SOLUTION TO LECTURE 15 PROBLEMS 57-60

$\boxed{P57}$ Let $\phi : \mathbb{Z}_n \longrightarrow \mathbb{Z}_k$ be a surjective homomorphism.
The first isomorphism theorem provides that

$$|\mathbb{Z}_n| \quad \mathbb{Z}_n / \ker\phi \approx \mathbb{Z}_k$$

Thus, using $|\mathbb{Z}_n| = n$, $|\mathbb{Z}_k| = k$ we obtain
from $|G/H| = \frac{|G|}{|H|}$ that $\frac{n}{|\ker\phi|} = k \Rightarrow n = k|\ker\phi|$
$\therefore k | n$.

Remark: my apologies once more for the original statement here...

$\boxed{P58}$ Let $m\mathbb{Z}_n = \{mx \mid x \in \mathbb{Z}_n\}$. Show that
$\mathbb{Z}_n / m\mathbb{Z}_n \approx \mathbb{Z}_m$ given an appropriate condition.

Consider $\psi([x]_n) = [x]_m$ since we want a surjection
it is clear we need $m|n$ (by $\boxed{P57}$). I'll show
ψ is well-defined, if $[x]_n = [x']_n$ then $x' = x + nk$
for some $k \in \mathbb{Z}$,

$$\begin{aligned} \psi([x']_n) &= [x+nk]_m \qquad \text{since } m|n \Rightarrow n = m\ell \\ &= [x + m\ell k]_m \\ &= [x]_m \qquad \therefore \psi \text{ is well-defined.} \end{aligned}$$

Next, ψ is clearly a homomorphism,

$$\begin{aligned} \psi([x]_n + [y]_n) &= \psi([x+y]_n) \\ &= [x+y]_m \\ &= [x]_m + [y]_m \\ &= \psi([x]_n) + \psi([y]_n). \end{aligned}$$

Moreover, $\ker\psi = \{[x]_n \mid [x]_m = 0\} = \{[x]_n \mid x = mk, k \in \mathbb{Z}\}$
then $\ker\psi = \{[mk]_n \mid k \in \mathbb{Z}\} = m\mathbb{Z}_n$. The 1st isomorphism
Th$^{\underline{m}}$ $\mathbb{Z}_n / m\mathbb{Z}_n \approx \mathbb{Z}_m$.

[P59] Let $\phi(\theta) = \cos\theta + i\sin\theta$.

Notice $\cos\theta$ and $\sin\theta$ are never both zero for the same θ, thus $\phi : \mathbb{R} \to \mathbb{C}^{\times}$. It remains to show ϕ is a homomorphism,

$$\begin{aligned}\phi(\theta+\beta) &= \cos(\theta+\beta) + i\sin(\theta+\beta)\\ &= \cos\theta\cos\beta - \sin\theta\sin\beta + i(\sin\theta\cos\beta + \sin\beta\cos\theta)\\ &= (\cos\theta + i\sin\theta)(\cos\beta + i\sin\beta)\\ &= \phi(\theta)\,\phi(\beta).\end{aligned}$$

Finally, observe $Im(\phi) = \{\phi(\theta) \mid \theta \in \mathbb{R}\} = \{\cos\theta + i\sin\theta \mid \theta \in \mathbb{R}\}$. thus $S^1 = \{\cos\theta + i\sin\theta \mid \theta \in \mathbb{R}\} \leq \mathbb{C}^{\times}$ as we have shown the image of a homomorphism is a subgroup.

[P60] Gallian #7 on page 169

$$HK = \{ xy \mid x \in H,\ y \in K \}$$

To count the # of objects in HK we must determine when $xy = x'y'$ for $x, x' \in H$ and $y, y' \in K$.
If $\exists\, x, x' \in H$ and $y, y' \in K$ for which $xy = x'y'$ then $(x')^{-1}x = y'y^{-1} \in H \cap K$ (since $(x')^{-1}x \in H$ and $y'y^{-1} \in K$)

Therefore, when distinct $(x,y), (x',y')$ yield $xy = x'y'$ it must be $xy = x'y' \in H \cap K$. Furthermore, for each $z \in H \cap K$ and $xy \in HK$ $(x \in H,\ y \in K)$ we calculate $xy = (xz)(z^{-1}y)$ thus every element in HK can be represented by at least $|H \cap K|$ products.
It follows by counting,

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

$\boxed{P60}$ (this proof from Dummit & Foote, pg. 93, Prop 13)

We note $HK = \bigcup_{h \in H} hK$. Each coset hK has $|K|$-elements.

Thus, to count the number of elements in HK we need to determine the # of distinct K-cosets in HK. Suppose $h_1, h_2 \in H$ and

$$
\begin{aligned}
h_1 K = h_2 K &\iff h_1 h_2^{-1} \in K \text{ where } h_1, h_2 \in H \\
&\iff h_1 h_2^{-1} \in H \cap K \\
&\iff h_1 (H \cap K) = h_2 (H \cap K)
\end{aligned}
$$

Thus, the # of distinct cosets of form hK is same as # of distinct $H \cap K$ cosets in H. We know $H \cap K \leq H$ hence by Lagrange's Th$^{\underline{m}}$ $[H : H \cap K] = \frac{|H|}{|H \cap K|}$.

Hence HK has $\frac{|H|}{|H \cap K|}$-cosets of the form hK each of which contains $|K|$-elements $\therefore$ $\underline{|HK| = \frac{|H||K|}{|H \cap K|}}$

Remark: HK need not be a group. Dummit & Foote provide $G = S_3$, $H = \langle (12) \rangle$, $K = \langle (23) \rangle$ as an example. Note $H \cap K = \{(1)\}$ and $|H| = |K| = 2$

So, $|HK| = \frac{2(2)}{1} = 4 \nmid 6 \therefore H \not\leq G$.