

25 #24 of pg. 83 Gallian

For any $a \in G$ (a group) show $\langle a \rangle \leq C(a)$

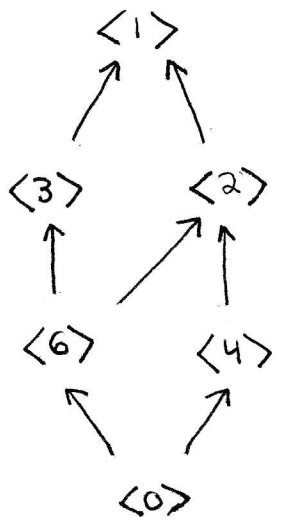
Notice $\langle a \rangle \leq G$ is known from our previous work. Thus, it suffices to demonstrate $\langle a \rangle \subseteq C(a)$ since we know $\langle a \rangle$ is non-empty and closed under the group operation. Let $a^k \in \langle a \rangle$ where $k \in \mathbb{Z}$

Notice, $a^k a = a^{k+1} = a a^k$ * thus $a^k \in C(a)$ and we conclude $\langle a \rangle \subseteq C(a)$. //

Remark: it's not a bad thing if you proved $\langle a \rangle \leq G$ in the process of solving **25**. I'm also operating under the assumption we know how to prove $a^k a^j = a^{k+j}$ hence we're free to use laws of exponents in (*)

26 Subgroup Lattice Diagrams # 32, 33 and 34 on pg. 84 Gallian

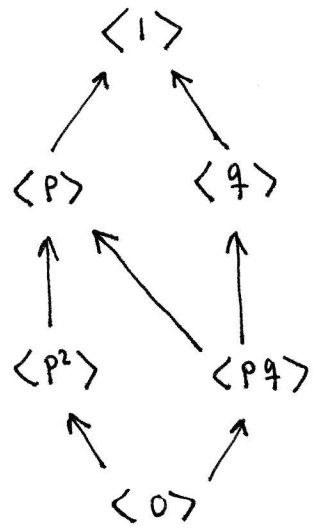
#32 \mathbb{Z}_{12} has 1, 2, 3, 4, 6, 12 as positive divisors



#34 \mathbb{Z}_8 has 1, 2, 4, 8 as divisors



#33 \mathbb{Z}_{p^2q} where $p \neq q$ are primes, have divisors p, p^2, q, pq, p^2q of \mathbb{Z}_{p^2q}



P27 Gallian exercise #41 of Chpt. 4

Suppose $a, b \in G$ (a group) and $|a|=m$ and $|b|=n$.

If $\langle a \rangle \cap \langle b \rangle = \{e\}$, prove that G contains an element whose order is $\text{lcm}(m, n)$ provided we assume $ab=ba$.
 Show this ^{may} fail when considering $a, b \in G$ with $ab \neq ba$.

I'll follow Gallian's hint.

Let $t = \text{lcm}(m, n)$ and $|ab| = s$. Note $t = mm'$ and $t = nn'$ for some m', n' .

Since $ab=ba$ we have $(ab)^t = a^t b^t = a^{mm'} b^{nn'} = (a^m)^{m'} (b^n)^{n'} = e$,

so $a^m = e$ and $b^n = e$ since $|a|=m$ and $|b|=n$ was assumed from the outset.

Observe $(ab)^t = e$ and $|ab| = s$ thus $s | t$.

Note also $e = (ab)^s = a^s b^s \Rightarrow a^s = b^{-s} \Rightarrow a^s, b^{-s} \in \langle a \rangle \cap \langle b \rangle = \{e\}$.

but $a^m = e$ and $a^s = e \Rightarrow m | s$

and $b^n = e$ and $b^{-s} = e \Rightarrow n | s$

Hence $t = \text{lcm}(m, n)$ also divides s . So, $s | t$ and $t | s$ hence $s = t$.

In summary ab has order $\text{lcm}(|a|, |b|)$ provided $\langle a \rangle \cap \langle b \rangle = \{e\}$ and $ab=ba$.

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Nonabelian case, consider $D_3 = \{1, x, x^2, y, xy, x^2y\}$
 $x^3=1$
 $y^2=1$
 $(xy)^2=1$

$a = xy$ and $b = x^2y$ have order two and

$\langle a \rangle = \{1, xy\}$ and $\langle b \rangle = \{1, x^2y\}$ so $\langle a \rangle \cap \langle b \rangle = \{1\}$.

Aww... this is not the one I want...

$a = x$ and $b = xy$ has $\text{lcm}(|a|, |b|) = 6$

$\langle a \rangle = \{1, x, x^2\}$ & $\langle b \rangle = \{1, xy\}$ have $\langle a \rangle \cap \langle b \rangle = \{1\}$

yet there is no element of order 6 in D_3 . In fact, 1 has order 1, $\{y, xy, x^2y\}$ order 2, $\{x, x^2\}$ order 3, that's it.

28] If G is an Abelian group which contains cyclic subgroups of orders 4 and 5 then what other sizes of cyclic subgroups must G contain?

Notice, G is not given to be cyclic so we cannot simply argue from Th^m 1.7.5. Some calculation is needed here. We have $a \in G$ for which $|a| = 4$ and

$\langle a \rangle = \{e, a, a^2, a^3\}$ the cyclic subgroup of order 4.

Likewise, $\exists b \in G$ for which $|b| = 5$ and $|\langle b \rangle| = 5$.

Consider $\langle a^2 \rangle = \{e, a^2\} \leq \langle a \rangle \leq G$ thus $|\langle a^2 \rangle| = 2$.

Also, $(ab)^{20} = a^{20} b^{20}$ as $ab = ba \Rightarrow (ab)^n = a^n b^n$ (prove it!)
thus $(ab)^{20} = (a^4)^5 (b^5)^4 = e$ and $(ab)^j = a^j b^j \neq e$ as
 $20 = \text{lcm}(4, 5)$ it is the first number to make a common multiple of both 4 and 5 hence $|ab| = 20$ and
we find $\langle ab \rangle$ is cyclic subgroup of order 20.

$\langle ab \rangle = \{e, ab, a^2 b^2, a^3 b^3, b^4, a, a^2 b, \dots, a^3 b^4\}$ (no particular order intended)

Notice $\text{lcm}(2, 5) = 10$ and $(a^2 b)^{10} = a^{20} b^{10} = (a^4)^5 (b^5)^2 = e$

and we have $\langle a^2 b \rangle$ a cyclic subgroup of order 10.

Finally, $\langle e \rangle = \{e\}$ is cyclic subgroup of order 1.

In summary, we also get $\langle a^2 \rangle$, $\langle a^2 b \rangle$, $\langle ab \rangle$ and $\langle e \rangle$
order: 2 10 20 1

—(this problem is way easier if G is cyclic!)—