \\ bx+ay & 2by+ax \end{bmatrix}}_{\text{in } H \text{ once again}}$$

thus H closed under multiplication. (you could show $H^{\times}$ is group in fact)

Thus,

$$\begin{aligned} \psi\big((a+b\sqrt{2})(x+y\sqrt{2})\big) &= \psi(ax+2by+(ay+bx)\sqrt{2}) \\ &= \begin{bmatrix} ax+2by & 2ay+2bx \\ bx+ay & ax+2by \end{bmatrix} \\ &= \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\begin{bmatrix} x & 2y \\ y & x \end{bmatrix} \\ &= \psi(a+b\sqrt{2})\,\psi(x+y\sqrt{2}) \end{aligned}$$

So, yes, ψ also preserves multiplication! (Neat)