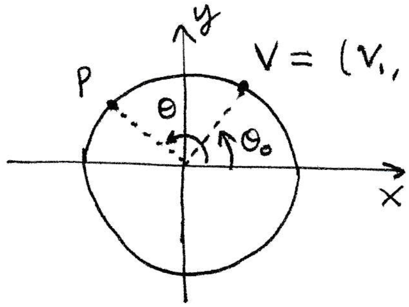
SOLUTION TO LECTURE 18 PROBLEMS 69 – 72

**P69** Let $G = O(2,\mathbb{R}) = \{ R \in \mathbb{R}^{2\times 2} \mid R^T R = I \}$ act on $\mathbb{R}^2$ via matrix multiplication; $R * v = Rv \quad \forall v \in \mathbb{R}^2, R \in G.$ Calculate $\mathcal{O}(v)$ for each $v \in \mathbb{R}^2$ and $G_v$ for each $v \in \mathbb{R}^2$

The orbit of $v = 0$ is easy to calculate as $R(0) = 0$ for all $R \in G$ $\therefore$ $\boxed{\mathcal{O}(0) = \{0\}}$. Suppose $v \neq 0$ consider $R * v = Rv$ has $\|Rv\|^2 = (Rv)\cdot(Rv) = (Rv)^T Rv$ hence $\|Rv\|^2 = v^T R^T R v = v^T I v = v^T v = \|v\|^2$. We find the points in the orbit of $v$ are the same distance from the origin $(0,0)$ as is $v$. Thus $\mathcal{O}(v)$ is a subset of the circle $x^2 + y^2 = \|v\|^2$.

Polar coordinates give $r = \|v\|$ hence an arbitrary point on the circle is $(\|v\|\cos\theta, \|v\|\sin\theta) = P$

$v = (v_1, v_2) = \|v\| (\cos\theta_0, \sin\theta_0)$




To obtain $P$ in $\mathcal{O}(v)$ we consider the rotation by $\theta - \theta_0 = \beta$.

Let $R = \begin{bmatrix} \cos(\theta-\theta_0) & -\sin(\theta-\theta_0) \\ +\sin(\theta-\theta_0) & \cos(\theta-\theta_0) \end{bmatrix}$ you can check $R^T R = I \therefore R \in G$

and $R * v = \begin{bmatrix} \cos(\theta-\theta_0) & -\sin(\theta-\theta_0) \\ +\sin(\theta-\theta_0) & \cos(\theta-\theta_0) \end{bmatrix} \begin{bmatrix} \|v\|\cos\theta_0 \\ \|v\|\sin\theta_0 \end{bmatrix}$

$= \left( \|v\| \left[\cos(\theta-\theta_0)\cos\theta_0 - \sin(\theta-\theta_0)\sin\theta_0\right], \|v\|\left[\sin(\theta-\theta_0)\cos\theta_0 + \cos(\theta-\theta_0)\sin\theta_0\right]\right)$

$= \|v\| \left( \cos(\theta - \theta_0 + \theta_0), \sin(\theta - \theta_0 + \theta_0)\right)$

$= \|v\| (\cos\theta, \sin\theta)$

$= P.$ $\therefore$ $\boxed{\mathcal{O}(v) = \text{circle centered about } (0,0) \text{ through } v}$

P69 continued

We've shown $\boxed{\mathcal{O}(0) = \{0\}}$ and $\boxed{\mathcal{O}(v) = \{ w \,/\, \|w\|^2 = \|v\|^2 \}}$ for $v \neq 0$.

Consider, $O(2,\mathbb{R})$ has normal subgroup $SO(2,\mathbb{R})$ where

$$SO(2,\mathbb{R}) = \operatorname{Ker}(\det) = \{ R \in O(2,\mathbb{R}) \mid \det(R) = 1 \} = H$$

Furthermore, $O(2,\mathbb{R}) = H \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H$ as $\det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1$.

Now, it can be shown $R \in SO(2,\mathbb{R})$ has form $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

for some $\theta \in \mathbb{R}$. Hence $R \in \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H$ has the form,

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

So in total, $R \in O(2,\mathbb{R})$ has the form $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\varepsilon\sin\theta & \varepsilon\cos\theta \end{bmatrix}$

where $\varepsilon = 1$ for $R \in SO(2,\mathbb{R})$ and $\varepsilon = -1$ for $R \in \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} SO(2,\mathbb{R})$.

Let $(x,y) \in \mathbb{R}^2$ be fixed point of $R = \begin{bmatrix} \cos\theta & \sin\theta \\ \varepsilon\sin\theta & \varepsilon\cos\theta \end{bmatrix}$,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\varepsilon\sin\theta & \varepsilon\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \implies \begin{aligned} x\cos\theta + y\sin\theta &= x \\ -\varepsilon x\sin\theta + \varepsilon\cos\theta\, y &= y \end{aligned}$$

Here think of $\cos\theta$, $\sin\theta$ as the unknowns and write,

$$\begin{bmatrix} x & y \\ \varepsilon y & -\varepsilon x \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad (*)$$

This linear system has $\det\begin{bmatrix} x & y \\ \varepsilon y & -\varepsilon x \end{bmatrix} = -\varepsilon(x^2+y^2)$.

When $x^2+y^2 = 0$ then $(x,y) = (0,0)$ and we find $R(0,0) = (0,0)$ $\forall R \in O(2)$.

When $x^2+y^2 \neq 0$ then $-\varepsilon(x^2+y^2) = \mp(x^2+y^2) \neq 0$ $\therefore$ for fixed $\varepsilon$ there exists a unique sol$^{\underline{n}}$ to $(*)$. Continued, P69 continued

$\varepsilon = 1$ $\begin{bmatrix} x & y \\ y & -x \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ for $(x,y) \neq (0,0)$

$$\Rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \frac{1}{-x^2-y^2}\begin{bmatrix} -x & -y \\ -y & x \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{x^2+y^2}\begin{bmatrix} x^2+y^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So $\cos\theta = 1$ and $\sin\theta = 0$ $\therefore$ $\underline{\theta = 0}$.

$$\therefore \underline{R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}.$$

$\varepsilon = -1$ $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ for $(x,y) \neq (0,0)$

$$\Rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \frac{1}{x^2+y^2}\begin{bmatrix} x & -y \\ y & x \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{x^2+y^2}\begin{bmatrix} x^2-y^2 \\ 2xy \end{bmatrix}$$

$$\therefore \cos\theta = \frac{x^2-y^2}{x^2+y^2} \quad \text{and} \quad \sin\theta = \frac{2xy}{x^2+y^2}$$

$$R_v = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \frac{1}{x^2+y^2}\left[\begin{array}{c|c} x^2-y^2 & 2xy \\ \hline 2xy & y^2-x^2 \end{array}\right]$$

Observe, $R_v\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{x^2+y^2}\begin{bmatrix} x(x^2-y^2) + 2xy^2 \\ 2x^2y + y(y^2-x^2) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Thus, $\boxed{G_v = \{ I, R_v \} \text{ for } v = (x,y) \neq (0,0)}$

P70 Gallian # 27 from pg. 146

Let $H, K$ be subgroups of a finite group $G$ and $H \leq K \leq G$. Prove $[G:H] = [G:K][K:H]$

By Lagrange's Th$^{m}$, $[G:K] = \#$ of $K$-cosets in $G = \dfrac{|G|}{|K|}$

likewise, $[K:H] = \#$ of $H$-cosets in $K = \dfrac{|K|}{|H|}$

and $[G:H] = \dfrac{|G|}{|H|} = \dfrac{|G|}{|K|} \cdot \dfrac{|K|}{|H|} = [G:K][K:H]$. //

Remark: we needed finiteness of $H, K, G$ to apply Lagrange Th$^{m}$.

P71 Find all homomorphisms from $\mathbb{Z}_5$ to $\mathbb{Z}_7$

Clearly $\phi : \mathbb{Z}_5 \longrightarrow \mathbb{Z}_7$ defined by $\phi(x) = 0 \ \forall x \in \mathbb{Z}_5$ defines a function with $\phi(x+y) = \phi(x) + \phi(y) \ \forall x, y \in \mathbb{Z}_5$. This $\phi$ is the trivial homomorphism.

Suppose $\psi : \mathbb{Z}_5 \longrightarrow \mathbb{Z}_7$ is a nontrivial map.

Observe
$$\left.\begin{aligned} 0 = \psi(0) = \psi(5(1)) &= \psi(1+1+1+1+1) \\ &= \psi(1) + \psi(1) + \psi(1) + \psi(1) + \psi(1) \\ &= 5\psi(1) \end{aligned}\right\} *$$

Hence, $5\psi(1) = 0$ in $\mathbb{Z}_7 \Rightarrow |\psi(1)| \mid 5 \Rightarrow |\psi(1)| = 1$ or $5$.

But, $\langle \psi(1) \rangle \leq \mathbb{Z}_7 \Rightarrow |\psi(1)| = 1$ or $|\psi(1)| = 7$ thus $|\psi(1)| = 1$

which implies $\psi(1) = 0$ and $\underbrace{\psi(x) = x\psi(1)}_{\text{using } * \text{ generalized}} = x(0) = 0$. Thus $\psi$ is the trivial homomorphism.

which we know from previous studies...

This contradicts $\psi \not\equiv \phi$. Thus,

$\phi \equiv 0$ is the only homomorphism from $\mathbb{Z}_5 \longrightarrow \mathbb{Z}_7$.

$\boxed{P72}$ Let $\phi : \mathbb{Z}_n \longrightarrow \mathbb{Z}_k$ be a homomorphism.

Consider, $\phi([1]_n) = [m]_k$ for some $m \in \mathbb{Z}$.

However, $\phi([0]_n) = \phi([n]_n) = 0 \Rightarrow 0 = \phi([n]_n) = n\,\phi([1]_n)$

or $n[m]_k = [nm]_k = 0$ thus $k \mid mn$. Moreover,

$$\begin{aligned} \phi([x]_n) &= \phi(x[1]_n) \\ &= x\,\phi([1]_n) \\ &= x[m]_k \\ &= [mx]_k \end{aligned}$$

$\phi$ a homomorphism.

$\therefore$ $\phi([x]_n) = [mx]_k$ for all $[x]_n \in \mathbb{Z}_n$ for some $m$ where $k \mid mn$.

For example,

$\phi : \mathbb{Z}_6 \longrightarrow \mathbb{Z}_4$ defined by

$\phi([x]_6) = [2x]_4$ defines homomorphism

since $2(6) = 12$ is multiple of $4$.

<u>Remark</u>: this result informs how to search for all homomorphisms from $\mathbb{Z}_n \longrightarrow \mathbb{Z}_k$ in terms of finding $m$ for which $k \mid mn$.