SOLUTIONS TO PROBLEMS 33–36 FROM LECTURE 9

P33) Gallian #1 on pg. 111

(1.) find the order of each of the following permutations

(a.) $(14)(14) = (1)$ $\therefore$ $\boxed{|(14)| = 2}$

(b.) $|(147)| = 3$

(c.) $|(14762)| = 5$

P34) Orders in S_n

(a.) what are the orders of the elements in S_5. Give example of each order

(1) order 1

(12) order 2

(123) order 3

(1234) order 4

(12345) order 5

$(12)(345)$ order 6 by Ruffini

(b.) Does S_{11} have an element of order 24? (If so find it...)

$(1,2,3)(4,5,6,7,8,9,10,11)$ has order 24.

(C.) Does S_{11} have an element of order 16?

$|\alpha\beta| = \text{lcm}(|\alpha|, |\beta|)$ for dijoint cycles

To obtain $\text{lcm}(|\alpha|, |\beta|) = 16$ we need a 16 cycle. Other factorizations of 16 like $4 \cdot 4$ or $2 \cdot 8$ have lcm's smaller than 16 thus we need a 16-cycle and there is no cycle of that length in S_{11}.

P 35 for $n > 1$ show $A_n \leq S_n$

Observe $(1) = (12)(12)$ for $n \geq 2$ thus $(1) \in A_n \neq \phi$.
Next, suppose $\alpha, \beta \in A_n$ then by definition, there exist an even # of 2-cycles for which,

$$\alpha = (a_1\, a_2)(a_3\, a_4) \cdots (a_{2k-1}\, a_{2k}) \qquad (k \text{ even})$$
$$\beta = (b_1\, b_2)(b_3\, b_4) \cdots (b_{2j-1}\, b_{2j}) \qquad (j \text{ even})$$

Notice $\beta^{-1} = (b_{2j}\, b_{2j-1}) \cdots (b_4 b_3)(b_2 b_1)$ by the extended socks-shoes inverse formula (can you prove it?)
Thus,

$$\alpha\beta^{-1} = \underbrace{(a_1\, a_2)(a_3 a_4) \cdots (a_{2k-1}\, a_{2k})}_{k\text{-transpositions}}\underbrace{(b_{2j}\, b_{2j-1}) \cdots (b_4 b_3)(b_2 b_1)}_{j\text{-transpositions}}$$

we find $\alpha\beta^{-1}$ has $(k+j)$-transpositions in its decomp. thus $\alpha\beta^{-1} \in A_n$ and we conclude $A_n \leq S_n$ by the one-step subgroup test. //

Remark: the proof above implicitly assumes Th^m 1.1.9 and you could argue Cor 1.1.10 immediately gives $A_n \leq S_n$ (I would be cool with that). However, on a test, you should ask if you intend such an argument as it may well be ☺.

$\boxed{P36}$ Gallian #40 pg. 113

$\boxed{\text{Prove } S_n \text{ is nonabelian for } n \geq 3}$

We can use our work on $\boxed{P29}$ to guide us. This is not hard.

Consider, $(13)(132) = (23)$ yet $(132)(13) = (12)$

Hence $(13)(132) \neq (132)(13)$ and $(13), (132) \in S_n$ for $n \geq 3$.

Thus S_n is nonabelian when $n \geq 3$.