SOLUTIONS TO LECTURE 30 PROBLEMS 127-132

P127 p. 272 # 5 Fraleigh | Show $\alpha = \sqrt{\sqrt[3]{2} - i}$ is algebraic over $\mathbb{Q}$

$$\alpha = \sqrt{\sqrt[3]{2} - i} \Rightarrow \alpha^2 = \sqrt[3]{2} - i$$

$$\Rightarrow \alpha^2 + i = \sqrt[3]{2}$$

$$\Rightarrow (\alpha^2 + i)^3 = 2$$

$$\Rightarrow \alpha^3 + 3i\alpha^2 + 3i^2\alpha + i^3 = 2$$

$$\Rightarrow 3i\alpha^2 - i = 2 - \alpha^3 + 3\alpha$$

$$\Rightarrow i(3\alpha^2 - 1) = 2 + 3\alpha - \alpha^3$$

$$\Rightarrow i^2(3\alpha^2 - 1)^2 = (2 + 3\alpha - \alpha^3)(2 + 3\alpha - \alpha^3)$$

$$\Rightarrow -1(9\alpha^4 - 6\alpha^2 + 1) = 4 + 6\alpha - 2\alpha^3 + 9\alpha^2 + 6\alpha - 3\alpha^4 - 2\alpha^3 - 3\alpha^4 + \alpha^6$$

Thus,

$$-9\alpha^4 + 6\alpha^2 - 1 = 4 + 12\alpha + 9\alpha^2 - 4\alpha^3 - 6\alpha^4 + \alpha^6$$

Hence,

$$\alpha^6 + 3\alpha^4 + 3\alpha^2 + 12\alpha + 5 = 0$$

in $\mathbb{Q}[x]$ as needed

Hence $P(x) = x^6 + 3x^4 + 3x^2 + 12x + 5$ has $P(\alpha) = 0$

thus $\alpha = \sqrt{\sqrt[3]{2} - i}$ is algebraic over $\mathbb{Q}$.

P128 #7 p. 272 Fraleigh | Find $\text{irr}(\alpha, \mathbb{Q})$ and $\deg(\alpha, \mathbb{Q})$ for $\alpha = \sqrt{\frac{1}{3} + \sqrt{7}}$

$$\alpha^2 = \frac{1}{3} + \sqrt{7} \Rightarrow \left(\alpha^2 - \frac{1}{3}\right)^2 = \left(\sqrt{7}\right)^2$$

$$\Rightarrow \alpha^4 - \frac{2\alpha^2}{3} + \frac{1}{9} = 7 = \frac{63}{9}$$

$$\Rightarrow \alpha^4 - \frac{2}{3}\alpha^2 - \frac{62}{9} = 0$$

$$\Rightarrow 9\alpha^4 - 6\alpha^2 - 62 = 0$$

Consider $g(x) = 9x^4 - 6x^2 - 62$

Note, modulo 4, $\overline{g(x)} = x^4 - 2x^2 - 2$

hence $\overline{g(0)} = -2$, $\overline{g(1)} = 1 - 2 - 2 = 1 = \overline{g(-1)} = \overline{g(3)}$

and $\overline{g(2)} = 16 - 2(4) - 2 = 8 - 2 = -2$ thus $\overline{g(x)}$

* is irreducible ~~over~~ in $\mathbb{Z}_4[x]$ * $\Rightarrow g(x)$ irreducible over $\mathbb{Q}$.

Hence, $\text{irr}\left(\sqrt{\frac{1}{3} + \sqrt{7}}, \mathbb{Q}\right) = x^4 - \frac{2}{3}x^2 - \frac{62}{9}$

and $\deg\left(\sqrt{\frac{1}{3} + \sqrt{7}}, \mathbb{Q}\right) = 4$.

P129 #12 p. 272 Fraleigh

Consider $\alpha = \sqrt{\pi}$ over $F = \mathbb{R}$. Classify α over $\mathbb{R}$ as algebraic or transcendental. If algebraic find $\deg(\alpha, \mathbb{R})$.

Note $\sqrt{\pi} \in \mathbb{R}$ and $P(x) = x - \sqrt{\pi}$ is irreducible over $\mathbb{R}$ with $P(\sqrt{\pi}) = \sqrt{\pi} - \sqrt{\pi} = 0$ hence $\deg(\sqrt{\pi}, \mathbb{R}) = 1$ and $\sqrt{\pi}$ is algebraic over $\mathbb{R}$.

OOPS! I NEED TO SHOW THIS QUARTIC CAN'T FACTOR IN $\mathbb{Z}_4[x]$. SEE OVER

P128 continued

Need to prove $\overline{q_5(x)} = x^4 - 2x^2 - 2$ does not have a quadratic factor in $\mathbb{Z}_4[x]$. We showed $\overline{q(x)} \neq 0$ $\forall x \in \mathbb{Z}_4$ hence $\overline{q(x)}$ has no linear factor by the Factor Th$^{\underline{m}}$. Note, if $\overline{g(x)} \mid \overline{q_5(x)}$ then $\overline{g(x)} \neq 0$ $\forall x \in \mathbb{Z}_4$.

$\underline{x^2 + \alpha x + \beta = g(x)}$ <u>gives 16 choices.</u>

$x^2 = x(x)$

① $x^2 + 1 \neq 0$

② $x^2 + 2 \neq 0$

$x^2 + 3 = x^2 - 1 = (x+1)(x-1)$

$x^2 + x = x(x+1)$

③ $x^2 + x + 1 \neq 0$

$x^2 + x + 2 = (x-1)(x-2)$

④ $x^2 + x + 3 \neq 0$

$x^2 + 2x = x(x+2)$

$x^2 + 2x + 1 = (x+1)^2$

$x^2 + 2x + 2 = (x-1)(x+2)$

⑤ $x^2 + 2x + 3 \neq 0$

$x^2 + 3x = x(x+3)$

⑥ $x^2 + 3x + 1 \neq 0$

$x^2 + 3x + 2 = (x+1)(x+2)$

⑦ $x^2 + 3x + 3 \neq 0$

By my calculations, $\exists$ 7 irreducible quadratics in $\mathbb{Z}_4[x]$.

P128 continued

$$\overline{q(x)} = x^4 - 2x^2 - 2 = x^4 + 2x^2 + 2$$

①

$$\begin{array}{r} x^2+1 \\ x^2+1\,\big)\overline{\,x^4+2x^2+2} \\ \underline{x^4+x^2} \\ x^2+2 \\ \underline{x^2+1} \\ ① \end{array}$$

②

$$\begin{array}{r} x^2 \\ x^2+2\,\big)\overline{\,x^4+2x^2+2} \\ \underline{x^4+2x^2} \\ 2 \end{array}$$

③

$$\begin{array}{r} x^2-x+2 \\ x^2+x+1\,\big)\overline{\,x^4+2x^2+2} \\ \underline{x^4+x^3+x^2} \\ -x^3+x^2+2 \\ \underline{-x^3-x^2-x} \\ 2x^2+x+2 \\ \underline{2x^2+2x+2} \\ \underline{-x} \end{array}$$

④

$$\begin{array}{r} x^2-x \\ x^2+x+3\,\big)\overline{\,x^4+2x^2+2} \\ \underline{x^4+x^3+3x^2} \\ -x^3-x^2+2 \\ \underline{-x^3-x^2-3x} \\ \underline{3x+2} \end{array}$$

⑤

$$\begin{array}{r} x^2-2x+3 \\ x^2+2x+3\,\big)\overline{\,x^4+2x^2+2} \\ \underline{x^4+2x^3+3x^2} \\ -2x^3-x^2+2 \\ \underline{-2x^3-4x^2-6x} \\ 3x^2+6x+2 \\ \underline{3x^2+6x+9} \\ \underline{1} \end{array}$$

⑥

$$\begin{array}{r} x^2-3x+2 \\ x^2+3x+1\,\big)\overline{\,x^4+2x^2+2} \\ \underline{x^4+3x^3+x^2} \\ -3x^3+x^2+2 \\ \underline{-3x^3-9x^2-3x} \\ 2x^2+3x+2 \\ \underline{2x^2+6x+2} \\ -3x \end{array}$$

⑦

$$\begin{array}{r} x^2-3x \\ x^2+3x+3\,\big)\overline{\,x^4+2x^2+2} \\ \underline{x^4+3x^3+3x^2} \\ -3x^3-x^2+2 \\ \underline{-3x^3-9x^2-9x} \\ 2+x \end{array}$$

thus $\overline{q(x)} = x^4+2x+2$ is irreducible in $\mathbb{Z}_4(x)$ as it has no quadratic factors by the above calculations.

$\boxed{P130}$ #16 of p. 272 Fraleigh

Consider $\alpha = \pi^2$, find $\deg(\pi^2, \mathbb{Q}(\pi^3))$

Notice $\alpha^3 = \pi^6 = \pi^3 \pi^3 = \pi^2 \pi^2 \pi^2 \in \mathbb{Q}(\pi^3)$

We argue $P(x) = x^3 - \pi^6 \in \mathbb{Q}(\pi^3)[x]$ and $P(\pi^2) = (\pi^2)^3 - \pi^6 = 0$. It remains to show $P(x)$ is irreducible over $\mathbb{Q}(\pi^3)$. Once that is shown, $\deg(\pi^2, \mathbb{Q}(\pi^3)) = \deg(P(x)) = \boxed{3}$

Suppose $P(x) = (x+\alpha)(x^2 + \beta x + \delta)$

$$\Rightarrow x^3 - \pi^6 = x^3 + x^2(\alpha+\beta) + x(\delta + \alpha\beta) + \alpha\delta$$

$$\Rightarrow \underbrace{\alpha + \beta = 0}_{\beta = -\alpha}, \quad \underbrace{\delta + \alpha\beta = 0}_{\delta = -\alpha\beta = \alpha^2}, \quad \underbrace{-\pi^6 = \alpha\delta}_{-\pi^6 = \alpha^3}$$

Thus $\alpha = -\pi^2 \notin \mathbb{Q}(\pi^3)$ $\therefore$ $P(x)$ irred. over $\mathbb{Q}(\pi^3)$.

P/31

TRUE/FALSE		
TRUE	a.	The number π is transcendental over $\mathbb{Q}$
TRUE	b.	$\mathbb{C}$ is simple extension of $\mathbb{R}$
TRUE	c.	every element of F is algebraic over F (a field)
TRUE	d.	$\mathbb{R}$ is an extension field of $\mathbb{Q}$
FALSE	e.	$\mathbb{Q}$ is an extension field of $\mathbb{Z}_2$
TRUE	f.	$\alpha \in \mathbb{C}$ algebraic over $\mathbb{Q}$ with degree n. If $f(\alpha)=0$ for $f(x) \neq 0 \in \mathbb{Q}[x]$ then $\deg(f(x)) \geq n$.
FALSE	g.	$\alpha \in \mathbb{C}$ alg. over $\mathbb{Q}$ with deg n. If $f(\alpha)=0$ for $f(x) \in \mathbb{R}[x]$, $f(x) \neq 0$ then $\deg(f(x)) \geq n$
TRUE	h.	every nonconstant poly. in $F[x]$ has zero in some ext. of F
FALSE	i.	every nonconstant poly. in $F[x]$ has a zero in every extension of F
TRUE	j.	If x is an indeterminant, $\mathbb{Q}[\pi] \cong \mathbb{Q}[x]$

(b.) $\mathbb{C} = \mathbb{R}(i)$

(c.) if $c \in F$ then $P(x) = x - c \in F[x]$ and $P(c) = 0$.

(d.) $\mathbb{Q} \leq \mathbb{R}$ (Same field ops.)

(e.) $\mathbb{Z}_2 \not\leq \mathbb{Q}$ (different addition, multiplication etc.)

(f.) if $f(\alpha) = 0$ for $\deg(f(x)) < n$ then this $\Rightarrow\!\Leftarrow$ the very def$^{\underline{n}}$ of $\deg(\alpha, \mathbb{Q}) = n$.

(g.) $\deg(\sqrt{2}, \mathbb{Q}) = 2$ but $f(x) = x - \sqrt{2} \in \mathbb{R}[x]$ and $f(\sqrt{2}) = 0$ yet $\deg(f(x)) = 1 < 2$.

(h.) Kronecker's Th$^{\underline{m}}$

(i.) $\mathbb{R}$ is it's own extension consider $f(x) = x^2 + 1$.

$\boxed{P132}$ # 25 pg. 273 of Fraleigh

(a.) Show $\underbrace{x^3+x^2+1}$ is irred. over $\mathbb{Z}_2$.

Note $f(x)$ has $f(0)=1$ and $f(1)=1+1+1=1$

thus $f(x)$ is irred. over $\mathbb{Z}_2$.

(b.) Let us extend $\mathbb{Z}_2$ to $\mathbb{Z}_2(\alpha)$ where $\alpha^3+\alpha^2+1=0$

factor x^3+x^2+1 in $(\mathbb{Z}_2(\alpha))[x]$.

$$\begin{array}{rl} & x^2+(1+\alpha)x-(\alpha^2+\alpha) \\ x-\alpha & \overline{)\,x^3+x^2+1} \\ & \underline{x^3-\alpha x^2} \\ & (1+\alpha)x^2+1 \\ & \underline{(1+\alpha)x^2-\alpha(1+\alpha)x} \\ & 1-(\alpha^2+\alpha)x \\ & \underline{(\alpha^2+\alpha)x-\alpha^3-\alpha^2} \\ & 1+\alpha^3+\alpha^2=0 \end{array}$$

Thus, $\underline{x^3+x^2+1=(x-\alpha)(x^2+(1+\alpha)x-\alpha^2-\alpha)}$.

It remains to determine if the quadratic piece also splits... indeed, after some calculation, I found

$$\boxed{x^3+x^2+1=(x-\alpha)(x-\alpha^2)(x+1+\alpha+\alpha^2)}$$

$$\underbrace{\qquad\qquad\qquad}$$

this could look quite different as $\alpha^2+1=-\alpha^3$ etc.