SOLUTION TO LECTURE 13 PROBLEMS 49 – 52

P49 $Q = \{1, -1, i, -i, j, -j, k, -k\}$

$\phantom{Q = \{}1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$

Find homomorphism, well isomorphism from $Q \to$ subgroup of S_8' by examining left-multiplications of Q.

l_{-1} $1 \mapsto -1$, $i \mapsto -i$, $j \mapsto -j$, $k \mapsto -k$

$(12) \qquad (34) \qquad (56) \qquad (78)$

so $l_{-1} \leftrightarrow (12)(34)(56)(78)$.

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l_i $1 \mapsto i \to -1 \to -i$

$j \mapsto k \mapsto -j \to -k \qquad l_i \leftrightarrow (1324)(57\,68)$.

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l_{-i} Note $l_{-i} l_i = l_{-i^2} = l_1 \Rightarrow l_{-i} = l_i^{-1}$ so $\to (4231)(8675)$

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l_j $1 \to j \to -1 \to -j$

$i \to -k \to -i \to k \qquad l_j \Leftrightarrow (1526)(3847)$

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l_{-j} since $l_{-j} = (l_j)^{-1} \longrightarrow (6251)(7483)$

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l_k $1 \mapsto k \to -1 \to -k$

$i \mapsto +j \to -i \to -j \qquad \longrightarrow (1728)(3546)$

$l_{-k} = (l_k)^{-1} \longrightarrow (8271)(6453)$.

In summary, these calculations support the homomorphism $\alpha \longrightarrow l_\alpha \longrightarrow$ cycle in S_8 corresponding to l_α

$$\boxed{Q \approx H = \{(1), (12)(34)(56)(78), (1324)(5768), (4231), (8675) \\ (1526)(3847), (6251)(7483), (1728)(3546), \\ (8271)(6453)\} \leq S_8}$$

$\boxed{P50}$ Let H, K be subgroups of G

(a.) Let $H, K \trianglelefteq G$. Consider $x \in g(H \cap K)g^{-1}$ for some $g \in G$.

Note, $x = gyg^{-1}$ where $y \in H \cap K$ hence $y \in H$ and $y \in K$.

Since $H \trianglelefteq G$ we have $gyg^{-1} \in gHg^{-1} \subseteq H$ $\therefore$ $gyg^{-1} \in H$.

Likewise, $K \trianglelefteq G$ thus $gyg^{-1} \in gKg^{-1} \subseteq K$ $\therefore$ $gyg^{-1} \in K$

Therefore, $gyg^{-1} \in H \cap K$ hence $x = gyg^{-1} \in H \cap K$

which shows $g(H \cap K)g^{-1} \subseteq H \cap K$ $\therefore$ $H \cap K \trianglelefteq G$.

-(I use Th$^{\underline{m}}$ 9.1 in Gallian for convenient characterization of normality)-

(b.) Suppose $|G| = 36$, $|H| = 12$, $|K| = 18$ using Lagrange's Th$^{\underline{m}}$ what are possible orders of $H \cap K$?

Notice $H \cap K \leq H \leq G$ and $H \cap K \leq K \leq G$

thus $|H \cap K| \mid |H| \mid |G|$ and $|H \cap K| \mid |K|$ & $|K| \mid |G|$.

We need $|H \cap K|$ divides both 12 and 18.

$\Rightarrow$ $\boxed{|H \cap K| = 1, 2, 3, 6}$ (can't do 4, 9, 12, 18)

[PS1] Let $H = \{1, x^3, x^6\} \subseteq D_9 = \{1, x, ..., x^8, y, xy, ..., x^8y\}$

Observe $|D_9| = 18$ hence $[D_9 : H] = \frac{18}{3} = 6$

The cosets of H in D_9 are,

$$\boxed{H,\ xH,\ x^2H,\ yH,\ xyH,\ x^2yH} - (*)$$

or explicitly, since $x^{-3} = x^6$ etc...

$\{1, x^3, x^6\}, \{x, x^4, x^7\}, \{x^2, x^5, x^8\}, \{y, yx^3, yx^6\} = \{y, x^6y, x^3y\}$
$, \{xy, xyx^3, xyx^6\} = \{xy, x^7y, x^4y\},$
$, \{x^2y, x^2yx^3, x^2yx^6\} = \{x^2y, x^8y, x^5y\}.$

Consider, $Hy = \{y, x^3y, x^6y\} = yH$
$Hxy = \{xy, x^3xy, x^6xy\} = xyH$
$Hx^2y = \{x^2y, x^3x^2y, x^6x^2y\} = x^2yH$

Likewise, $xH = Hx$, $x^2H = Hx^2$ thus $H \trianglelefteq D_9$.

In retrospect, D_9/H forms a group $(*)$.

[PS2] Let G & H be groups.

Remark: I find easier path

(a.) $\{e\} \times H = \{(e,h) \mid h \in H\}$. Consider

$$(x,y)(\{e\}\times H) = \{(x,y)(e,h) \mid h \in H\}$$
$$= \{(xe, yh) \mid h \in H\}$$
$$= \{(x, k) \mid k \in H\}$$

Can you see why yh for any $h \in H$ gives all of H?

$$(\{e\}\times H)(x,y) = \{(e,h)(x,y) \mid h \in H\}$$
$$= \{(ex, hy) \mid h \in H\}$$
$$= \{(x, k) \mid k \in H\}$$

$\therefore (x,y)(\{e\}\times H) = (\{e\}\times H)(x,y) \Rightarrow \underline{\{e\}\times H \trianglelefteq G \times H}$.

(a.) to see $\{e\} \times H \leq G \times H$ we can identify that $\{e\} \times H = \text{Ker}(\pi_1)$ as $\pi_1(x,y) = x$ has $\pi_1(x,y) = e \Rightarrow x = e$ and $y \in H$.

$$\pi_1((x,y)(a,b)) = \pi_1((xa, yb)) = xa = \pi_1(x,y)\,\pi_1(a,b)$$

so π_1 is a homomorphism and we conclude $\text{Ker}(\pi_1) = \{e\} \times H \trianglelefteq G \times H$

Remark: you can skip what I did on the last page. To show $\{e\} \times H$ normal it suffices to show it is the kernel of a homomorphism! (I should be lazier on last page)

(b.) to show $G \times H \approx H \times G$ consider $\psi : G \times H \longrightarrow H \times G$ def^n by $\psi(g,h) = (h,g)$

Notice, if $(h,g) \in H \times G$ then $\psi(g,h) = (h,g) \therefore \psi$ onto.

Also, $\psi(x,y) = \psi(a,b) \Rightarrow (y,x) = (b,a) \Rightarrow \underline{y = b} \;\&\; \underline{x = a}$ thus $(x,y) = (a,b)$ and we find ψ injective. It remains to show ψ a homomorphism,

$$\begin{aligned}
\psi((x,y)(a,b)) &= \psi((xa, yb)) &&: \text{def}^n \text{ of product in } G \times H \\
&= (yb, xa) &&: \text{def}^n \text{ of } \psi \\
&= (y,x)(b,a) &&: \text{def}^n \text{ of product in } H \times G \\
&= \psi(x,y)\,\psi(a,b) &&: \text{def}^n \text{ of } \psi
\end{aligned}$$

Thus ψ is a bijective homomorphism and so, $G \times H \approx H \times G$.