SOLUTION TO LECTURE 11 PROBLEMS 41 – 44

$\boxed{P41}$ Show $Inn(G) \leq Aut(G)$.

Recall $Inn(G) = \{ \phi_g : G \to G \mid \phi_g(x) = gxg^{-1} \ \forall x \in G \}$ and $Aut(G) = \{ f : G \to G \mid f \text{ an isomorphism} \}$ is a group w.r.t function composition. Consider,

$$\phi_e(x) = exe^{-1} = x \quad \Rightarrow \quad \phi_e = Id$$

thus $Inn(G)$ contains the identity of $Aut(G)$ and we find $Inn(G) \neq \emptyset$. Next, we should show $Inn(G) \subseteq Aut(G)$. Observe, for $x, y \in G$,

$$\phi_g(xy) = g(xy)g^{-1} = gxg^{-1}gyg^{-1} = \phi_g(x)\phi_g(y)$$

hence $\phi_g : G \to G$ is homomorphism. Moreover,

$$Ker\,\phi_g = \{ x \in G \mid \underbrace{gxg^{-1} = e}_{x = g^{-1}eg = g^{-1}g = e} \} \quad \therefore Ker\,\phi_g = \{e\}.$$

If $h \in G$ then $\phi_g(g^{-1}hg) = g(g^{-1}hg)g^{-1} = h \ \therefore \phi_g$ is onto and we have shown $\phi_g : G \to G$ is isomorphism $\therefore \phi_g \in Aut(G)$.

Next, suppose $\phi_g, \phi_h \in Inn(G)$ then

$$(\phi_g \circ \phi_h)(x) = \phi_g(hxh^{-1}) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = \phi_{gh}(x).$$

thus $\phi_g \circ \phi_h = \phi_{gh} \in Inn(G)$. Also, $\phi_g \in Inn(G)$ then calculate $(\phi_{g^{-1}} \circ \phi_g)(x) = \phi_{g^{-1}}(gxg^{-1}) = g^{-1}(gxg^{-1})(g^{-1})^{-1} = x$ hence $\phi_{g^{-1}} \circ \phi_g = Id$ and likewise $\phi_g \circ \phi_{g^{-1}} = Id$ thus $(\phi_g)^{-1} = \phi_{g^{-1}} \in Inn(G)$. Therefore, by two-step-subgroup test we find $Inn(G) \leq Aut(G)$.

$Z(D_4) = \{1, x^2\}$ we have $x^2 z = z x^2 \ \forall z \in D_4$

$$\phi_g(z) = gzg^{-1} = x^4 gzg^{-1} = (x^2 g) z (g^{-1} x^2)$$
$$= (x^2 g) z (x^2 g)^{-1}$$
$$= \phi_{x^2 g}(z)$$

Thus $Inn(D_4) = \{\phi_1, \phi_x, \phi_{xy}, \phi_y\}$

	ϕ_1	ϕ_x	ϕ_y	ϕ_{xy}
ϕ_1	ϕ_1	ϕ_x	ϕ_y	ϕ_{xy}
ϕ_x	ϕ_x	ϕ_1	ϕ_{xy}	ϕ_y
ϕ_y	ϕ_y	ϕ_{xy}	ϕ_1	ϕ_x
ϕ_{xy}	ϕ_{xy}	ϕ_y	ϕ_x	ϕ_1

$$\phi_y \phi_x = \phi_{yx} = \phi_{x^{-1}y} = \phi_{x^3 y} = \phi_{xy}.$$

Apparently, $Inn(D_4) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$.
$Inn(D_4)$ has 3 elements of order 2 etc...

<u>Remark</u>: I originally did this problem a much more tedious way. For example,

$$\phi_x(x^m) = x x^m x^{-1} = x^3 x^m x^{-3} = \phi_{x^3}(x^m)$$
$$\phi_x(x^m y) = x(x^m y)x^{-1} = x^3(x^m y)x^2 x^{-1} = x^3(x^m y)x^{-3} = \phi_{x^3}(x^m y)$$

Thus $\phi_x(z) = \phi_{x^3}(z) \ \forall z \in D_4 \ \therefore \ \phi_x = \phi_{x^3}$. Etc...

However, with my new insight, I think we can easily calculate $Inn(G)$ given $Z(G)$. There is a nice connection with $Inn(G)$ and $Z(G)$ — coset-representatives.

— (see P44) — !

P43 Gallian #30 on pg. 131

Suppose G is finite Abelian group with no element of order 2. Show the mapping $g \mapsto g^2$ is an automorphism of G. Show, by example, it G is infinite then $g \mapsto g^2$ need not be an automorphism.

Let $\varphi(g) = g^2$ then $\varphi(gh) = (gh)^2 = ghgh = g^2h^2$ provided G is abelian. Thus $\varphi(gh) = \varphi(g)\varphi(h)$ $\forall g, h \in G$. Hence φ is a homomorphism from $G \to G$. Consider, $\text{Ker}\,\varphi = \{g \in G \mid g^2 = e\} = \{e\}$ provided $\nexists$ an element of order 2 in G. In the case $|G| < \infty$ we know $\varphi : G \to G$ is injective iff φ is surjective $\therefore$ $\varphi \in \text{Aut}(G)$. However, in the $|G| = \infty$ case we only know φ is an injective homomorphism. So, surjectivity will fail in our example:

Consider, $\varphi : \mathbb{Z} \to \mathbb{Z}$ the map $g \mapsto g^2$ is simply $g \mapsto 2g = g + g$ in additive context. Of course $\text{Im}\,\varphi = 2\mathbb{Z}$ so, for example $1 \notin 2\mathbb{Z}$ thus $\varphi : \mathbb{Z} \to \mathbb{Z}$ is not surjective $\therefore$ $\varphi \notin \text{Aut}(G)$.

(to be clear, $\mathbb{Z}$ is abelian and $2g = 0 \Rightarrow g = 0$ so $\nexists$ an element of order 2 in $\mathbb{Z}$.)

$\boxed{P44}$ # 33 on p. 131 Gallian (in retrospect, this makes $\boxed{42}$ easier !)

Suppose g and h induce the same inner automorphism of a group G. Prove $h^{-1}g \in Z(G)$

$$\phi_g(x) = \phi_h(x) \quad \forall x \in G$$

$$\Rightarrow gxg^{-1} = hxh^{-1} \quad \forall x \in G$$

$$\Rightarrow h^{-1}gxg^{-1} = h^{-1}hxh^{-1} \quad \forall x \in G$$

$$\Rightarrow h^{-1}gxg^{-1}h = h^{-1}hxh^{-1}h = x \quad \forall x \in G.$$

$$\Rightarrow h^{-1}gxg^{-1}hh^{-1}g = xh^{-1}g \quad \forall x \in G$$

$$\Rightarrow (h^{-1}g)x = x(h^{-1}g) \quad \forall x \in G$$

$$\Rightarrow \underline{h^{-1}g \in Z(G)}.$$

OR, MORE TO THE POINT,

If $gxg^{-1} = hxh^{-1}$ then $h^{-1}gxg^{-1}g = h^{-1}hxh^{-1}g$ thus $h^{-1}gx = xh^{-1}g$. So, $gxg^{-1} = hxh^{-1}$ $\forall x \in G$ implies $h^{-1}gx = xh^{-1}g$ $\forall x \in G$ $\therefore$ $h^{-1}g \in Z(G)$.