SOLUTIONS TO LECTURE 17 PROBLEMS 65–68

P65 Gallian #3 from pg. 169

Def^{n}/ the <u>commutator subgroup</u> G' of G is by $G' = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$ the notation $\langle \ \rangle$ indicates we use $x^{-1}y^{-1}xy$ to <u>generate</u> G'. So, $a_1^{i_1}a_2^{i_2}\cdots a_n^{i_n}$ is a typical element of G' where $a_i = x_i^{-1}y_i^{-1}x_iy_i$ for each $i = 1, 2, \dots, n$ and $x_i, y_i \in G$ and $i_j = \pm 1$ for $j = 1, 2, \dots, n$. Prove G' is <u>characteristic</u> subgroup of G. (#1 tells us $N \leq G$ is <u>characteristic subgroup</u> if $\phi(N) = N$ for all $\phi \in Aut(G)$)

Let $\phi \in Aut(G)$. Consider, for $x, y \in G$,

$$\begin{aligned}
\phi(x^{-1}y^{-1}xy) &= \phi(x^{-1})\phi(y^{-1})\phi(x)\phi(y) &&: \phi \text{ homom.} \\
&= \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) &&: \phi \text{ homomorphism} \\
&= a^{-1}b^{-1}ab &&\text{where } a = \phi(x),\ b = \phi(y).
\end{aligned}$$

Thus ϕ sends generator of G' to generator of G'. If $a_1^{i_1}a_2^{i_2}\cdots a_n^{i_n} \in G'$ where $a_i = x_i^{-1}y_i^{-1}x_iy_i$ then $\phi(a_1^{i_1}\cdots a_n^{i_n}) = \phi(a_1)^{i_1}\phi(a_2)^{i_2}\cdots\phi(a_n)^{i_n} \in G'$ thus $\phi(G') \subseteq G'$. However, by the same argument $\phi^{-1}(G') \subseteq G'$ and we deduce $G' \subseteq \phi(G')$ and so $\phi(G') = G'$ $\therefore$ G' is characteristic.

$\boxed{P66}$ Gallian #6 from pg. 169

Let $H, K \leq G$ and $HK = \{hk \mid h \in H, k \in K\}$ and $KH = \{kh \mid k \in K, h \in H\}$. Prove that HK is a group $\Leftrightarrow HK = KH$

$\Leftarrow$] Suppose $HK = KH$. Let $a, b \in HK$ thus there exist $h_1, h_2 \in H$ and $k_1, k_2 \in K$ for which $a = h_1 k_1 = k_2 h_2$ and $\exists\, h_3, h_4 \in H$ and $k_3, k_4 \in K$ for which $b = h_3 k_3 = k_4 h_4$.

Consider
$$ab^{-1} = (h_1 k_1)(h_3 k_3)^{-1}$$
$$= h_1 k_1 k_3^{-1} h_3^{-1} \quad : \text{socks-shoes in group } G \text{ for which } HK \subseteq G.$$
$$= h_1 h_5 k_6$$

noting $k_1 k_3^{-1} \in K$ and $h_3^{-1} \in H$ we have $k_1 k_3^{-1} h_3^{-1} \in KH$ and as $KH = HK$ we know $k_1 k_3^{-1} h_3^{-1} \in HK$ which implies $\exists\, h_5 \in H$, $k_6 \in K$ s.t. $h_5 k_6 = k_1 k_3^{-1} h_3^{-1}$.

Thus $ab^{-1} \in HK$. Moreover, $e \in H$ and $e \in K$ thus $ee \in HK \neq \emptyset$ thus by one-step-subgroup test we deduce $HK \leq G$.

$\Rightarrow$] Suppose $HK \leq G$. Observe $x \in H$ has $x = xe \in HK$ thus $H \subseteq HK$ and as $H \leq G$ we have $\underline{H \leq HK}_{*}$. Likewise $\underline{K \leq HK}_{**}$.

Suppose $z \in KH$ then $z = kh$ for $k \in K$ and some $h \in H$ thus z is formed by product of elements in $HK \leq G$ using $*$ & $**$. and thus $z \in HK \Rightarrow KH \subseteq HK$. Conversely, suppose $u \in HK$ so $\exists\, h_2 \in H, k_2 \in K$ s.t. $u = h_2 k_2$. Since $HK \leq G$ $\exists\, a \in HK$ for which $a^{-1} = h_2 k_2$. Suppose $a = h_3 k_3$ for $h_3 \in H$ and $k_3 \in K$ then $h_2 k_2 = (h_3 k_3)^{-1} = k_3^{-1} h_3^{-1} \in KH$

$\therefore u \in KH$ and $HK \subseteq KH$ and we conclude $HK = KH$. //

Remark: one can write less. See pg. 94 of Dummit & Foote.

$\boxed{P67}$ Gallian # 54 from pg. 164 / Show $U(144) \approx U(140)$

$$U(144) = U(12^2) = U(3^2 4^2) \approx U(3^2) \times U(4^2) \quad : \gcd(3^2, 4^2) = 1.$$

$$\approx \mathbb{Z}_{9-3} \times U(2^4) \quad : U(p^n) = \mathbb{Z}_{p^n - p^{n-1}} \text{ odd prime}$$

$$\approx \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \quad : U(2^n) \approx \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{n-2}} \approx \mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^n}$$

$$\approx \underline{\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_4}.$$

Likewise,

$$U(140) = U(20 \cdot 7) = U(4 \cdot 5 \cdot 7)$$

$$\approx U(4) \times U(5) \times U(7)$$

$$\approx \underline{\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6}$$

But, $U(144) \approx \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \approx \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6 \approx U(140)$. //

$\boxed{\text{Remark: can you prove } G \times H \approx H \times G \text{ ?}}$

$\boxed{P68}$ (a.) Suppose G, H, K are finite groups. Suppose $G \oplus H = G \oplus K$ which means $G \cap H = \{e\}$ & $G \cap K = \{e\}$ and $G, H, K \trianglelefteq G' = G \oplus H = G \oplus K$. Moreover, we have unique $g \in G$, $h \in H$ to represent $z \in G \oplus H$ that is, $z = gh$. Likewise $y = gk$ for each $y \in G \oplus K$ for unique $g \in G$ and $k \in K$. We have all of this to work with here. I'll use an 1st isomorphism argument,

$$\phi : G \oplus H \to G \oplus H$$

$\phi(gh) = h$ has $\operatorname{Im} \phi = H$ and $\operatorname{Ker} \phi = G$

Hence, as ϕ is homomorphism, $G \oplus H / G \approx H$. By nearly the same argument $G \oplus K / G \approx K$. Hence,

$$H \approx \frac{G \oplus H}{G} = \frac{G \oplus K}{G} \approx K \quad \therefore \quad \underline{H \approx K}.$$

P68 continued:

(b.) find G, H, K possibly infinite for which
$$G \oplus H = G \oplus K \not\Rightarrow H \approx K$$

Well, I think this part is bogus.
Sorry, but, I see no reason our proof from (a.) does not extend here. This is probably what I meant,
$GH = GK \not\Rightarrow H \approx K$. For example,
$$(\mathbb{R}^\times)(\mathbb{R}^\times) = (\mathbb{R}^\times)(\{1\}) \quad \text{yet} \quad \mathbb{R}^\times \not\approx \{1\}.$$

(c.) Show $G \oplus H = G \oplus K \not\Rightarrow H = K$

Infinite example, $\mathbb{R}^2 = e_1\mathbb{R} \oplus e_2\mathbb{R} = e_1\mathbb{R} \oplus (e_1+e_2)\mathbb{R}$
yet $e_2\mathbb{R} \neq (e_1+e_2)\mathbb{R}$. (from linear algebra)

Finite Example: $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$

observe,

$$G = \langle (0,1) \rangle = \{(0,0), (0,1)\}$$
$$H = \langle (1,0) \rangle = \{(0,0), (1,0)\}$$
$$K = \langle (1,1) \rangle = \{(0,0), (1,1)\}$$

has $G \cap H = \{(0,0)\} = G \cap K$ and
$G + H = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G + K = \mathbb{Z}_2 \times \mathbb{Z}_2$
yet $H \neq K$ despite $G \oplus H = G \oplus K$.
(it is true that $H \approx K$ in this example, indeed part (a.) makes this inevitable)