SOLUTION TO PROBLEMS 29–32 (LECTURE 8)

$\boxed{P29}$

S_3	(1)	(123)	(132)	(13)	(12)	(23)
(1)	(1)	(123)	(132)	(13)	(12)	(23)
(123)	(123)	(132)	(1)	(23)	(13)	(12)
(132)	(132)	(1)	(123)	(12)	(23)	(13)
(13)	(13)	(12)	(23)	(1)	(123)	(132)
(12)	(12)	(23)	(13)	(132)	(1)	(123)
(23)	(23)	(13)	(12)	(123)	(132)	(1)

(*)

Note, $(132) = (321)$

so $(123)^{-1} = (321)$.

(correction to 1st draft of notes)

$(132)(132) = (123)$

$(13)(132) = (1)(23) = (23)$

$(12)(132) = (13)(2) = (13)$

$(23)(132) = (12)(3) = (12)$

$(132)(13) = (12)$

$(12)(13) = (132)$

$(23)(13) = (32)(31) = (312) = (123)$

(*) $(132)(12) = (1)(23) = (23)$

$(13)(12) = (123)$

$(12)(12) = (1)$

$(23)(12) = (132)$

P30 For τ given below find

(i.) disjoint-cycle presentation of τ

(ii) Find inverse of τ

(iii) Find order of τ

(iv.) write τ as product of transpositions, is τ even/odd?

(v.) conjugate τ by $\sigma = (123)(45)$ (ie. calculate $\sigma\tau\sigma^{-1}$)

(vi.) Calculate τ^{99}

(a.) $\tau = (124)(35)(24)(132) = (153)(2)(4)(= \boxed{(153) = \tau}$ - (i.)

$\tau^{-1} = (153)^{-1} = \boxed{(351)}$ (ii.)

(iii.) The $\boxed{\text{order of } \tau \text{ is } 3}$ since τ is a 3-cycle and you can easily verify $\tau^3 = (1)$ but $\tau^2 \neq (1)$.

(iv.) $\tau = (153) = \boxed{(13)(15) \text{ hence } \tau \text{ is even.}}$

(v.) $\sigma = (123)(45)$

$\sigma^{-1} = (45)^{-1}(123)^{-1} = (54)(321)$.

$$\begin{aligned}\sigma\tau\sigma^{-1} &= (123)(45)(153)(45)(321) \\ &= (124)(3)(5) \\ &= \boxed{(124) = \sigma\tau\sigma^{-1}}\end{aligned}$$

(vi.) $\tau^{99} = (\tau^3)^{33} = (1)^{33} = (1)$. $\therefore$ $\boxed{\tau^{99} = (1)}$

P30 continued ($\tau = (1253)(354)(135)$) part (b.)

(i.) $\tau = (1253)(354)(135)$

$= (134)(25) \quad \therefore \quad \boxed{\tau = (134)(25)}$ disjoint cycles.

(ii.) $\tau^{-1} = ((134)(25))^{-1}$

$= (25)^{-1}(134)^{-1}$

$= \boxed{(25)(431)}$

(iii.) By Ruffini, order$(\tau) = \text{lcm}(3, 2) = \boxed{6}$

(iv.) $\tau = (134)(25) = \boxed{(14)(13)(25) = \tau} \quad \therefore$ τ is odd.

(v.) $\sigma \tau \sigma^{-1} = (123)(45)(134)(25)(54)(321)$

$= \boxed{(152)(34)}$

(vi.) $\tau^{99} = (\tau^6)^{16}\tau^3$

$= \tau^3$

$= [(134)(25)]^3$

$= (134)^3(25)^3$

$= (25)^3$

$= (25)^2(25)$

$= \boxed{(25)}$

disjoint cycles commute hence $(ab)^n = a^n b^n$. ($ab = ba$ for $a, b \in S_5$)

$$\begin{array}{r} 16 \\ 6\overline{)99} \\ \underline{6} \\ 39 \\ \underline{36} \\ \boxed{3} \end{array}$$

$99 = 6(16) + 3$

- SHOUT OUT TO LONG DIVISION -

(fun.)

PROBLEM 30 continued, part (c.)

(i.) $\tau = (12435)(134)(45) = (1524)(3) = \boxed{(1524) = \tau}$

(ii) $\boxed{\tau^{-1} = (4251)}$.

(iii) order of τ is 4 since τ is a 4-cycle.

(iv.) $\tau = (1524) = \boxed{(14)(12)(15) = \tau}$ $\therefore$ τ is odd.

(v.) $\sigma\tau\sigma^{-1} = (123)(45)(1524)(45)(321)$

$= (1)(2435)$

$= \boxed{(2435) = \sigma\tau\sigma^{-1}}$

(vi.) $\tau^{99} = \tau^{96+3} = \underbrace{(\tau^4)}_{(1)}{}^{24}\tau^3 = \tau^3$

But, $\tau^3\tau = (1)$ $\therefore$ $\tau^{-1} = \tau^3$ so, $\boxed{\tau^{99} = (4251)}$

$= \underline{(1425)}$.

P31 Gallian #10 on pg. 112 / Suppose S is finite set. Show $f: S \to S$ is one-one iff it is onto. Is this claim true for S an infinite set?

Let $S = \{s_1, s_2, \dots, s_n\}$. Suppose $f: S \to S$ is one-one. Since $f(a) = f(b) \Rightarrow a = b$ we have the elements $f(s_1), f(s_2), \dots, f(s_n)$ are distinct. Thus

$$f(S) = \{f(s_1), \dots, f(s_n)\} \subseteq S$$

and, by counting, $f(S) = \{s_1, s_2, \dots, s_n\}$ (there are no other subsets of S which have n-elements except S itself). Thus 1-1 $\Rightarrow$ onto.

Conversely, suppose $f(S) = S$.
If $\exists a, b \in S$ for which $f(a) = f(b)$ but $a \neq b$ then $f(S) = \{f(s_1), \dots, f(a), \widehat{f(b)}, \dots, f(s_n)\}$ has $(n-1)$-elements which clearly $\not\leftrightarrow$ $f(S) = S$ $\leftarrow$ has n-elements. Thus f is injective and we conclude f is injective iff f is surjective when f is a function on a finite set.

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The claim is false in context of infinite sets.
Notice $\frac{d}{dx} : P[\mathbb{R}] \longrightarrow P[\mathbb{R}]$ yet $\frac{df}{dx} = \frac{dg}{dx} \not\Rightarrow f = g$ we only have $f = g + c$. There are many examples.

[P32] Gallian exercise #35 pg. 113

How many elements of order 5 are there in A_6?

$A_6 \leq S_6$ it is set of even permutations on $\{1,2,3,4,5,6\}$.

An element of order 5 is a 5-cycle. We cannot form it from product of smaller cycles. Ruffini would need a, b s.t. $\operatorname{lcm}(a,b) = 5$ but this requires $a=1$ or $b=1$. Furthermore, $\alpha = (a_1\, a_2\, a_3\, a_4\, a_5)$ a 5-cycle gives,

$$\alpha = (a_1, a_5)(a_1, a_4)(a_1, a_3)(a_1, a_2) \in A_6$$

as α is an even permutation. It remains to count how many 5-cycles are in S_6,

$$\begin{array}{ccccc} (a & b & c & d & e) \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 6 & 5 & 4 & 3 & 2 \end{array} \text{ choices}$$

But, $(abcde) = (bcdea) = (cdeab) = (deabc) = (eabcd)$ hence the above overcounts. In summary,

$$\frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{5} = 6\cdot 4\cdot 3\cdot 2 = \boxed{144 \text{ elements of order 5 in } A_6}$$

Same counting for 2-cycles in S_6 would be $\frac{6\cdot 5}{2} = 15$. Those are:
$(12), (13), (14), (15), (16), (23), (24), (25), (26), (34), (35), (36), (45), (46), (56)$.