LECTURE 3 PROBLEM SOLUTIONS

(this tells us f 1-1 and onto and it also indicated f^{-1} exists with a little work.)

$\boxed{P9}$ Let $S \neq \emptyset$ and let $G = \{f : S \to S \,/\, f \text{ is bijection}\}$

Let $Id_S = 1_S$ be defined by $1_S(x) = x \ \forall x \in S$.

Clearly $f \circ 1_S = 1_S \circ f$ for $f : S \to S$ as

$$(f \circ 1_S)(x) = f(1_S(x)) = f(x)$$

and $(1_S \circ f)(x) = 1_S(f(x)) = f(x) \ \therefore (f \circ 1_S)(x) = (1_S \circ f)(x) \ \forall x \in S$

and we've shown $f \circ 1_S = 1_S \circ f$. Thus 1_S serves as and identity for fnct. composition. To see 1_S is a bijection simply note,

(i.) $1_S(a) = 1_S(b) \Rightarrow a = b \ \therefore 1_S$ is injective.

(ii.) if $x \in S$ then $1_S(x) = x \ \therefore 1_S$ is surjective.

Thus $1_S \in G \neq \emptyset$.

Continuing, suppose $f, g, h \in G$ then $\underbrace{(f \circ g) \circ h = f \circ (g \circ h)}$ since fnct. composition is associative.

(true even for non-bijective fncts.)

Consider, $f, g \in G$ then we need to show $f \circ g : S \to S$ is a bijection.

(i.) $(f \circ g)(a) = (f \circ g)(b) \Rightarrow f(g(a)) = f(g(b))$
$\Rightarrow g(a) = g(b) \quad : f \text{ is } 1\text{-}1$
$\Rightarrow a = b \quad : g \text{ is } 1\text{-}1.$
$\Rightarrow f \circ g$ is 1-1.

(ii.) If $x \in S$ then $(f \circ g)(g^{-1}(f^{-1}(x)) = f(g(g^{-1}(f^{-1}(x))))$
$= f(f^{-1}(x))$
$= x \quad \therefore f \circ g$ is onto.

Thus $f \circ g \in G$ as $f \circ g$ is bijection.

Likewise, the inverse of $f \in G$ is $f^{-1} : S \to S$ and $f \circ f^{-1} = 1_S = f^{-1} \circ f \ \therefore f^{-1} \in G$. In summary $(G, \circ)$ is associative, closed, has identity and inverses.

$\therefore (G, \circ)$ forms a group.

$\boxed{P10}$ Prove: $\text{Isom}(\mathbb{R}^n) = \{\phi \mid \phi \text{ an isometry of } \mathbb{R}^n\}$ forms a group w.r.t. fnct. comp. Also, $\text{Orth}(n,\mathbb{R}) \leq \text{Isom}(\mathbb{R}^n)$

- We have Th$^{\underline{m}}$ 1.3.7: it states every isometry of $\mathbb{R}^n$ is a bijection. Moreover, every isometry fixing zero is a nonsingular linear transformation.
- Th$^{\underline{m}}$ 1.3.7 tells us $\text{Isom}(\mathbb{R}^n) \subseteq G = $ permutations on $\mathbb{R}^n$ thus we are free to use subgroup test in view of $\overbrace{S = \mathbb{R}^n}^{}$ for $\boxed{P9}$.

Consider, $\phi = 1$ is an isometry since

$$\|1(x) - 1(y)\| = \|x-y\| \quad \forall x, y \in \mathbb{R}^n \therefore \quad 1 \in \text{Isom}(\mathbb{R}^n) \neq \emptyset.$$

Suppose $\phi_1, \phi_2 \in \text{Isom}(\mathbb{R}^n)$ then suppose $x, y \in \mathbb{R}^n$

$$\begin{aligned} \|(\phi_1 \circ \phi_2)(x) - (\phi_1 \circ \phi_2)(y)\| &= \|\phi_1(\phi_2(x)) - \phi_1(\phi_2(y))\| \quad \phi_1 \text{ isometry.} \\ &= \|\phi_2(x) - \phi_2(y)\| \quad \phi_2 \text{ isometry.} \\ &= \|x-y\| \end{aligned}$$

thus $\phi_1 \circ \phi_2 \in \text{Isom}(\mathbb{R}^n)$. It remains to show $\text{Isom}(\mathbb{R}^n)$ closed under inverses. Consider $\phi \in \text{Isom}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$, note ϕ^{-1} exists as ϕ is bijection. Also,

$$\begin{aligned} \|\phi^{-1}(x) - \phi^{-1}(y)\| &= \|\phi(\phi^{-1}(x)) - \phi(\phi^{-1}(y))\| \quad : \phi \text{ isom.} \\ &= \|x-y\| \quad : \phi \circ \phi^{-1} = 1_{\mathbb{R}^n}. \end{aligned}$$

thus $\phi^{-1} \in \text{Isom}(\mathbb{R}^n)$ and by two-step subgroup test we deduce $\text{Isom}(\mathbb{R}^n) \leq G$. It remains to show $\text{Orth}(n,\mathbb{R}) \leq \text{Isom}(\mathbb{R}^n)$

Recall $\text{Orth}(n,\mathbb{R}) = \{\phi \in \text{Isom}(\mathbb{R}^n) \mid \phi(0) = 0\}$.

P10 continued

Observe $1(0) = 0$ thus $1 \in \text{Orth}(n, \mathbb{R}) \neq \phi$.

Suppose $\phi_1, \phi_2 \in \text{Orth}(n, \mathbb{R})$ then $\phi_1, \phi_2 : \mathbb{R}^n \to \mathbb{R}^n$ are isometries and $\phi_1(0) = 0$ and $\phi_2(0) = 0$.

Consider,
$$\begin{aligned}(\phi_1 \circ \phi_2)(0) &= \phi_1(\phi_2(0)) \quad \phi_2(0) = 0 \\ &= \phi_1(0) \quad \phi_1(0) = 0 \\ &= 0\end{aligned}$$

thus $\phi_1 \circ \phi_2 \in \text{Orth}(n, \mathbb{R})$. Likewise,

$\phi \in \text{Orth}(n, \mathbb{R})$ where $\phi(0) = 0$
$$\Rightarrow \phi^{-1}(\phi(0)) = \phi^{-1}(0)$$
$$\Rightarrow \underline{0 = \phi^{-1}(0)}. \ \therefore \underline{\phi^{-1} \in \text{Orth}(n, \mathbb{R})}.$$

thus $\text{Orth}(n, \mathbb{R}) \leq \text{Isom}(\mathbb{R}^n)$ by two-step subgroup test.

$\boxed{P11}$ Show $\text{Orth}(n, \mathbb{R})$ gives rise to $O(n, \mathbb{R})$

$O(n, \mathbb{R}) = \{ R \in \mathbb{R}^{n \times n} \mid R^T R = I \}$

where $O(n, \mathbb{R}) \overset{\text{def}^n}{=\!=} \{ [T] \mid T \in \text{Orth}(n, \mathbb{R}) \}$.

Recall Th$^{\underline{m}}$ 1.3.4, $\phi(0) = 0 \iff \phi(x) \cdot \phi(y) = x \cdot y$ where ϕ is an isometry. Thus $T \in \text{Orth}(n, \mathbb{R})$ has

$$T(x) \cdot T(y) = x \cdot y \quad \forall x, y \in \mathbb{R}^n$$

Remark: I expand this partial sol$^{\underline{n}}$

$$\iff (T(x))^T T(y) = x^T y$$
$$\iff ([T]x)^T [T]y = x^T y \iff x^T [T]^T [T] y = x^T y$$
$$\iff [T]^T [T] = I.$$

$\boxed{P11}$ Let $R \in O(n, \mathbb{R})$ thus $\exists\, T \in Isom(\mathbb{R}^n)$ with $T(0) = 0$ and $[T] = R$. By Th$^{\underline{m}}$ 1.3.4 T preserves dot-products. Hence, for $x, y \in \mathbb{R}^n$,

$$T(x) \cdot T(y) = x \cdot y \Rightarrow (Rx) \cdot (Ry) = x \cdot y$$

$\Rightarrow (Rx)^T Ry = x^T y$: recall dot-product is row-column product

$\Rightarrow x^T R^T R y = x^T y$: socks-shoes for transpose

$\Rightarrow x^T R^T R y = x^T I y$: $Iy = y$ for identity matrix

$\Rightarrow x^T (R^T R - I) y = 0$

Thus $x^T(R^T R - I)y = 0 \ \forall x, y \in \mathbb{R}^n$. Choose $x = e_i$ and $y = e_j$ where $(e_i)_k = \delta_{ik}$ defines the standard basis for $\mathbb{R}^n$, note, $e_i^T A e_j = A_{ij}$ from Math 321,

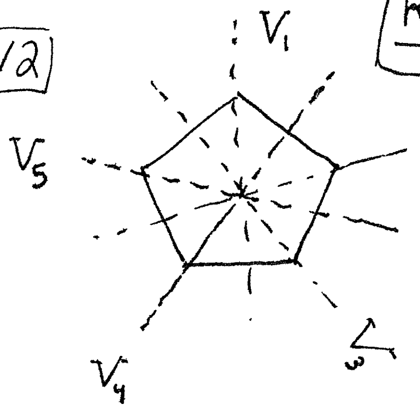
$$(e^i)^T (R^T R - I) e_j = (R^T R - I)_{ij} = 0$$

Thus $R^T R - I = 0 \Rightarrow R^T R = I \therefore R \in \{R \in \mathbb{R}^{n \times n} \mid R^T R = I\}$.
We have shown $O(n, \mathbb{R}) \subseteq \{R \in \mathbb{R}^{n \times n} \mid R^T R = I\}$.

Let $R \in \{\bar{R} \in \mathbb{R}^{n \times n} \mid \bar{R}^T \bar{R} = I\}$. Let $T(x) = Rx \ \forall x \in \mathbb{R}^n$. Observe, $T(x) \cdot T(y) = (Rx) \cdot (Ry) = x^T R^T R y = x^T I y = x \cdot y$ for all $x, y \in \mathbb{R}^n$. Thus T preserves dot-products and by Th$^{\underline{m}}$ 1.3.4, $T(0) = 0$ and $T \in Isom(\mathbb{R}^n)$ thus $T \in Orth(n, \mathbb{R})$. Therefore, $[T] = R \in O(n, \mathbb{R})$.
Hence, ~~ORdeR~~ $\{R \in \mathbb{R}^{n \times n} \mid R^T R = I\} \subseteq O(n, \mathbb{R})$.
We conclude, $\underline{O(n, \mathbb{R}) = \{R \in \mathbb{R}^{n \times n} / R^T R = I\}}$.

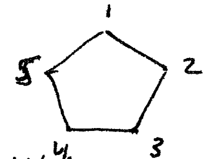
– (Remark: often the above is used to define $O(n, \mathbb{R})$ when a longer discussion of isometries is to be avoided ☺.) –

P12

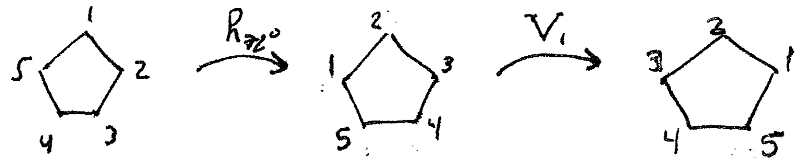


Remark: I give geom. solⁿ here, next time I'll use generators r, f and frf = r^{-1} etc...

- $V_1, \dots, V_5$: reflections about pictured axes.
- $R_{0°}, R_{72°}, \dots, R_{288°}$: rotations by indicated CCW degrees.

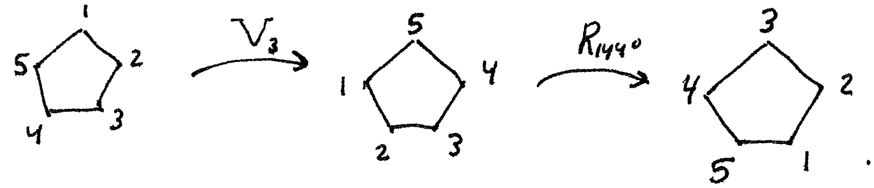
To calculate geometrically,  label and draw pictures. (you could label differently... sorry grader...)

(a.) $V_1 R_{72°}$:

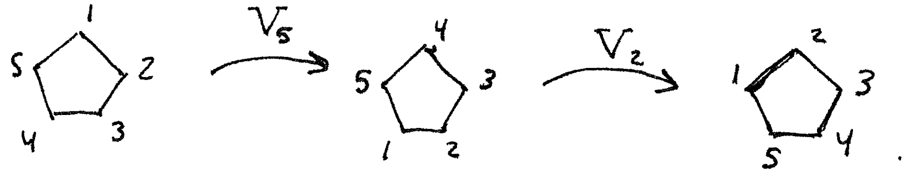


(*)

$R_{144°} V_3$:

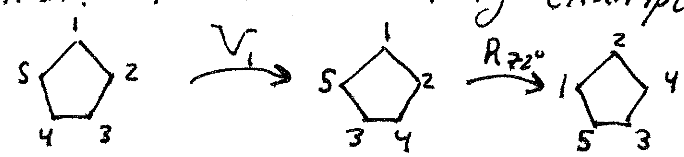


$V_2 V_5$:



(b.) No D_5 is nonabelian. There are many examples.

Consider $R_{72°} V_1$:



(**)

comparing (*) and (**) we see $R_{72°} V_1 \neq V_1 R_{72°}$.

(c.)

By geometry here.

$V_1 V_1 = V_2 V_2 = V_3 V_3 = V_4 V_4 = V_5 V_5 = e = R_{0°}$

$R_{72°} R_{288°} = R_{0°} \quad \therefore \quad (R_{72°})^{-1} = R_{288°} \quad \& \quad (R_{288°})^{-1} = R_{72°}$

$R_{144°} R_{216°} = R_{0°} \quad \therefore \quad (R_{144°})^{-1} = R_{216°} \quad \& \quad (R_{216°})^{-1} = R_{144°}$

and naturally $R_{0°} R_{0°} = R_{0°} \quad \therefore (R_{0°})^{-1} = R_{0}$.

(d.) $|V_j| = 2$ for $j = 1,2,3,4,5$. $|R_{\theta°}| = 5$ for $\theta = 72°, 144°, 216°, 288°$ and $|R_{0°}| = 1$.