Solution to Lecture 1 Problems

[P1] If group then state identity and typical inverse & determine if abelian. If it is not a group then explain which axioms fail and explain how via counter-examples.

(a.) $(\mathbb{Z}_{\geq 0}, +)$ the non-negative integers with addition is NOT a group. It is associative with identity 0. However, it is not closed under inverses.
For example, $2 + x = 0 \Rightarrow x = -2 \notin \mathbb{Z}_{\geq 0}$.
Note, $(\mathbb{Z}_{\geq 0}, +)$ does have closure.

(b.) $(3\mathbb{Z}, +)$ does form a group with identity 0 and typical element $3k$ has additive inverse $-3k = 3(-k)$. It is abelian.

(c.) $(\mathbb{R}_{<0}, \cdot)$ negative reals with multiplication.
Not closed, $(-2)(-3) = 6 \notin \mathbb{R}_{<0}$.
No identity, $1 \notin \mathbb{R}_{<0}$.
Inverses, $xy = 1 \Rightarrow y = \frac{1}{x}$
Well, actually, $x \in \mathbb{R}_{<0} \Rightarrow \frac{1}{x} \in \mathbb{R}_{<0}$
So $\mathbb{R}_{<0}$ is closed under inverses despite the fact $1 \notin \mathbb{R}_{<0}$

Some folks argue that $1 \notin G \Rightarrow$ inverses can't exist. For future reference, in such a case we will side on side of saying inverses exist. Look at the table, what sense is a Quasigroup if this isn't an option?

$\boxed{P1}$

(d) $(\mathbb{R}_{\neq 0}, \div)$ nonzero reals with division.

Closure holds since $\frac{a}{b} \neq 0$ provided $a, b \neq 0$.

Associativity fails! $\underbrace{(3 \div 2)}\div 4 \neq 3 \div (2 \div 4)$

$$\frac{3/2}{4} \neq \frac{3}{2/4}$$

~~Identity exists~~, $\frac{x}{1} = x$ oops! $\frac{1}{x} \neq x$

It is not possible to solve $\frac{x}{e} = \frac{e}{x} = x$
for all x, no such $e \in \mathbb{R}_{\neq 0}$ exists.
These equations imply $x^2 - e^2 = 0$
thus $(x+e)(x-e) = 0 \Rightarrow \underbrace{x = -e \text{ or } x = e}$

for fixed e it is impossible for these to hold $\forall x \in \mathbb{R}_{\neq 0}$.

(e.) $(\mathbb{Q}_{>0}, \cdot)$

positive rationals with multiplication, do form a group. Identity is $1 \in \mathbb{Q}_{>0}$.
Notice $\frac{a}{b} \in \mathbb{Q}_{>0}$ has $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1$
thus $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \in \mathbb{Q}_{>0}$. This is
an abelian group.

$\boxed{P2}$ $(\mathbb{Z}, -)$ is not a group since

$$(a-b)-c \neq a-(b-c) \quad \forall a,b,c \in \mathbb{Z}.$$

For example, $\underbrace{(3-4)-5}_{-6} \neq \underbrace{3-(4-5)}_{4}$

Subtraction is not associative.

$\#$

$(\mathbb{Z}, \cdot)$ is not a group since $1X = X1 = X$ $\forall X \in \mathbb{Z}$ shows $1 \in \mathbb{Z}$ serves as the identity in $(\mathbb{Z}, \cdot)$ yet $2X = 1 \Rightarrow X = \frac{1}{2} \notin \mathbb{Z}$ thus $2^{-1} \notin \mathbb{Z}$. $(\mathbb{Z}, \cdot)$ fails to be invertible.

$\boxed{P3}$ To prove $Gl(2, \mathbb{R})$ is nonabelian we need only exhibit $A, B \in Gl(2, \mathbb{R})$ for which $AB \neq BA$. __MANY__ choices exist, here's mine,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in Gl(2, \mathbb{R}) \quad \text{as} \quad \det(A) = 1 \neq 0,$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in Gl(2, \mathbb{R}) \quad \text{as} \quad \det(B) = 1 \neq 0.$$

Calculate,
$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Thus, $AB \neq BA$ for some $A, B \in Gl(2, \mathbb{R})$ and we deduce $Gl(2, \mathbb{R})$ is nonabelian.

P4 Solve # 37 from pg. 56 of Gallian.

$G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| \ a \in \mathbb{R}, a \neq 0 \right\}$. Show G is group w.r.t. matrix multiplication. Also explain how it is $A \in G$ has $A^{-1} \in G$ despite $\det A = 0$

Suppose $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$ and $B = \begin{bmatrix} b & b \\ b & b \end{bmatrix}$ for $a, b \neq 0$.

Notice $AB = \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix}$ and as $ab \neq 0$ we find $AB \in G$. This shows closure. Observe $A, B, C \in G \Rightarrow (AB)C = A(BC)$ as matrix mult. is associative. Next consider identity,

$$Ae = eA = A \Rightarrow \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} x & x \\ x & x \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2ax & 2ax \\ 2ax & 2ax \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

thus $2ax = a$ for all $a \neq 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$.

Apparently $e = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ has $Ae = eA = A$ for all $\begin{bmatrix} a & a \\ a & a \end{bmatrix} \in G$. Thus G has identity.

Inverses: observe $\begin{bmatrix} a & a \\ a & a \end{bmatrix} \in G$ has

$$\begin{bmatrix} 1/(4a) & 1/(4a) \\ 1/(4a) & 1/(4a) \end{bmatrix}\begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} 1/(4a) & 1/(4a) \\ 1/(4a) & 1/(4a) \end{bmatrix}$$

and $\begin{bmatrix} \frac{1}{4a} & \frac{1}{4a} \\ \frac{1}{4a} & \frac{1}{4a} \end{bmatrix} \in G$ as $a \neq 0 \Rightarrow \frac{1}{4a} \neq 0$. Thus G is closed under inversion $\therefore$ G is group.

However, the group operation is not $GL(2, \mathbb{R})$. Inverse for G not same inverse!

P4 continued

G and $Gl(2,\mathbb{R})$ both use matrix multiplication. However, $G \not\subseteq Gl(2,\mathbb{R})$ and they do not have the same identity or inverse.

<u>Remark</u>: I hope you learned to think a bit about what "identity" means for a given example. We should take care to solve

$$ae = ea = a$$

before we make too many assumptions about "e". Of course, in many examples $e = 1$ or $e = 0$ as we expect... but... there are exceptions.