SOLUTION TO LECTURE 12 PROBLEMS 45–48

P45 Gallian # 7 pg. 145

Find all left cosets of $\{1, 11\}$ in $U(30)$

Remark: $|U(30)| = \varphi(30) = \varphi(6)\varphi(5) = (2)(4) = 8.$
I know this, but, we prove it in Later Lecture...

$U(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}$ hooray, 8 things

$1\{1, 11\} = \{1, 11\} = 11\{1, 11\}$ Let $H = \{1, 11\}$.

$7H = \{7, 77\} = \{7, 17\} = 17H.$

$13H = \{13, 143\} = \{13, 23\} = 23H.$

$19H = \{19, 209\} = \{19, 29\} = 29H.$

In summary, $U(30)/H = \{\underbrace{\{1, 11\}}_{H}, \underbrace{\{7, 17\}}_{7H}, \underbrace{\{13, 23\}}_{13H}, \underbrace{\{19, 29\}}_{19H}\}$

P46 Prove the following isomorphisms

(a.) $U(7) \approx \mathbb{Z}_6$

[Note $\langle a\rangle \approx \langle b\rangle$ for $|a| = |b| = k$ via the isomorphism $\psi(a^k) = b^k$] [Details in Ex. 2.2.1 on pg. 67]

Note $U(7) = \{1, 2, 3, 4, 5, 6\} = \langle 3\rangle$
since mod 7, $3, 3^2, 3^3, 3^4, 3^5, 3^6 = 3, 2, 6, 4, 5, 1.$

Hence $U(7)$ is cyclic group of order 6 as is $\mathbb{Z}_6$

$\therefore U(7) \approx \mathbb{Z}_6$ (By Ex. 2.2.1)

Alternatively, write Cayley Tables for both and arrange elements in $U(7)$ as to match pattern.

$$\begin{array}{ccccccl} 1, & 3, & 2, & 6, & 4, & 5 & : U(7) \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 1 & 2 & 3 & 4 & 5 & : \mathbb{Z}_6 \end{array}$$

I see 5 is the other generator of $U(7)$ from this.

P46 continued (I'm giving two sol$^{\underline{n}}$s for (a.))

$U(7)$	1	3	2	6	4	5
1	1	3	2	6	4	5
3	3	2	6	4	5	1
2	2		4	5	1	3
6	6	4	5	1	3	2
4	4	5	1	3	2	6
5	5	1	3	2	6	4

$\mathbb{Z}_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Same pattern indicates natural isomorphism:

$$\underline{U(7)} \qquad \begin{array}{ccc} 1 & \longmapsto & 0 \\ 3 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \\ 6 & \longmapsto & 3 \\ 4 & \longmapsto & 4 \\ 5 & \longmapsto & 5 \end{array} \qquad \underline{\mathbb{Z}_6}$$

(b.) $H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \,\middle|\, n \in \mathbb{Q} \right\} \approx \mathbb{Q}$

Let $\left\{ \psi(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \ \forall n \in \mathbb{Q} \right\}$ so $\psi : \mathbb{Q} \longrightarrow H$.

Observe, for $m, n \in \mathbb{Q}$,

$$\psi(m+n) = \begin{bmatrix} 1 & m+n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \psi(m)\psi(n)$$

Hence ψ is homomorphism from $(\mathbb{Q}, +)$ to $H \leq Gl(2, \mathbb{Q})$

ψ is onto, if $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in H$ then $\psi(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

ψ is one-one as $\psi(a) = \psi(b) \Rightarrow \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \Rightarrow a = b$.

Thus ψ is an isomorphism which establishes $\mathbb{Q} \approx H$.

$\boxed{P47}$

(a.) Consider $(\mathbb{Z}_5)^{2\times 2}$ and $Gl_2(\mathbb{R})$. These are not isomorphic since $|(\mathbb{Z}_5)^{2\times 2}| = 5^4 = 625$ whereas $Gl_2(\mathbb{R}) = \{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mid ad-bc \neq 0\}$ has ∞-ly many elements (for example, $cI = \left[\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right]$ for $c \neq 0$) and $\psi : (\mathbb{Z}_5)^{2\times 2} \longrightarrow Gl_2(\mathbb{R})$ a bijection is <u>impossible</u>.

<u>Remark</u>: since bijections preserve cardnality we first need isomorphic groups to have the same size. A <u>finite</u> group cannot be isomorphic to an <u>infinite</u> group.

(b.) $\mathbb{Z}_{222}$ and D_{111} have the same size $|\mathbb{Z}_{222}| = |D_{111}| = 222$. However, $\mathbb{Z}_{222}$ is cyclic; $\mathbb{Z}_{222} = \langle 1 \rangle$ whereas D_{111} only has x with $x^{111} = 1$ as the highest order in D_{111}. In other words, D_{111} not cyclic

$\therefore D_{111} \not\cong \mathbb{Z}_{222}$.

(we could also observe xy and x^2y have order 2 in D_{111} $\therefore$ D_{111} not cyclic)

$$(x^2y)(x^2y) = x^2x^{-2}yy = y^2 = 1.$$

$\boxed{P47}$ continued,

(c.) $\underline{A_4 \not\approx D_6 \text{ why?}}$ 1st note $|A_4| = \frac{4!}{2} = 12 = |D_6|$ so order is not enough to see the distinction. Consider, A_4 consists of <u>even</u> permutations in S_4 hence:

$$A_4 = \{(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$$

In particular, A_4 has 8-elements of order 3.

In contrast, $D_6 = \{1, x, x^2, \underset{\substack{\uparrow \\ \text{order } 2}}{x^3}, x^4, x^5, \underbrace{y, xy, x^2y, x^3y, x^4y, x^5y}_{\text{order } 2}\}$

and $|1| = 1$, $|x| = |x^5| = 6$, $|x^2| = |x^4| = 3$ so as you can see, $D6$ does not have 8-elements of order 3

$\therefore$ $A_4 \not\approx D_6$ as they do not have same # of elements of order 3. (you could compare elements of other orders and find the same disparity)

(d.) $\underline{\mathbb{R}^\times \not\approx \mathbb{C}^\times}$

In $\mathbb{C}^\times$ we have i with $i^2 = -1$ and $i^4 = 1$.

if $\psi : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$ was an isomorphism

then $\psi(i^2) = \psi(ii) = \psi(i)\psi(i) = \psi(-1)$ sol$^{\underline{t}}$'s are ± 1

But, $\psi(1) = \psi(-1)\psi(-1) = 1 \Rightarrow (\psi(-1))^2 = 1$ where

$\psi(-1) \in \mathbb{R}$ and $\psi(-1) \neq 1$ since $\psi(1) = 1 \Rightarrow \underline{\psi(-1) = -1}$.

Thus $(\psi(i))^2 = \psi(-1) = -1$ which is impossible in $\mathbb{R}$.

$\nexists\ x \in \mathbb{R}$ such that $x^2 = -1$. Hence $\psi : \mathbb{C}^\times \to \mathbb{R}^\times$ d.n.e.

and we find $\mathbb{R}^\times \not\approx \mathbb{C}^\times$. Alternatively, see Ex. 2.2.9 in my notes.

P48 Prove $(\mathbb{Q}, +)$ is not isomorphic to any proper subgroup of $\mathbb{Q}$. Gallian, p. 132, # 42

Consider $\frac{p}{q} \in \mathbb{Q}$. We have $\frac{p}{q} = \underbrace{\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}}_{p\text{-fold summands}}$

If $\psi : \mathbb{Q} \to H \leq \mathbb{Q}$ then,

$$\begin{aligned} \psi\left(\frac{p}{q}\right) &= \psi\left(\frac{1}{q} + \cdots + \frac{1}{q}\right) \\ &= \psi\left(\frac{1}{q}\right) + \cdots + \psi\left(\frac{1}{q}\right) \\ &= p\,\psi\left(\frac{1}{q}\right). \end{aligned}$$

Furthermore,

$$\psi\left(\frac{q}{q}\right) = \psi\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right) = \psi\left(\frac{1}{q}\right)$$
$$\Rightarrow \psi(1) = q\,\psi\left(\frac{1}{q}\right)$$
$$\Rightarrow \psi\left(\frac{1}{q}\right) = \frac{1}{q}\psi(1)$$

All together, $\psi\left(\frac{p}{q}\right) = p\,\psi\left(\frac{1}{q}\right) = \frac{p}{q}\psi(1)$.

Or, $*$ $\psi(x) = x\,\psi(1)$ for an isomorphism from $\underbrace{\mathbb{Q} \text{ to } H \leq \mathbb{Q}}_{\text{both under } +}$

If $H \leq \mathbb{Q}$ and $H \neq \{0\}$ and $H \neq \mathbb{Q}$ then $\exists z \in \mathbb{Q}$ for which $z \notin H$.
Also, $H \neq \{0\}$ and $\psi : \mathbb{Q} \longrightarrow H$ an isomorphism implies $\exists y \in H$ for which some $x \in \mathbb{Q}$ has $\psi(x) = y$.
by $(*)$ $\psi(x) = x\,\psi(1) = y$ hence $\psi(1) = y/x$.

Consider, $\psi\left(\frac{zx}{y}\right) = \left(\frac{zx}{y}\right)\psi(1) = \left(\frac{zx}{y}\right)\left(\frac{y}{x}\right) = z \notin H$

thus ψ is not onto H $\therefore \nexists$ an isomorphism from $(\mathbb{Q},+)$ to a proper, nontrivial subgroup of $\mathbb{Q}$ under $+$.