SOLUTION TO LECTURE 16 PROBLEMS 61 – 64

$\boxed{P61}$ Let $H_1, H_2, \dots, H_K \trianglelefteq G$. Show $H_1 H_2 \cdots H_k \trianglelefteq G$.

Let $H_1, \dots, H_k \trianglelefteq G$. Let $a, b \in H_1 H_2$ where $a = a_1 a_2$ and $b = b_1 b_2$ for $a_1, b_1 \in H_1$ and $a_2, b_2 \in H_2$. Since $H_2 \trianglelefteq G$ we have $b_1^{-1} H_2 = H_2 b_1^{-1}$ and so $a_2 b_2^{-1} \in H_2 \Rightarrow a_2 b_2^{-1} b_1^{-1} \in H_2 b_1^{-1}$ and $\exists\, h_2 \in H_2$ such that $\underline{a_2 b_2^{-1} b_1^{-1} = b_1^{-1} h_2}$ $*$. Thus,

$$\begin{aligned} ab^{-1} &= (a_1 a_2)(b_1 b_2)^{-1} \\ &= a_1 a_2 b_2^{-1} b_1^{-1} \quad &&\text{: socks - shoes} \\ &= a_1 b_1^{-1} h_2 &&\text{: by } * \end{aligned}$$

thus $ab^{-1} \in H_1 H_2$ as $a_1 b_1^{-1} \in H_1$ and $h_2 \in H_2$. Moreover, $e \in H_1 \cap H_2$ and $ee = e$ $\therefore$ $e \in H_1 H_2 \neq \emptyset$. It remains to show $H_1 H_2 \trianglelefteq G$. Consider $z \in g H_1 H_2 g^{-1}$ for some $g \in G$. Notice $z = g x_1 x_2 g^{-1}$ for some $x_1 \in H_1$ and $x_2 \in H_2$. Moreover, as $gH_1 = H_1 g$ and $H_2 g^{-1} = g^{-1} H_2$ there exist $x_1' \in H_1$ and $x_2' \in H_2$ such that $g x_1 = x_1' g$ and $x_2 g^{-1} = g^{-1} x_2'$ thus,

$$z = g x_1 x_2 g^{-1} = x_1' g g^{-1} x_2' = x_1' x_2' \in H_1 H_2$$

Hence, $g H_1 H_2 g^{-1} \subseteq H_1 H_2$ and we find $H_1 H_2 \trianglelefteq G$.

Consider $H_1 H_2 H_3 = H' H_3$ and as $H' \trianglelefteq G$ we may apply our argument above with $H' \to H_1$, $H_3 \to H_2$ and deduce $H' H_3 \trianglelefteq G$ thus $H_1 H_2 H_3 \trianglelefteq G$. Continue in this fashion until we obtain $H_1 H_2 \cdots H_k \trianglelefteq G$.

(we could clean this up with an induction, if $H_1 \cdots H_n \trianglelefteq G$ then $H_1 \cdots H_n H_{n+1} = H' H_{n+1} \trianglelefteq G$ $\therefore$ by induction true for $n \le k$.)

P62 Chpt. 8 # 32 pg. 163

$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \ a, b \in \mathbb{Z}_3 \right\}$$

Show that H is an Abelian group of order 9.
Is $H \approx \mathbb{Z}_9$ or is $H \approx \mathbb{Z}_3 \times \mathbb{Z}_3$?

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+a & y+b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a+x & b+y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

thus H is abelian. Since $a \in \mathbb{Z}_3$ and $b \in \mathbb{Z}_3$ there are 9 choices for $a, b \Rightarrow |H| = 9$. It is convenient to express H as $\mathbb{Z}_3^2$ with $*$

$$(a, b) * (x, y) = (a+x, b+y)$$

quite clearly $H \approx \mathbb{Z}_3 \times \mathbb{Z}_3$. We could show $\psi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = (a, b)$ is an isomorphism of H & $\mathbb{Z}_3 \times \mathbb{Z}_3$.. However, perhaps it's easier to simply exhibit two subgroups of order 3.

$$\left\langle \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$\left\langle \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Hence H is <u>not</u> cyclic as cyclic groups have just one subgroup of each possible order, $\therefore$ $H \not\approx \mathbb{Z}_9$.

[P63] Chpt. 8 # 41 from pg. 163 Gallian

Determine # of elements of order 15 and # of cyclic subgroups of order 15 in $\mathbb{Z}_{30} \times \mathbb{Z}_{20}$

Key concept, $|(a,b)| = \text{lcm}(|a|, |b|)$ and $|a|$ must divide 30 whereas $|b| \,/\, 20$ as $a \in \mathbb{Z}_{30}$ and $b \in \mathbb{Z}_{20}$. Cases,

1.) $\underbrace{|a| = 15}$, $|b| = \underbrace{1, 5}_{1+4 \text{ choices.}}$

$\phi(15) = \phi(3)\phi(5) = 2(4) = 8$ choices $\Big\}$ $5(8) = 40$ choices.

2.) $\underbrace{|a| = 3}_{2 \text{ such elements}}$, $\underbrace{|b| = 5}_{4 \text{ choices}}$ $\longrightarrow$ 8 more possibilities.

There are no other ways to obtain 15 from the allowed a, b ($|a| = 5$, $|b| = 3$ no good as $3 \nmid 20$.)

In summary, $\boxed{48 \text{ elements of order } 15}$

Since $\phi(15) = 8$ each subgroup of order 15 contains 8-elements of order 15 and we find $48/8 = \boxed{6 \text{ subgroups of order } 15.}$

P64 Gallian Chpt. 11 # 8, pg. 219

Show that $\exists$ two Abelian groups of order 108 that have exactly 13 subgroups of order 3.

Consider, $108 = 4(27) = 4(3)^3$ hence the Fun. Th$^{\underline{m}}$ of finite abelian groups,

$\mathbb{Z}_4 \times \mathbb{Z}_{27}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27}$
$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9$
$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9$

are the distinct rep. of non-isomorphic groups of order 108. Subgroup of order 3,

$(a, b) \in \mathbb{Z}_4 \times \mathbb{Z}_{27} \Rightarrow |a| = 1,\ |b| = 3$ ... no good.

Consider $(a, b, c) \in \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9$ to obtain $\text{lcm}(|a|, |b|, |c|) = 3$ need $|a| = 1,\ |b| = 3, 1,\ |c| = 3, 1$

So I count $(2+1)(2+1) = 9$ elements of order 3.

Next, $(a, b, c, d) \in \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ with order 3 requires $|a| = 1,\ |b| = 1, 3,\ |c| = 1, 3,\ |d| = 1, 3$. Break it down,

1.) $|b| = 1$ then $|c| = 1$ then $|d| = 3 \Rightarrow$ 2 choices
$|b| = 1$ and $|c| = 3$ then $|d| = 1$ or $3 \Rightarrow 2(3) = 6$ choices
$|b| = 3$ then $|c| = 1, 3$ and $|d| = 1$ or $3 \Rightarrow 2(3)(3) = 18$ choices

Hence, $\exists$ 26 elements of order 3 in $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ likewise for nearly the same reasons $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ has 26 order 3 elements. Since $\phi(3) = 2$ it follows each subgroup of order 3 has 2 elements of order 3 $\therefore \frac{26}{2} = 13$ sub. groups of order 3 in both $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.