SOLUTION TO LECTURE 14 PROBLEMS 53 – 56

$\boxed{P53}$ Let $|G| = 6$. Prove $\exists x \in G$ with $|x| = 2$.
However, prove there cannot be 5 such elements of order 2.

If $G = \langle a \rangle = \{1, a, a^2, a^3, a^4, a^5\}$ then $(a^3)^2 = 1$ so $|a^3| = 2$.

If G is not cyclic then $e, a, b \in G$ where e is identity and $b \neq a^k$ for any k. The $\langle a \rangle \leq G$ has either 2 or 3 elements by Lagrange's Th$\underline{m}$. If $|\langle a \rangle| = 2$ then $a^2 = 1$ or $|a| = 2$. If $|\langle a \rangle| = 3$ then $G = \{e, a, a^2, b, x, y\}$ and $\langle b \rangle \leq G$ has, by Lagrange's Th $|\langle b \rangle| = 2$ or 3 so $\langle b \rangle = \{e, b\}$ or $\langle b \rangle = \{e, b, b^2\}$

If $|\langle b \rangle| = 2$ then G has element of order 2.

If $|\langle b \rangle| = 3$ then $G = \{e, a, a^2, b, b^2, y\}$ and as y is not a power of a or b we forced to see $|\langle y \rangle| = 2$.

Suppose $|G| = 6$ and all elements besides e have order 2. Consider, $a \neq b \neq e$ with $a^2 = e, b^2 = e$ then ab is another element of G hence $(ab)^2 = e$.

Observe, $\{e, a, b, ab\} \leq G$ but $4 \nmid 6$ hence $\nexists\, G$ of order 6 with all elements order 2 except e.

P54 Let G be order 6 group with $a, b \in G$ and $|a| = 3$, $|b| = 2$. Show either G is cyclic or $ab \neq ba$

Consider, $G = \{e, a, a^2, b, x, y\}$. What can we set x, y to be?

1.) $x = ab = ba$

note,
$$x^2 = a^2 b^2 = a^2$$
$$x^3 = aba^2 = a^3 b = b$$
$$x^4 = abb = a$$
$$x^5 = aba = a^2 b$$
$$x^6 = (ab)(a^2 b) = a^3 b^2 = e.$$

Hence $|x| = 6$ $\therefore$ G is cyclic.

2.) $x = ab \neq ba = y$.

P55 Let G be group of order 6. Suppose G not cyclic. Show the Cayley Table of G matches S_3

G	e	a	a^2	b	ab	ba
e	e	a	a^2	b	ab	ba
a	a	a^2	e	ab	ba	b
a^2	a^2	e	a	ba	b	ab
b	b	ba	ab	e	a^2	a
ab	ab	b	ba	a	e	a^2
ba	ba	ab	b	a^2	a	e

forced by need to not repeat ba in column 6 and the a, a^2, e, ab already in row 2. Hence, b must then go

S_3	(1)	(123)	(132)	(13)	(23)	(12)
(1)	(1)	(123)	(132)	(13)	(23)	(12)
(123)	(123)	(132)	(1)	(23)	(12)	(13)
(132)	(132)	(1)	(123)	(12)	(13)	(23)
(13)	(13)	(12)	(23)	(1)	(123)	(123)
(23)	(23)	(13)	(12)	(123)	(1)	(132)
(12)	(12)	(23)	(13)	(132)	(123)	(1)

We can see
$$\psi(a) = (123)$$
$$\psi(b) = (13)$$
defines an isomorphism.

$\boxed{P55}$ <u>continued</u>

we found $a(ab) = ba$ and $a(ba) = b$

notice $a^2 b = ba \Rightarrow a^2 b a^2 = ba^3 = b$ etc...

I'll let the reader confirm it, but we can simplify products of a, b to e, a, a^2, b, ab, ba given $a^3 = e$, $b^2 = e$ and $a^2 b = ba$.

$$S_3 \approx \underbrace{\langle a, b \mid a^3 = e,\ b^2 = e,\ a^2 b = ba \rangle}$$

this notation means to form all possible products of a & b subject the <u>relations</u> $a^3 = e$, $b^2 = e$, $a^2 b = ba$.

If $a = (123)$, $b = (13)$ then we can replace $\approx$ with $=$ in the above; $S_3 = \langle a, b \mid a^3 = e,\ b^2 = e,\ a^2 b = ba \rangle$.

$\boxed{P56}$ Suppose $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$

Recall $\pi_1 : G_1 \to G_1/N_1$ where $\pi_1(x) = xN_1$

and $\pi_2 : G_2 \to G_2/N_2$ defined by $\pi_2(y) = yN_2$

are the <u>natural</u> <u>homomorphisms</u> with

$\text{Ker}\, \pi_1 = N_1$ and $\text{Ker}\, \pi_2 = N_2$.

Moreover, $\pi_1(G_1) = G_1/N_1$ & $\pi_2(G_2) = G_2/N_2$

that is, π_1 & π_2 are surjections. Construct,

$$\psi = \pi_1 \times \pi_2 : G_1 \times G_2 \to (G_1/N_1) \times (G_2/N_2)$$

$$\psi(x,y) = (\pi_1(x), \pi_2(y))$$

Remark: showing ψ a homomorphism onto $\frac{G_1}{N_1} \times \frac{G_2}{N_2}$ with kernel $N_1 \times N_2$ solves the problem by 1st iso. Th^{m}

Notice, ψ is a homomorphism since:

$$\begin{aligned}
\psi((x,y)(a,b)) &= \psi((xa, yb)) \\
&= (\pi_1(xa), \pi_2(yb)) \\
&= (\pi_1(x)\pi_1(a), \pi_2(y)\pi_2(b)) \\
&= (\pi_1(x), \pi_2(y))(\pi_1(a), \pi_2(b)) \\
&= \psi(x,y)\,\psi(a,b)
\end{aligned}$$

Also, ψ is a surjection as π_1, π_2 are surjections and

$$\begin{aligned}
\text{Ker}\, \psi &= \{(x,y) \in G_1 \times G_2 \mid (\pi_1(x), \pi_2(y)) = (N_1, N_2)\} \\
&= \{(x,y) \in G_1 \times G_2 \mid x \in N_1,\ y \in N_2\} \\
&= N_1 \times N_2
\end{aligned}$$

Thus <u>$N_1 \times N_2 \trianglelefteq G_1 \times G_2$</u>. and by 1st isom. Th^{m}, $\underline{\dfrac{G_1 \times G_2}{N_1 \times N_2} \cong \dfrac{G_1}{N_1} \times \dfrac{G_2}{N_2}}$