Name: ______________________________

PHYSICS 231, SPRING 2023.	TEST II (150PTS+10PTS)

Your solutions should be neat, correct and complete. Full credit is not necessarily attained from the correct answer, you can lose points if the solution is not readable. **Box answer** where appropriate. You may omit units on calculations, however, answers must include proper units for full-credit. You are allowed a 3" × 5" card. By default, problems based on earth with $g = 9.8m/s^2$.

Problem 1 (20pts) It is observed that a particular spring obeys a modified form of Hooke's Law; $F(x) = -k(x + \beta x^2)$ where k and β are constants.

(a) find a potential energy function for $F = -\frac{dU}{dx}$

$$\frac{dU}{dx} = k(x + \beta x^2) \xrightarrow[\text{integrate}]{} \boxed{U = U_0 + k\left(\tfrac{1}{2}x^2 + \tfrac{1}{3}\beta x^3\right)}$$

(b) omit units and suppose (just for this part) that $k = \beta = 1$. Find the work done by the force from $x = 0$ to $x = 1$.

$$W = \int_0^1 F(x)dx = -\int_0^1 \frac{dU}{dx}dx = -U(1) + U(0)$$

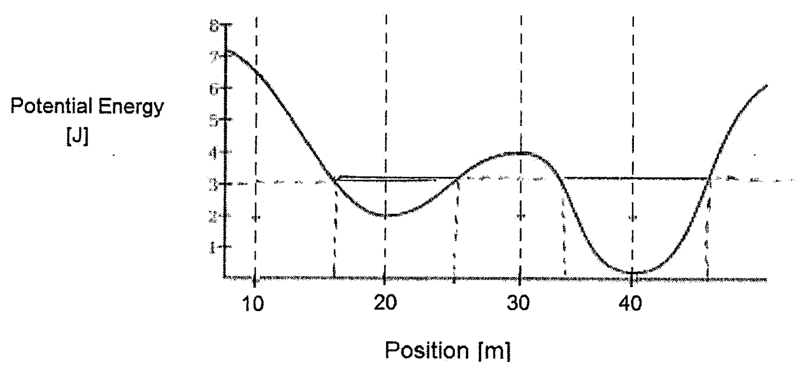
$$\Rightarrow \boxed{W = \frac{-1}{2} - \frac{1}{3} = \frac{-5}{6} \cong -0.8333}$$

Problem 2 (10pts) You are given the graph of potential energy for a particle under the influence of a particular conservative force. **If the total energy of the particle of** $3.0\,J$ then approximate the the possible range of motion in the following cases:

(a) initial position of $20\,m$: ANSWER HERE: $16m \leq x \leq 25m$

(b) initial position of $40\,m$: ANSWER HERE: $33m \leq x \leq 46m$

approximate



$E \geq U(x)$ for classically permissable motions.

Problem 3 (20pts) Suppose two carts slide along an essentially frictionless track. Cart A has mass $M_A = 20\,kg$ and cart B has mass $M_B = 60\,kg$. Suppose A travels rightward with an initial speed of $v_o = 13\,m/s$. Furthermore, after the carts collide, cart A moves to the left at speed $6\,m/s$ cart B moves right at speed $v_B = 15\,m/s$. **Find the initial speed and direction of cart B before the collision. Also, was the collision elastic?**

$$M_A(13\,m/s) + M_B V_{Bo} = M_A(-6\,m/s) + M_B(15\,m/s)$$

$$V_{Bo} = \frac{20kg(-6\,m/s) + (60kg)(15\,m/s) - 20kg(13\,m/s)}{60kg}$$

Can argue ① or ②,

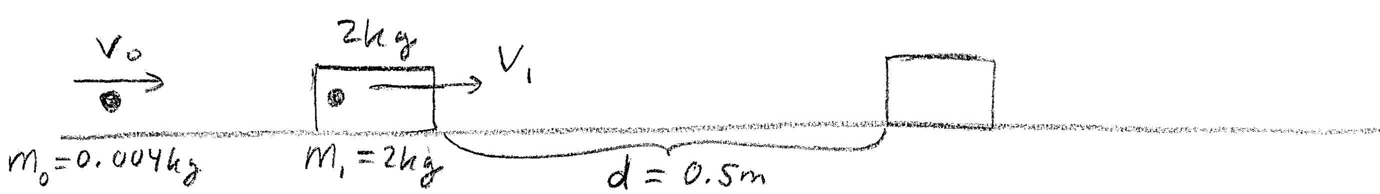
$$\boxed{V_{Bo} \cong 8.667\,m/s}$$

① $V_{Ao} - V_{Bo} \cong 4.333\,m/s$ yet $V_{Af} - V_{Bf} = -21\,m/s \Rightarrow$ not elastic.

② $KE_o = \frac{1}{2}M_A V_o^2 + \frac{1}{2}M_B V_{Bo}^2 \cong 3944\,J$ vs. $KE_f = \frac{1}{2}M_A V_{Af}^2 + \frac{1}{2}M_B V_{Bf}^2 = 7110\,J$

$KE_o \neq KE_f \quad \therefore$ not elastic.

Problem 4 (20pts) A 4.00 gram bullet is fired horizontally into a 2.00 kg wooden block resting on a horizontal surface. The coefficient of kinetic friction between block and surface is 0.30. The bullet remains embedded in the block, which is observed to slide 0.50 m along the surface before stopping. What was the initial speed of the bullet?



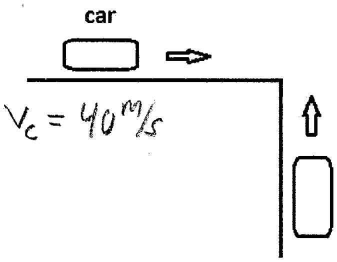
$$m_o V_o = (m_o + m_1)V_1 \quad \therefore \quad V_o = \frac{(m_o + m_1)V_1}{m_o}$$

$$\frac{1}{2}(m_o + m_1)V_1^2 = F_f d = \mu(m_o + m_1) g d \Rightarrow V_1 = \sqrt{2\mu g d}$$

$$V_o = \left(\frac{m_o + m_1}{m_o}\right)\sqrt{2\mu g d}$$

$$\boxed{V_o \cong 859.04\,m/s}$$

Problem 5 (20pts) Suppose a 1000 kg car travelling 40.0 m/s and a 2000 kg truck travelling 25.0 m/s collide and stick together after the crash. See diagram for the geometry of the problem. Find the velocity (magnitude and standard angle) of the car and truck immediately after the crash.



$V_c = 40\,m/s$ $\qquad V_T = 25\,m/s$

$m_c = 1000\,kg$

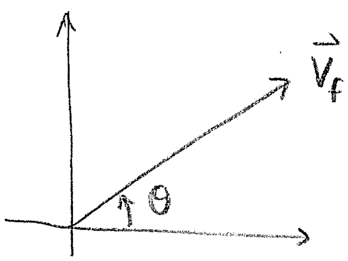
$m_T = 2000\,kg$

$$\vec{P}_0 = m_c \langle V_c, 0\rangle + m_T \langle 0, V_T\rangle = (m_c + m_T)\,\vec{V}_f$$

$$\vec{V}_f = \frac{1}{m_c + m_T}\langle m_c V_c ,\ m_T V_T\rangle$$

$$\vec{V}_f = \frac{1}{3000\,kg}\langle 1000\,kg\,(40\,m/s),\ 2000\,kg\,(25\,m/s)\rangle$$

$$\vec{V}_f = \langle 13.33\,m/s ,\ 16.67\,m/s\rangle$$



$$V_f = \sqrt{(13.33\,m/s)^2 + (16.67\,m/s)^2}$$

$$\boxed{V_f = 21.34\,m/s}$$

$$\boxed{\theta = 51.35^\circ}$$

$$\theta = \tan^{-1}\left(\frac{16.67}{13.33}\right)$$

Problem 6 (15pts) Hypothetically(as in, I'm not saying this is the correct result from the above problem), if the crash in the previous problem gave the car and truck crashed together and initial speed 10m/s and there was a coefficient of kinetic friction of 0.4 then how far would it take for the car and truck to skid to a stop?

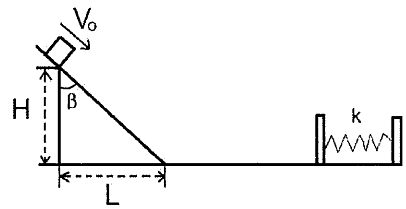
$$\frac{1}{2} m V_0^2 = F_f\, d \qquad (\text{energy lost to friction} = \text{intial energy})$$

$$\frac{1}{2} m V_0^2 = \mu m g d$$

$$d = \frac{V_0^2}{2\mu g} = \frac{(10\,m/s)^2}{2(0.4)(9.8\,m/s^2)} = \boxed{12.76\,m}$$

$$\left(\text{If you used } V_0 = 21.34\,m/s \text{ then } d = 58.09\,m\right)$$

Problem 7 (20pts) Suppose the inclined plane and horizontal plane pictured below have no friction and the spring and board have very small mass compared to mass $M = 2.0\,kg$ pictured at the top of the incline. Suppose $L = 14.3\,m$ and $\beta = 40^o$. It takes a force of $20\,N$ to compress the spring a distance of $1.00\,cm$ from it's equilbrium. If the mass M slides down the incline, across the plane and then compresses the spring a distance $1.2\,m$ (just before bouncing back) then what is the initial speed V_o which was given to M ?



$$20\,N = F_{spring} = kx_c = k(0.01m)$$

$$\therefore \; \underline{k = 2000 \frac{N}{m}}\,.$$

$$\tan\beta = \frac{L}{H} \quad \therefore \; H = \frac{L}{\tan\beta}\,.$$

Conservation of mechanical energy, we know $X_f = 1.2\,m$

$$MgH + \frac{1}{2}MV_o^2 = \frac{1}{2}kX_f^2$$

$$V_o^2 = \frac{kX_f^2 - 2MgH}{M}$$

$$V_o = \sqrt{\frac{kX_f^2}{M} - \frac{2gL}{\tan\beta}}$$

$$V_o = \sqrt{\frac{(2000\,N/m)(1.2m)^2}{2.0\,kg} - \frac{2(9.8\,m/s^2)(14.3m)}{\tan(40^\circ)}}$$

$$\boxed{V_o = 33.26\,m/s}$$

Problem 8 (15pts) A mass $m_1 = M$ is at $\overbrace{(7,4,-5)}^{\vec{r}_1}cm$ and $m_2 = 2M$ is at $\overbrace{(4,4,4)}^{\vec{r}_2}cm$. If $m_3 = 3M$ is at (a,b,c) and the center of mass is at $\underbrace{(3,3,2)}_{\vec{R}}cm$ then find the values of a, b and c.

$$m_1 + m_2 + m_3 = 6M$$

$$\vec{R} = \frac{1}{m_1+m_2+m_3}\left(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3(a,b,c)\right)$$

$$(a,b,c) = \frac{1}{m_3}\left(6M\vec{R} - m_1\vec{r}_1 - m_2\vec{r}_2\right)$$

$$(a,b,c) = \frac{1}{3M}\left(6M\vec{R} - M\vec{r}_1 - 2M\vec{r}_2\right)$$

$$\begin{aligned}(a,b,c) &= 2\vec{R} - \frac{1}{3}\vec{r}_1 - \frac{2}{3}\vec{r}_2 \\ &= 2(3,3,2) - \frac{1}{3}(7,4,-5) - \frac{2}{3}(4,4,4) \\ &= (1,2,3) \qquad \therefore \boxed{a = 1cm,\ b = 2cm,\ c = 3cm}\end{aligned}$$

Problem 9 (10pts) Find the work done by the force $\vec{F} = \langle e^{ax}, b\sin(ky), 2cz\rangle$ along the path from $(0,0,0)$ to (x_1, y_1, z_1) given $a, b, k, c \neq 0$.

$$\vec{F} = -\nabla U = \left\langle -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right\rangle = \langle e^{ax}, b\sin(ky), 2cz\rangle$$

We need

$$\frac{\partial U}{\partial x} = -e^{ax} \longrightarrow U = \frac{-1}{a}e^{ax} + C_1(y,z)$$

$$\frac{\partial U}{\partial y} = -b\sin(ky) \longrightarrow U = \frac{b}{k}\cos(ky) + C_2(x,z)$$

$$\frac{\partial U}{\partial z} = -2cz \longrightarrow U = -cz^2 + C_3(x,y)$$

Thus deduce, $\underline{U = \frac{-1}{a}e^{ax} + \frac{b}{k}\cos(ky) - cz^2}$.

$$W = \int_C \vec{F}\cdot d\vec{r} = -\int_C (\nabla U)\cdot d\vec{r} = -U(x_1,y_1,z_1) + U(0,0,0)$$

$$\boxed{W = \frac{1}{a}\left(e^{ax_1} - 1\right) + \frac{b}{k}\left(1 - \cos(ky_1)\right) + cz_1^2}$$

Problem 10 (10pts) Let $\vec{F}(x,y) = \frac{1}{x^2+y^2}\langle -y, x\rangle$. Directly calculate:

$$\oint_C \vec{F} \bullet d\vec{r}$$

where $C = \{(x,y) \mid x^2+y^2 = R^2$ for $R > 0$ and we assume C is given the CCW-orientation. Then, find a potential energy function for $\vec{F}$. Is $\vec{F}$ a conservative force? Discuss. Explain these seemingly contradictory results.

① $C: \; x = R\cos t, \quad y = R\sin t \quad$ for $0 \le t \le 2\pi$

$$\vec{F}(R\cos t, R\sin t) = \frac{1}{(R\cos t)^2 + (R\sin t)^2}\langle -R\sin t, R\cos t\rangle = \frac{1}{R}\langle -\sin t, \cos t\rangle$$

$$\frac{d\vec{r}}{dt} = \left\langle \frac{d}{dt}(R\cos t), \frac{d}{dt}(R\sin t)\right\rangle = \langle -R\sin t, R\cos t\rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \frac{1}{R}\langle -\sin t, \cos t\rangle \bullet \langle -R\sin t, R\cos t\rangle\, dt$$

$$= \int_0^{2\pi} \frac{R}{R}(\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} dt$$

$$= \boxed{2\pi}$$

* I should say, loops whose interior is contained within D.

②

$$\frac{\partial U}{\partial x} = \frac{y}{x^2+y^2} = \left(\frac{1}{1+(y/x)^2}\right)\left(\frac{y}{x^2}\right) = \frac{\partial}{\partial x}\left[-\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$\frac{\partial U}{\partial y} = \frac{-x}{x^2+y^2} = \frac{1}{1+(y/x)^2}\left(\frac{-x}{x^2}\right) = \frac{\partial}{\partial y}\left[-\tan^{-1}\left(\frac{y}{x}\right)\right]$$

We find $U(x,y) = -\tan^{-1}\left(\frac{y}{x}\right)$ gives $-\nabla U = \vec{F}$

We note C is a loop yet $\oint_C \vec{F} \bullet d\vec{r} = 2\pi \neq 0$

If $\vec{F}$ is conservative on $D \subseteq \mathbb{R}^2$ then $\oint_C \vec{F} \bullet d\vec{r} = 0$ for all loops inside D* thus $\vec{F}$ is not conservative, yet... it seems we found a PE function $U(x,y) = -\tan^{-1}(y/x)$.

Notice U is only defined for $x \neq 0$, in fact $\vec{F}$ is conservative on $\mathbb{R}^2 - \{(0,0)\}$ (the punctured plane)