SOLUTION TO LECTURE 31 PROBLEMS 133 – 138

P133

#3 of §31 Fraleigh find degree and basis over $\mathbb{Q}$

$$F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{18}) = (\mathbb{Q}(\sqrt{2}, \sqrt{3}))(\sqrt{18})$$

Notice $\sqrt{18} = 3\sqrt{2} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$ thus extending $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ by $\sqrt{18}$ does nothing; $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{18}) = \underline{\mathbb{Q}(\sqrt{2}, \sqrt{3}) = F}$

$$\begin{aligned} F = (\mathbb{Q}(\sqrt{2}))(\sqrt{3}) &= \{ c_1 + c_2\sqrt{3} \mid c_1, c_2 \in \mathbb{Q}(\sqrt{2}) \} \; * \\ &= \{ a_1 + b_1\sqrt{2} + (a_2 + b_2\sqrt{2})\sqrt{3} \mid a_1, b_1, a_2, b_2 \in \mathbb{Q} \} \\ &= \{ a_1 + b_1\sqrt{2} + a_2\sqrt{3} + b_2\sqrt{6} \mid a_1, b_1, a_2, b_2 \in \mathbb{Q} \} \end{aligned}$$

* note $\text{irr}(\sqrt{3}, \mathbb{Q}(\sqrt{2})) = x^2 - 3$ as $P(x) = x^2 - 3 \in \mathbb{Q}(\sqrt{2})[x]$ with $P(\sqrt{3}) = 0$ and $\nexists$ smaller degree polynomial which annihilates $\sqrt{3}$. Suppose otherwise, $q(x) = x + \beta$ with $\beta \in \mathbb{Q}(\sqrt{2})$ $q(\sqrt{3}) = \sqrt{3} + \beta = 0 \Rightarrow \underline{\sqrt{3} = a + b\sqrt{2}}$ for $a, b \in \mathbb{Q}$ which is impossible hence $\deg(\sqrt{3}, \mathbb{Q}(\sqrt{2})) = 2$

Continuing, we've shown $\text{span}_{\mathbb{Q}}\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\} = F$. To see this forms a basis we can prove LI of $S = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ over $\mathbb{Q}$. That seems unfun. Instead, notice

$$\begin{array}{c} (\mathbb{Q}(\sqrt{2}))(\sqrt{3}) = F \\ \Big| \, 2 \\ \mathbb{Q}(\sqrt{2}) \\ \Big| \, 2 \\ \mathbb{Q} \end{array}$$

$$\begin{aligned} [F : \mathbb{Q}] &= [\mathbb{Q}(\sqrt{2})(\sqrt{3}) : \mathbb{Q}(\sqrt{2})] \underbrace{[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]} \\ &= 2(2) \\ &= 4. \end{aligned}$$

$P(x) = x^2 - 2$ is irred, $P(\sqrt{2}) = 0$ by Eisenstein with 2.

Thus F has degree 4

In conclusion, $\boxed{[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{18}) : \mathbb{Q}] = 4 \text{ with basis } \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}}$

$\boxed{P134}$

$\boxed{\#6 \text{ of } §31 \text{ Fraleigh}}$ find degree and basis for $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ over $\mathbb{Q}$

Consider, $\alpha = \sqrt{2}+\sqrt{3} \Leftrightarrow$

$$\Leftrightarrow \alpha - \sqrt{3} = \sqrt{2}$$

$$\Leftrightarrow (\alpha - \sqrt{3})^2 = 2$$

$$\Leftrightarrow \alpha^2 - 2\alpha\sqrt{3} + 3 = 2$$

$$\Leftrightarrow \alpha^2 + 1 = 2\alpha\sqrt{3}$$

$$\Leftrightarrow (\alpha^2+1)^2 = 4\alpha^2(3) = 12\alpha^2$$

$$\Leftrightarrow \alpha^4 + 2\alpha^2 + 1 = 12\alpha^2$$

$$\Leftrightarrow \alpha^4 - 10\alpha^2 + 1 = 0$$

Consider $P(x) = x^4 - 10x^2 + 1$ has $P(\alpha) = 0$ by above algebra.
Notice, modulo 2, $\overline{P(0)} = 1$ and $\overline{P(1)} = 1 - 0 + 1 = 0$ (no help)
Then, modulo 3, $\overline{P(x)} = \overline{x^4 - x^2 + 1}$

$\overline{P(0)} = 1$, $\overline{P(1)} = 1 - 1 + 1 = 1$, $\overline{P(-1)} = \overline{P(2)} = 1$

hence $P(x)$ is irreducible over $\mathbb{Q}$ and we find
$\deg(\alpha, \mathbb{Q}) = 4$ as $\text{irred}(\alpha, \mathbb{Q}) = x^4 - 10x^2 + 1$.
Thus $\{1, \alpha, \alpha^2, \alpha^3\}$ forms basis for $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ over $\mathbb{Q}$.

<u>Remark:</u> can also prove $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ for which the natural basis choice is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.

$\boxed{P135}$

$\boxed{\text{\#7 of §31 Fraleigh}}$ find deg. and basis for $\mathbb{Q}(\sqrt{2}\sqrt{3})$ over $\mathbb{Q}$

Observe $\sqrt{2}\sqrt{3} = \sqrt{6}$ and $P(x) = x^2 - 6$ is irred by Eisenstein with prime 2 thus $\deg(\sqrt{6}, \mathbb{Q}) = 2$ and hence $\{1, \sqrt{6}\}$ forms basis for $\mathbb{Q}(\sqrt{2}\sqrt{3})$.

$\boxed{P136}$

$\boxed{\text{§31 \#9 Fraleigh}}$ $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{6}, \sqrt[3]{24})$ over $\mathbb{Q}$: find deg & basis

Ok, sorry to be boring, this is #3 all over again.
Notice $\sqrt[3]{2}\sqrt[3]{2}\sqrt[3]{6} = \sqrt[3]{24}$ $\therefore$ $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{6}) = \underbrace{\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{6}, \sqrt[3]{24})}_{\text{call this } \mathbb{F}}$.

Consider, $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{6}) = (\mathbb{Q}(\sqrt[3]{2}))(\sqrt[3]{6})$

we argue $f(x) = x^3 - 6 \in \mathbb{Q}(\sqrt[3]{2})[x]$ is irreducible and $f(\sqrt[3]{6}) = 0$ thus $[\mathbb{Q}(\sqrt[3]{2})(\sqrt[3]{6}) : \mathbb{Q}(\sqrt[3]{2})] = 3$.

To prove $f(x)$ is irreducible note $x^3 - 6$ is irred. over $\mathbb{Q}$ by Eisenstein $p = 2$. Thus a factorization of $f(x)$ must involve nonrational coefficients,

$$(x^3 - 6) = (x + \alpha)(x^2 + \beta x + \gamma)$$
$$= x^3 + (\alpha + \beta)x^2 + (\alpha\beta + \gamma)x + \alpha\gamma$$

We need $\alpha \notin \mathbb{Q}$ ~~and~~ or at least one of $\beta, \gamma \notin \mathbb{Q}$.

Equate coeff.,

$$\underbrace{\alpha + \beta = 0}_{\alpha = -\beta}, \quad \underbrace{\alpha\beta + \gamma = 0}_{\gamma = -\alpha\beta = \alpha^2}, \quad \underbrace{\alpha\gamma = -6}_{\alpha\gamma = \alpha^3 = -6}$$

Hence $\alpha = -\sqrt[3]{6} \notin \mathbb{Q}(\sqrt[3]{2}) \Rightarrow x^3 - 6 \in \mathbb{Q}(\sqrt[3]{2})[x]$ is irreducible over $\mathbb{Q}(\sqrt[3]{2})$

<u>Remark</u>: on 2nd thought, this is <u>NOT</u> #3 over again. This has more fun.

Continuing $\boxed{P136}$:

Since $\text{irr}(\sqrt[3]{6}, \mathbb{Q}(\sqrt[3]{2})) = x^3 - 6$ we find

$$[(\mathbb{Q}(\sqrt[3]{2}))(\sqrt[3]{6}) : \mathbb{Q}(\sqrt[3]{2})] = 3$$

Hence,

$$(\mathbb{Q}(\sqrt[3]{2}))(\sqrt[3]{6}) = \{c_1 + c_2\sqrt[3]{6} + c_3(\sqrt[3]{6})^2 \mid c_1, c_2, c_3 \in \mathbb{Q}(\sqrt[3]{2})\}$$

$$= \{a_1 + a_2 2^{1/3} + a_3 2^{2/3} + (a_4 + a_5 2^{1/3} + a_6 2^{2/3})6^{1/3} + (a_7 + a_8 2^{1/3} + a_9 2^{2/3})6^{2/3}$$
$$a_1, a_2, a_3, \dots, a_9 \in \mathbb{Q}\}$$

$$= \{a_1 + a_2 2^{1/3} + a_3 2^{2/3} + a_4 6^{1/3} + a_5 2^{1/3}6^{1/3} + a_6 2^{2/3}6^{1/3} +$$
$$+ a_7 6^{2/3} + a_8 2^{1/3}6^{2/3} + a_9 2^{2/3}6^{2/3} \mid a_1, \dots, a_9 \in \mathbb{Q}\}$$

Notice $[\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{6}) : \mathbb{Q}] = \underbrace{[\mathbb{Q}(\sqrt[3]{2})(\sqrt[3]{6}) : \mathbb{Q}(\sqrt[3]{2})]}_{3}\underbrace{[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]}_{3} = 9$

thus

as $x^3 - 2$ is irred. by Eisen. $p = 2$.

$$\{1, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt[3]{6}, \sqrt[3]{12}, \sqrt[3]{24}, \sqrt[3]{36}, \sqrt[3]{72}, \sqrt[3]{144}\}$$

gives basis for $(\mathbb{Q}(\sqrt[3]{2}))(\sqrt[3]{6})$ over $\mathbb{Q}$.

$\boxed{P137}$

$\boxed{\text{\# 10 of §31 Fraleigh}}$ find degree and basis for $\mathbb{Q}(\sqrt{2}, \sqrt{6})$ over $\mathbb{Q}(\sqrt{3})$

Notice $\sqrt{6} = \sqrt{2}\sqrt{3}$ thus $(\mathbb{Q}(\sqrt{3}))(\sqrt{2})$ contains $\sqrt{6}$

thus $\mathbb{Q}(\sqrt{2}, \sqrt{6})$ is just $\mathbb{Q}(\sqrt{3})$ extended by $\sqrt{2}$

we argued in $\boxed{P133}$ that x^2-3 is irreducible over $\mathbb{Q}(\sqrt{2})$

and the same argument here shows x^2-2 is irred. over $\mathbb{Q}(\sqrt{3})$

Thus $[\mathbb{Q}(\sqrt{3})(\sqrt{2}) : \mathbb{Q}(\sqrt{3})] = 2$

$$\Rightarrow \quad \mathbb{Q}(\sqrt{3})(\sqrt{2}) = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q}(\sqrt{3}) \}$$

and thus $\{1, \sqrt{2}\}$ serves as the basis.

$\boxed{P138}$ $\boxed{\text{\# 29 of p. 292 Fraleigh}}$

Let E be finite extension of F, and let $P(x) \in F[x]$ be irred. over F and $\deg(P(x))$ not a divisor of $[E : F]$. Show $P(x)$ has no zeros in E