More than Complex Analysis

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purpose and origins

These notes are a first step towards a book I wish to write on $\mathcal{A}$-Calculus. These are about 70 percent about complex analysis and 30 percent about the generalization of complex analysis to other algebras of finite dimension. The non-standard material is largely adapted from several papers I have written on $\mathcal{A}$-Calculus in recent years.

I often refer to the non-standard material as $\mathcal{A}$-Calculus because it is calculus where $\mathbb{R}$ has been replaced by $\mathcal{A}$ which denotes a real associative unital algebra of finite dimension. Some would call this *hypercomplex analysis*. I am reluctant to use that term because it is not really a generalization of complex analysis. In fact, complex analysis is a special case of $\mathcal{A}$-Calculus which has arguably the most beautiful and elegant properties. Much of the reason I am introducing some rudimentary $\mathcal{A}$-Calculus in these notes is to draw your attention to just how special complex analysis is compared to other cases. I think in the traditional path some of that is lost because all we do is think about $\mathbb{C}$. Don't worry, we will think a lot about $\mathbb{C}$, just not all the time.

In terms of Complex Analysis, much of what I say stems from an in-depth study I made of Gamelin's *Complex Analysis* in a previous offering of Math 331 at Liberty in 2014-2015. The You Tube videos on Complex Analysis by me in 2015 are tied to that study as are the notes I entitled *Guide to Gamelin*. In those notes I cover is the basic core of undergraduate complex analysis. My understanding of these topics began with a study of the classic text of Churchill as I took Math 513 at NCSU a few years ago. My advisor Dr. R.O. Fulp taught the course and added much analysis which was not contained in Churchill. Churchill is a good book, but, the presentation of analysis and computations is more clear in Gamelin. I also have learned a great amount from Reinhold Remmert's *Complex Function Theory* [R91]. The history and insight of that book will bring me to say a few dozen things this semester, it's a joy to read, but, it's not a first text in complex analysis so I have not required you obtain a copy. There are about a half-dozen other books I consult for various issues and I will comment on those as we use them.

Remark: many of the quotes given in this text are from [R91] or [N91] where the original source is cited. I decided to simply cite those volumes rather than add the original literature to the bibliography for several reasons. First, I hope it prompts some of you to read the literature of Remmert. Second, the original documents are hard to find in most libraries.

For your second read through complex analysis I recommend [R91] and [RR91] or [F09] for the student of pure mathematics. For those with an applied bent, I recommend [A03].

format of this guide

These notes were prepared with LaTeX. I tend to use green for definitions, blue for theorems, red for remarks and black for just about everything else. This is a work in progress, my apologies for mistakes, just email me if one is troubling. I am here to help,

James Cook, August 27, 2018 version 1.0

Notations:

Some of the notations below are from Gamelin, however, others are from [R91] and elsewhere.

Symbol	terminology	Definition in Guide
$\mathbb{C}$	complex numbers	[1.1.1]
$\mathbf{Re}(z)$	real part of z	[1.1.1]
$\mathbf{Im}(z)$	imaginary part of z	[1.1.1]
$\bar{z}$	complex conjugate of z	[1.1.3]
$\lvert z\rvert$	modulus of z	[1.1.3]
$\mathbb{C}^{\times}$	nonzero complex numbers	[1.1.6]
$\mathbb{C}[z]$	polynomials in z with coefficients in $\mathbb{C}$	[1.1.8]
$\mathbb{R}[z]$	polynomials in z with coefficients in $\mathbb{R}$	
$Arg(z)$	principle argument of z	[1.3.1]
$arg(z)$	set of arguments of z	[1.3.1]
$e^{i\theta}$	imaginary exponential	[1.3.4]
$\lvert z\rvert e^{i\theta}$	polar form of z	[1.3.4]
ω	primitive root of unity	[1.3.12]
$\mathbb{C}^*$	extended complex plane	[**??**]
$\mathbb{C}^-$	slit plane $\mathbb{C}-(-\infty,0]$	[2.1.1]
$\mathbb{C}^+$	slit plane $\mathbb{C}-[0,\infty)$	[2.1.1]
$f\lvert_U$	restriction of f to U	[2.1.2]
$\sqrt[n]{z}$	n-th principal root	[2.1.4]
Arg_α	α-argument of	[2.1.5]
e^z	complex exponential	[2.2.1]
$Log(z)$	principal logarithm	[2.3.1]
$log(z)$	set of logarithms	[2.3.2]
z^α	set of complex powers	[2.4.1]
$\sin(z), \cos(z)$	complex sine and cosine	[2.5.1]
$\sinh(z), \cosh(z)$	complex hyperbolic functions	[2.5.2]
$\tan(z)$	complex tangent	[2.5.3]
$\tanh(z)$	complex hyperbolic tangent	[2.5.3]
$\lim_{n\to\infty} a_n$	limit as $n\to\infty$	[10.1.1]
$\lim_{z\to z_o} f(z)$	limit as $z\to z_o$	[4.3.1]
$C^0(U)$	continuous functions on U	[4.3.4]
$D_\varepsilon(z_o)$	open disk radius ε centered at z_o	[4.2.1]
∂S	boundary of S	[4.2.3]
$[p,q]$	line segment from p to q	[4.2.4]
$f'(z)$	complex derivative	[7.1.1]
J_F	Jacobian matrix of F	[**??**]
$u_x = v_y$, $u_y = -v_x$	CR-equations of $f = u+iv$	[7.2.1]
$\mathcal{O}(C)$	entire functions on $\mathbb{C}$	[7.2.5]
$\mathcal{O}(D)$	holomorphic functions on D	[7.2.9]

You can also use the search function within the pdf-reader.

Contents

Chapter 1

Complex Numbers

1.1 Algebra and Geometry of Complex Numbers

I set aside the question of *existence* for now. Rest assured there is such a thing as $\mathbb{C}$ which merits all the definitions and constructions shared in this section.

Definition 1.1.1. *Let $a, b, c, d \in \mathbb{R}$. A* **complex number** *is an expressions of the form $a + ib$. By assumption, if $a + ib = c + id$ we have $a = c$ and $b = d$. We define the* **real part** *of $a + ib$ by* $\textbf{Re}(a+ib) = a$ *and the* **imaginary part** *of $a+ib$ by* $\textbf{Im}(a+ib) = b$*. The set of all complex numbers is denoted $\mathbb{C}$. Complex numbers of the form $a + i(0)$ are called* **real** *whereas complex numbers of the form $0 + ib$ are called* **imaginary***. The set of imaginary numbers is denoted $i\mathbb{R} = \{iy \mid y \in \mathbb{R}\}$.*

It is customary to write $a + i(0) = a$ and $0 + ib = ib$ as the 0 is superfluous. Furthermore, the notation[1] $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ compactly expresses the fact that each complex number is written as the sum of a real and pure imaginary number. There is also the assumption $\mathbb{R} \cap i\mathbb{R} = \{0\}$. In words, the only complex number which is both real and pure imaginary is 0 itself.

We add and multiply complex numbers in the usual fashion:

Definition 1.1.2. *Let $a, b, c, d \in \mathbb{R}$. We define* **complex addition** *and* **multiplication** *as follows:*

$$(a + ib) + (c + id) = (a + c) + i(b + d) \qquad \& \qquad (a + ib)(c + id) = ac - bd + i(ad + bc).$$

Often the definition is recast in pragmatic terms as $i^2 = -1$ and proceed as usual. Let me remind the reader what is "usual". Addition and multiplication are commutative and obey the usual distributive laws: for $x, y, z \in \mathbb{C}$

$$x + y = y + x, \qquad \& \qquad xy = yx, \qquad \& \qquad x(y + z) = xy + xz,$$

associativity of addition and multiplication can also be derived:

$$(x + y) + z = x + (y + z), \qquad \& \qquad (xy)z = x(yz).$$

The additive identity is 0 whereas the multiplicative identity is 1, in particular:

$$z + 0 = z \qquad \& \qquad 1 \cdot z = z$$

[1] see the discussion of $\oplus$ (the direct sum) in my linear algebra notes. Here I view $\mathbb{R} \leq \mathbb{C}$ and $i\mathbb{R} \leq \mathbb{C}$ as independent $\mathbb{R}$-subspaces whose direct sum forms $\mathbb{C}$.

for all $z \in \mathbb{C}$. Notice, the notation $1z = 1 \cdot z$. Sometimes we like to use a $\cdot$ to emphasize the multiplication, however, usually we just use **juxtaposition** to denote the multiplication. Finally, using the notation of Definition 1.1.2, let us check that $i^2 = ii = (0+i)(0+i) = -1$. Take $a = 0, b = 1, c = 0, d = 1$:

$$i^2 = ii = (0+1i)(0+1i) = (0 \cdot 0 - 1 \cdot 1) + i(0 \cdot 1 + 1 \cdot 0) = -1.$$

In view of all these properties (which the reader can easily prove follow from Definition 1.1.2) we return to the multiplication of $a+ib$ and $c+id$:

$$\begin{aligned} (a+ib)(c+id) &= a(c+id) + ib(c+id) \\ &= ac + iad + ibc + i^2bd \\ &= ac - bd + i(ad+bc). \end{aligned}$$

Of course, this is precisely the rule we gave in Definition 1.1.2. It is convenient to define the **modulus** and **conjugate** of a complex number before we work on fractions of complex numbers.

Definition 1.1.3. *Let $a, b \in \mathbb{R}$. We define* **complex conjugation** *as follows:*

$$\overline{a+ib} = a - ib.$$

We also define the **modulus** *of $a+ib$ which is denoted $|a+ib|$ where*

$$|a+ib| = \sqrt{a^2+b^2}.$$

The complex number $a+ib$ is naturally identified[2] with (a, b) and so we have the following geometric interpretations of conjugation and modulus:

(i.) conjugation reflects points over the real axis.

(ii.) modulus of $a+ib$ is the distance from the origin to $a+ib$.

Let us pause to think about the problem of two-dimensional vectors. This gives us another view on the origin of the modulus formula. We call the x-axis the **real axis** as it is formed by complex numbers of the form $z = x$ and the y-axis the **imaginary axis** as it is formed by complex numbers of the form $z = iy$. In fact, we can identify 1 with the unit-vector $(1, 0)$ and i with the unit-vector $(0, 1)$. Thus, 1 and i are orthogonal vectors in the plane and if we think about $z = x + iy$ we can view (x, y) as the coordinates[3] with respect to the basis $\{1, i\}$. Let $w = a+ib$ be another *vector* and note the standard dot-product of such vectors is simply the sum of the products of their horizontal and vertical components:

$$\langle z, w \rangle = xa + yb$$

You can calculate that $\mathbf{Re}(z\overline{w}) = xa + yb$ thus a formula for the dot-product of two-dimensional vectors written in complex notation is just:

$$\langle z, w \rangle = \mathbf{Re}(z\overline{w}).$$

You may also recall from calculus III that the length of a vector $\vec{A}$ is calculated from $\sqrt{\vec{A} \bullet \vec{A}}$. Hence, in our current complex notation the length of the vector z is given by $|z| = \sqrt{\langle z, z \rangle} = \sqrt{z\bar{z}}$.

[2] Euler 1749 had this idea, see [N] page 60.

[3] if you've not had linear algebra yet then you may read on without worry

If you are a bit lost, read on for now, we can also simply understand the $|z| = \sqrt{z\bar{z}}$ formula directly:

$$(a+ib)(\overline{a+ib}) = (a+ib)(a-ib) = a^2+b^2 \qquad \Rightarrow \qquad |z| = \sqrt{z\bar{z}}.$$

Properties of conjugation and modulus are fun to work out:

$$\overline{z+w} = \overline{z}+\overline{w} \qquad \& \qquad \overline{z\cdot w} = \overline{z}\cdot\overline{w} \qquad \& \qquad \overline{\overline{z}} = z \qquad \& \qquad |zw| = |z||w|.$$

We will make use of the following throughout our study:

$$|z+w| \leq |z|+|w|, \qquad |z-w| \geq |z|-|w| \qquad \& \qquad |z| = 0 \text{ if and only if } z=0.$$

also, the geometrically obvious:

$$\mathbf{Re}(z) \leq |z| \qquad \& \qquad \mathbf{Im}(z) \leq |z|.$$

We now are ready to work out the formula for the reciprocal of a complex number. Suppose $z \neq 0$ and $z = a+ib$ we want to find $w = c+id$ such that $zw = 1$. In particular:

$$(a+ib)(c+id) = 1 \quad \Rightarrow \quad ac-bd = 1, \;\; \& \;\; ad+bc = 0$$

You can try to solve these directly, but perhaps it will be more instructive[4] to discover the formula for the reciprocal by a formal calculation:

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} \qquad \Rightarrow \qquad \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}.$$

I said *formal* as the calculation in some sense assumes properties which are not yet justified. In any event, it is simple to check that the reciprocal formula is valid: notice, if $z \neq 0$ then $|z| \neq 0$ hence

$$z\cdot\left(\frac{\bar{z}}{|z|^2}\right) = z\cdot\left(\frac{\bar{z}}{|z|^2}\right) = z\cdot\left(\frac{1}{|z|^2}\cdot\bar{z}\right) = \frac{1}{|z|^2}(z\bar{z}) = \frac{1}{|z|^2}|z|^2 = 1.$$

The calculation above proves $z^{-1} = \bar{z}/|z|^2$.

Example 1.1.4.

$$\frac{1}{i} = \frac{-i}{|i|^2} = \frac{-i}{1} = -i.$$

Of course, this can easily be seen from the basic identity $ii = -1$ *which gives* $1/i = -i$.

Example 1.1.5.

$$(1+2i)^{-1} = \frac{1-2i}{|1+2i|^2} = \frac{1-2i}{1+4} = \frac{1-2i}{5}.$$

A more pedantic person would insist you write the standard **Cartesian form** $\frac{1}{5} - i\frac{2}{5}$.

The only complex number which does not have a multiplicative inverse is 0. This is part of the reason that $\mathbb{C}$ forms a **field**. A field is a set which allows addition and multiplication such that the only element without a multiplicative inverse is the additive identity (aka "zero"). There is a more precise definition given in abstract algebra texts, I'll leave that for you to discover. That said, it is perhaps useful to note that $\mathbb{Z}/p\mathbb{Z}$ for p prime, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields. Furthermore, it is sometimes useful to have notation for the set of complex numbers which admit a multicative inverse;

[4] this calculation is how to find $(a+ib)^{-1}$ for explicit examples

Definition 1.1.6. *The group of nonzero complex numbers is denoted* $\mathbb{C}^\times$ *where* $\mathbb{C}^\times = \mathbb{C} - \{0\}$*.*

If we envision $\mathbb{C}$ as the plane, this is the plane with the origin removed. For that reason $\mathbb{C}^\times$ is also known as the **punctured plane**. The term **group** is again from abstract algebra and it refers to the multiplicative structure paired with $\mathbb{C}^\times$. Notice that $\mathbb{C}^\times$ is not closed under addition since $z \in \mathbb{C}^\times$ implies $-z \in \mathbb{C}^\times$ yet $z + (-z) = 0 \notin \mathbb{C}^\times$. I merely try to make some connections with your future course work in abstract algebra.

The complex conjugate gives us nice formulas for the real and imaginary parts of $z = x + iy$. Notice that if we add $z = x + iy$ and $\bar{z} = x - iy$ we obtain $z + \bar{z} = 2x$. Likewise, subtraction yields $z - \bar{z} = 2iy$. Thus as (by definition) $x = \mathbf{Re}(z)$ and $y = \mathbf{Im}(z)$ we find:

$$\mathbf{Re}(z) = \frac{1}{2}(z + \bar{z}) \qquad \& \qquad \mathbf{Im}(z) = \frac{1}{2i}(z + \bar{z})$$

In summary, for each $z \in \mathbb{C}$ we have $z = \mathbf{Re}(z) + i\mathbf{Im}(z)$.

Example 1.1.7.

$$|z| = |\mathbf{Re}(z) + i\mathbf{Im}(z)| \leq |\mathbf{Re}(z)| + |i\mathbf{Im}(z)| = |\mathbf{Re}(z)| + |i||\mathbf{Im}(z)| = |\mathbf{Re}(z)| + |\mathbf{Im}(z)|.$$

An important basic type of function in complex function theory is a polynomial. These are sums of power functions. Notice that z^n is defined inductively just as in the real case. In particular, $z^0 = 1$ and $z^n = zz^{n-1}$ for all $n \in \mathbb{N}$. The story of $n \in \mathbb{C}$ waits for a future section.

Definition 1.1.8. *A* **complex polynomial of degree** $n \geq 0$ *is a function of the form:*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_o$$

for $z \in \mathbb{C}$*. The set of all polynomials in* z *is denoted* $\mathbb{C}[z]$*.*

The theorem which follows makes complex numbers an indispensable tool for polynomial algebra.

Theorem 1.1.9. Fundamental Theorem of Algebra *Every complex polynomial* $p(z) \in \mathbb{C}[z]$ *of degree* $n \geq 1$ *has a factorization*

$$p(z) = c(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k},$$

where $z_1, z_2, \ldots, z_k$ *are distinct and* $m_j \geq 1$ *for all* $j \in \mathbb{N}_k$*. Moreover, this factorization is unique upto a permutation of the factors.*

I prefer the statement above (also given on page 4 of Gamelin) to what is sometimes given in other books. The other common version is: *every nonconstant complex polynomial has a zero.* Let us connect this to our version. Recall[5] the **factor theorem** states that if $p(z) \in \mathbb{C}[z]$ with $deg(p(z)) = n \geq 1$ and z_o satisfies $p(z_o) = 0$ then $(z - z_o)$ is a **factor** of $p(z)$. This means there exists $q(z) \in \mathbb{C}[z]$ with $deg(q(z)) = n - 1$ such that $p(z) = (z - z_o)q(z)$. It follows that we may completely factor a polynomial by repeated application of the alternate version of the Fundamental Theorem of Algebra and the factor theorem.

[5] I suppose this was only presented in the case of real polynomials, but it also holds here. See Fraleigh or Dummit and Foote or many other good abstract algebra texts for how to build polynomial algebra from scratch. That is not our current purpose so I resist the temptation.

Example 1.1.10. *Let* $p(z) = (z+1)(z+2-3i)$ *note that* $p(z) = z^2+(3-3i)z-3i$. *This polynomial has zeros of* $z_1 = -1$ *and* $z_2 = -2+3i$. *These are not in a* **conjugate pair** *but this is not surprising as* $p(z) \notin \mathbb{R}[z]$. *The notation* $\mathbb{R}[z]$ *denotes polynomials in* z *with coefficients from* $\mathbb{R}$.

Example 1.1.11. *Suppose* $p(z) = (z^2+1)((z-1)^2+9)$. *Notice* $z^2+1 = z^2 - i^2 = (z+i)(z-i)$. *We are inspired to do likewise for the first factor which is already in completed-square format:*

$$(z-1)^2 + 9 = (z-1)^2 - 9i^2 = (z-1-3i)(z-1+3i).$$

Thus, $p(z) = (z+i)(z-i)(z-1-3i)(z-1+3i)$. *Notice* $p(z) \in \mathbb{R}[z]$ *is clear from the initial formula and we do see the complex zeros of* $p(z)$ *are arranged in conjugate pairs* $\pm i$ *and* $1 \pm 3i$.

The example above is no accident: complex algebra sheds light on real examples. Since $\mathbb{R} \subseteq \mathbb{C}$ it follows we may naturally view $\mathbb{R}[z] \subseteq \mathbb{C}[z]$ thus the Fundamental Theorem of Algebra applies to polynomials with real coefficients in this sense: to solve a real problem we enlarge the problem to the corresponding complex problem where we have the mathematical freedom to solve the problem in general. Then, upon finding the answer, we drop back to the reals to present our answer. I invite the reader to derive the Fundamental Theorem of Algebra for $\mathbb{R}[z]$ by applying the Fundamental Theorem of Algebra for $\mathbb{C}[z]$ to the special case of real coefficients. Your derivation should probably begin by showing a complex zero for a polynomial in $\mathbb{R}[z]$ must come with a conjugate zero.

The importance of taking a complex view was supported by Gauss throughout his career. From a letter to Bessel in 1811 [R91](p.1):

> At the very beginning I would ask anyone who wants to introduce a new function into analysis to clarify whether he intends to confine it to real magnitudes [real values of its argument] and regard the imaginary values as just vestigial - or whether he subscribes to my fundamental proposition that in the realm of magnitudes the imaginary ones $a+b\sqrt{-1} = a+bi$ have to be regarded as enjoying equal rights with the real ones. We are not talking about practical utility here; rather analysis is, to my mind, a self-sufficient science. It would lose immeasurably in beauty and symmetry from the rejection of any fictive magnitudes. At each stage truths, which otherwise are quite generally valid, would have to be encumbered with all sorts of qualifications.

Gauss used the complex numbers in his dissertation of 1799 to prove the Fundamental Theorem of Algebra. Gauss offered four distinct proofs over the course of his life. See Chapter 4 of [N91] for a discussion of Gauss' proofs as well as the history of the Fundamental Theorem of Algebra. Many original quotes and sources are contained in that chapter which is authored by Reinhold Remmert.

1.2 On the Existence of Complex Numbers

Euler's work from the eigthteenth century involves much calculation with complex numbers. It was Euler who in 1777 introduced the notation $i = \sqrt{-1}$ to replace $a + b\sqrt{-1}$ with $a + ib$ (see [R91] p. 10). As is often the case in this history of mathematics, we used complex numbers long before we had a formal construction which proved the existence of such numbers. In this subsection I add some background about how to **construct** complex numbers. In truth, my true concept of complex numbers is already given in what was already said in this section in the discussion up to Definition 1.1.3 (after that point I implicitly make use of Model I below). In particular, I would claim a mature viewpoint is that a complex number is defined by it's properties. That said, it is good to give a construction which shows such objects do exist. However, it's also good to realize the construction is not written in stone as it may well be replaced with some *isomorphic* copy. There are three main models:

Model I: complex numbers as points in the plane: Gauss proposed the following construction: $\mathbb{C}_{Gauss} = \mathbb{R}^2$ paired with the multiplication $\star$ and addition rules below:

$$(a,b)+(c,d)=(a+c,b+d) \qquad (a,b)\star(c,d)=(ac-bd,ad+bc)$$

for all $(a,b),(c,d)\in\mathbb{C}_{Gauss}$. What does this have to do with $\sqrt{-1}$? Consider,

$$(1,0)\star(a,b)=(a,b)$$

Thus, multiplication by $(1,0)$ is like multiplying by 1. Also,

$$(0,1)\star(0,1)=(-1,0)$$

It follows that $(0,1)$ is like i. We can define a mapping $\Psi:\mathbb{C}_{Gauss}\to\mathbb{C}$ by $\Psi(a,b)=a+ib$. This mapping has $\Psi(z+w)=\Psi(z)+\Psi(w)$ as well as $\Psi(z\star w)=\Psi(z)\Psi(w)$. We observe that Ψ is a one-one correspondence of $\mathbb{C}_{Gauss}$ and $\mathbb{C}$ which preserves multiplication and addition. Intuitively, the existence of Ψ means that $\mathbb{C}$ and $\mathbb{C}_{Gauss}$ are the *same* object viewed in different notation[6].

Model II: complex numbers as matrices of a special type: perhaps Cayley was the first to [7] propose the following construction:

$$\mathbb{C}_{matrix}=\left\{\begin{bmatrix} a & b \\ -b & a\end{bmatrix} \;\middle|\; a,b\in\mathbb{R}\right\}$$

Addition is matrix addition and we multiply in $\mathbb{C}_{matrix}$ using the standard matrix multiplication:

$$\begin{bmatrix} a & b \\ -b & a\end{bmatrix}\begin{bmatrix} c & d \\ -d & c\end{bmatrix}=\begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd\end{bmatrix}.$$

In matrices, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$ serves as the multiplicative identity (it is like 1) whereas the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$ is analogus to i. Notice, $\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}=-\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$. The mapping $\Phi:\mathbb{C}_{matrix}\to\mathbb{C}$ defined by $\Phi\left(\begin{bmatrix} a & b \\ -b & a\end{bmatrix}\right)=a+ib$ is a one-one correspondence for which the algebra of matrices transfers to the algebra of complex numbers.

Model III: complex numbers as an extension field of $\mathbb{R}$: The set of real polynomials in x is denoted $\mathbb{R}[x]$. If we define $\mathbb{C}_{extension}=\mathbb{R}[x]/<x^2+1>$ then the multiplication and addition in this set is essentially that of polynomials. However, strict polynomial equality is replaced with congruence modulo x^2+1. Suppose we use $[f(x)]$ to denote the equivalence class of $f(x)$ modulo x^2+1 then as a point set:

$$[f(x)]=\{f(x)+(x^2+1)h(x)\mid h(x)\in\mathbb{R}[x]\}.$$

[6]the careful reader is here frustrated by the fact I have yet to say what $\mathbb{C}$ is as a point set

[7]I asked this at the math stackexchange site and it appears Cayley knew of these in 1858, see the link for details.

More to the point, $[x^2+1]=[0]$ and $[x^2]=[-1]$. From this it follows:

$$[a+bx][c+dx] = [(a+bx)(c+dx)] = [ac+(ad+bc)x+bdx^2] = [ac-bd+(ad+bc)x].$$

In $\mathbb{C}_{extension}$ the constant polynomial class $[1]$ serves as the multiplicative identity whereas $[x]$ is like i. Furthermore, the mapping $\Xi([a+bx]) = a+bi$ gives a one-one correspondence which preserves the addition and multiplication of $\mathbb{C}_{extension}$ to that of $\mathbb{C}$. The technique of field extensions is discussed in some generality in the second course of a typical abstract algebra sequence. Cauchy found this formulation in 1847 see [N91] p. 63.

Conclusion: as point sets $\mathbb{C}_{Gauss}, \mathbb{C}_{matrix}, \mathbb{C}_{extension}$ are not the same. However, each one of these objects provides the algebraic structure which (in my view) defines $\mathbb{C}$. We could use any of them as the complex numbers. For the sake of being concrete, I will by default use $\mathbb{C} = \mathbb{C}_{Gauss}$. But, I hope you can appreciate this is merely a **choice**. But, it's also a good choice since geometrically it is natural to identify the plane with $\mathbb{C}$. You might take a moment to appreciate we face the same foundational issue when we face the question of what is $\mathbb{R}, \mathbb{Q}, \mathbb{N}$ etc. I don't think we ever constructed these in our course work. You have always worked formally in these systems. It sufficed to accept truths about $\mathbb{N}, \mathbb{Q}$ or $\mathbb{R}$, you probably never required your professor to show you such a system could indeed exist. Rest assured, they exist.

Remark: it will be our custom whenever we write $z = x+iy$ it is understood that $x = \mathbf{Re}(z) \in \mathbb{R}$ and $y = \mathbf{Im}(z) \in \mathbb{R}$. If we write $z = x+iy$ and intend $x, y \in \mathbb{C}$ then it will be our custom to make this explicitly known. This will save us a few hundred unecessary utterances in our study.

1.3 Polar Representations

Polar coordinates in the plane are given by $x = r\cos\theta$ and $y = r\sin\theta$ where we define $r = \sqrt{x^2+y^2}$. Observe that $z = x+iy$ and $r = |z|$ hence:

$$z = |z|(\cos\theta + i\sin\theta).$$

The **standard angle** is measured CCW from the positive x-axis. There is considerable freedom in our choice of θ. For example, we identify geometrically $-\pi/2, 3\pi/2, 7\pi/2, \dots$. It is useful to have a notation to express the totality of this ambiguity as well as to remove it by a standard choice:

Definition 1.3.1. *Let $z \in \mathbb{C}$ with $z \neq 0$.* **Principle argument** *of z is the $\theta_o \in (-\pi, \pi]$ for which $z = |z|(\cos\theta_o + i\sin\theta_o)$. We denote the principle argument by $Arg(z) = \theta_o$. The* **argument** *of z is denoted $arg(z)$ which is the* **set** *of all $\theta \in \mathbb{R}$ such that $z = |z|(\cos\theta + i\sin\theta)$.*

From basic trigonometry we find: for $z \neq 0$,

$$arg(z) = Arg(z) + 2\pi\mathbb{Z} = \{Arg(z) + 2\pi k \mid k \in \mathbb{Z}\}.$$

Notice that $arg(z)$ is not a function on $\mathbb{C}$. Instead, $arg(z)$ is a **multiply-valued function**. You should recall a function is, by definition, **single-valued**. In contrast, the Principle argument is a function from the punctured plane $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$ to $(-\pi, \pi]$.

Example 1.3.2. *Let $z = 1-i$ then $Arg(z) = -\pi/4$ and $arg(z) = \{-\pi/4 + 2\pi k \mid k \in \mathbb{Z}\}$.*

Example 1.3.3. *Let $z = -2 - 3i$. We can calculate $\tan^{-1}(-3/-2) \approxeq 0.9828$. Furthermore, this complex number is found in quadrant III hence the standard angle is approximately $\theta = 0.9828 + \pi = 4.124$. Notice, $\theta \neq Arg(z)$ since $4.124 \notin (-\pi, \pi]$. We substract 2π from θ to obtain the approximate value of $Arg(z)$ is -2.159. To be precise, $Arg(z) = \tan^{-1}(3/2) - \pi$ and*

$$arg(z) = \tan^{-1}(3/2) - \pi + 2\pi\mathbb{Z}.$$

At this point it is useful to introduce a notation which simultaneously captures sine and cosine and their appearance in the formulas at the beginning of this section. What follows here is commonly known as **Euler's formula**. Incidentally, it is mentioned in [E91] (page 60) that this formula appeared in Euler's writings in 1749 and the manner in which he wrote about it implicitly indicates that Euler already understood the geometric interpretation of $\mathbb{C}$ as a plane. It fell to nineteenth century mathematicians such as Gauss to clarify and demystify $\mathbb{C}$. It was Gauss who first called $\mathbb{C}$ **complex numbers** in 1831 [E91](page 61). This is what Gauss had to say about the term "imaginary" in a letter from 1831 [E91](page 62)

> It could be said in all this that so long as imaginary quantities were still based on a fiction, they were not, so to say, fully accepted in mathematics but were regarded rather as something to be tolerated; they remained far from being given the same status as real quantities. There is no longer any justification for such discrimination now that the metaphysics of imaginary numbers has been put in a true light and that it has been shown that they have just as good a real objective meaning as the negative numbers.

I only wish the authority of Gauss was properly accepted by current teachers of mathematics. It seems to me that the education of precalculus students concerning complex numbers is far short of where it ought to reach. Trigonometry and two dimensional geometry are both greatly simplified by the use of complex notation.

Definition 1.3.4. *Let $\theta \in \mathbb{R}$ and define the* **imaginary exponential** *denoted $e^{i\theta}$ by:*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

For $z \neq 0$, if $z = |z|e^{i\theta}$ then we say $|z|e^{i\theta}$ is a **polar form** *of z.*

The polar form is not unique unless we restrict the choice of θ.

Example 1.3.5. *Let $z = -1 + i$ then $|z| = \sqrt{2}$ and $Arg(z) = \frac{3\pi}{4}$. Thus, $-1 + i = \sqrt{2}e^{i\frac{3\pi}{4}}$.*

Example 1.3.6. *If $z = i$ then $|z| = 1$ and $Arg(z) = \frac{\pi}{2}$ hence $i = e^{i\frac{\pi}{2}}$.*

Properties of the imaginary exponential follow immediately from corresponding properties for sine and cosine. For example, since sine and cosine are never zero at the same angle we know $e^{i\theta} \neq 0$. On the other hand, as $\cos(0) = 1$ and $\sin(0) = 0$ hence $e^0 = \cos(0) + i\sin(0) = 1$ (if this were not the case then the notation of $e^{i\theta}$ would be dangerous in view of what we know for exponentials on $\mathbb{R}$). The imaginary exponential also supports the law of exponents:

$$e^{i\theta}e^{i\beta} = e^{i(\theta+\beta)}.$$

This follows from the known adding angle formulas $\cos(\theta + \beta) = \cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta)$ and $\sin(\theta + \beta) = \sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta)$. However, the imaginary exponential does not behave exactly the same as the real exponentials. It is far from injective[8] In particular, we have 2π-periodicity of the imaginary exponential function: for each $k \in \mathbb{Z}$,

$$e^{i(\theta+2\pi k)} = e^{i\theta}.$$

This follows immediately from the definition of the imaginary exponential and the known trigonometric identities: $\cos(\theta + 2\pi k) = \cos(\theta)$ and $\sin(\theta + 2\pi k) = \cos(\theta)$ for $k \in \mathbb{Z}$. Given the above, we have the following modication of the $1-1$ principle from precalculus:

$$e^{i\theta} = e^{i\beta} \quad \Rightarrow \quad \theta - \beta \in 2\pi\mathbb{Z}.$$

Example 1.3.7. *To solve $e^{3i} = e^{i\theta}$ yields $3 - \theta = 2\pi k$ for some $k \in \mathbb{Z}$. Therefore, the solutions of the given equation are of the form $\theta = 3 - 2\pi k$ for $k \in \mathbb{Z}$.*

In view of the addition rule for complex exponentials the multiplication of complex numbers in polar form is very simple:

Example 1.3.8. *Let $z = re^{i\theta}$ and $w = se^{i\beta}$ then*

$$zw = re^{i\theta}se^{i\beta} = rse^{i(\theta+\beta)}.$$

We learn from the calculation above that the product of two complex numbers has a simple geometric meaning in the polar notation. The magnitude of $|zw| = |z||w|$ and the angle of zw is simply the sum of the angles of the products. To be careful, we can show:

$$arg(zw) = arg(z) + arg(w)$$

where the addition of sets is made in the natural manner[9]:

$$arg(z) + arg(w) = \{\theta' + \beta' \mid \theta' \in arg(z),\ \beta' \in arg(w)\}.$$

If we multiply $z \neq 0$ by $e^{i\beta}$ then we **rotate** $z = |z|e^{i\theta}$ to $ze^{i\beta} = |z|e^{i(\theta+\beta)}$. It follows that multiplication by imaginary exponentials amounts to rotating points in the complex plane.
The formulae below can be derived by an inductive argument and the addition law for imaginary exponentials.

Theorem 1.3.9. de Moivere's formulae *let $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$ then $(e^{i\theta})^n = e^{in\theta}$.*

To appreciate this I'll present $n = 2$ as Gamelin has $n = 3$.

Example 1.3.10. *De Moivere gives us $(e^{i\theta})^2 = e^{2i\theta}$ but $e^{i\theta} = \cos\theta + i\sin\theta$ thus squaring yields:*

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta.$$

However, the definition of the imaginary exponential gives $e^{2i\theta} = \cos(2\theta) + i\sin(2\theta)$. Thus,

$$\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta = \cos(2\theta) + i\sin(2\theta).$$

Equating the real and imaginary parts separately yields:

$$\cos^2\theta - \sin^2\theta = \cos(2\theta), \qquad \& \qquad 2\cos\theta\sin\theta = \sin(2\theta).$$

[8] or 1-1 if you prefer that terminology, the point is multiple inputs give the same output.

[9] Let $S, T \subseteq \mathbb{C}$ and $c \in \mathbb{C}$ then we define

$$cS = \{cs \mid s \in S\} \qquad c + S = \{c + s \mid s \in S\} \qquad S + T = \{s + t \mid s \in S, t \in T\}.$$

These formulae of de Moivere were discovered between 1707 and 1738 by de Moivere then in 1748 they were recast in our present formalism by Euler [R91] see p. 150. Incidentally, page 149 of [R91] gives a rather careful justification of the polar form of a complex number which is based on the application of *function theory*[10]. I have relied on your previous knowledge of trigonometry which may be very non-rigorous. In fact, I should mention, at the moment $e^{i\theta}$ is simply a convenient notation with nice properties, but, later it will be the inevitable extension of the real exponential to complex values. That mature viewpoint only comes much later as we develop a large part of the theory, so, in the interest of not depriving us of exponentials until that time I follow Gamelin and give a transitional definition. It is important we learn how to calculate with the imaginary exponential as it is ubiquitous in examples throughout our study.

Definition 1.3.11. *Suppose $n \in \mathbb{N}$ and $w, z \in \mathbb{C}$ such that $z^n = w$ then z is an* **n-th root of w**. *The* **set of all n-th roots** *of w is (by default) denoted $w^{1/n}$.*

The polar form makes quick work of the algebra here. Suppose $w = \rho e^{i\phi}$ and $z = re^{i\theta}$ such that $z^n = w$ for some $n \in \mathbb{N}$. Observe, $z^n = (re^{i\theta})^n = r^n(e^{i\theta})^n = r^n e^{in\theta}$ hence we wish to find all solutions of:

$$r^n e^{in\theta} = \rho e^{i\phi} \qquad \star .$$

Take the modulus of the equation above to find $r^n = \rho$ hence $r = \sqrt[n]{\rho}$ where we use the usual notation for the (unique) n-th positive root of $r > 0$. Apply $r = \sqrt[n]{\rho}$ to $\star$ and face what remains:

$$e^{in\theta} = e^{i\phi}.$$

We find $n\theta - \phi \in 2\pi\mathbb{Z}$. Thus, $\theta = \frac{2\pi k + \phi}{n}$ for some $k \in \mathbb{Z}$. At first glance, you might think there are **infinitely** many solutions ! However, it happens[11] as k ranges over $\mathbb{Z}$ notice that $e^{i\theta}$ simply we cycles back to the same solutions over and over. In particular, if we restrict to $k \in \{0, 1, 2, \dots, n-1\}$ it suffices to cover all possible n-th roots of w:

$$(\rho e^{i\phi})^{1/n} = \left\{ \sqrt[n]{\rho} e^{i\frac{\phi}{n}}, \sqrt[n]{\rho} e^{i\frac{2\pi+\phi}{n}}, \dots, \sqrt[n]{\rho} e^{i\frac{2\pi(n-1)+\phi}{n}} \right\} \qquad \star^2 .$$

We can clean this up a bit. Note that $\frac{2\pi k+\phi}{n} = \frac{2\pi k}{n} + \frac{\phi}{n}$ hence

$$e^{i\frac{2\pi k+\phi}{n}} = e^{i\left(\frac{2\pi k}{n} + \frac{\phi}{n}\right)} = e^{i\frac{2\pi k}{n}} e^{i\frac{\phi}{n}} = \left(e^{i\frac{2\pi}{n}}\right)^k e^{i\frac{\phi}{n}}$$

The term raised to the k-th power is important. Notice that once we have one element in the set of n-roots then we may **generate** the rest by repeated multiplication by $e^{i\frac{2\pi}{n}}$.

Definition 1.3.12. *Suppose $n \in \mathbb{N}$ then $\omega = e^{i\frac{2\pi}{n}}$ is an* **primitive n-th root of unity**. *If $z^n = 1$ then we say z is an* **n-th root of unity**.

In terms of the language above, every n-th root of unity can be generated by raising the primitive root to some power between 0 and $n - 1$. Returning once more to $\star^2$ we find, using $\omega = e^{i\frac{2\pi}{n}}$:

$$(\rho e^{i\phi})^{1/n} = \left\{ \sqrt[n]{\rho} e^{i\frac{\phi}{n}}, \ \sqrt[n]{\rho} e^{i\frac{\phi}{n}}\omega, \ \sqrt[n]{\rho} e^{i\frac{\phi}{n}}\omega^2, \ \dots, \ \sqrt[n]{\rho} e^{i\frac{\phi}{n}}\omega^{n-1} \right\}.$$

We have to be careful with some real notations at this juncture. For example, it is no longer ok to conflate $\sqrt[n]{x}$ and $x^{1/n}$ even if $x \in (0, \infty)$. The quantity $\sqrt[n]{x}$ is, by definition, $w \in \mathbb{R}$ such that $w^n = x$. However, $x^{1/n}$ is a **set** of values ! (unless we specify otherwise for a specific problem)

[10] in Remmert's text the term "function theory" means complex function theory

[11] it is very likely I prove this assertion in class via the slick argument found on page 150 of [R91].

Example 1.3.13. *The primitive fourth root of unity is* $e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = \cos\pi/2 + i\sin\pi/2 = i$. *Thus, noting that* $1 = 1e^0$ *we find:*

$$1^{1/4} = \{1,\ i,\ i^2,\ i^3\} = \{1, i, -1, -i\}$$

Geometrically, these are nicely arranged in perfect symmetry about the unit-circle.

Example 1.3.14. *Building from our work in the last example, it is easy to find* $(3+3i)^{1/4}$. *Begin by noting* $|3+3i| = \sqrt{18}$ *and* $Arg(3+3i) = \pi/4$ *hence* $3+3i = \sqrt{18}e^{i\pi/4}$. *Thus, note* $\sqrt[4]{\sqrt{18}} = \sqrt[8]{18}$

$$(3+3i)^{1/4} = \{\sqrt[8]{18}e^{i\pi/16}, i\sqrt[8]{18}e^{i\pi/16}, -\sqrt[8]{18}e^{i\pi/16}, -i\sqrt[8]{18}e^{i\pi/16}\}.$$

which could also be expressed as:

$$(3+3i)^{1/4} = \{\sqrt[8]{18}e^{i\pi/16}, \sqrt[8]{18}e^{5i\pi/16}, \sqrt[8]{18}e^{9i\pi/16}, \sqrt[8]{18}e^{13i\pi/16}\}.$$

Example 1.3.15. $(-1)^{1/5}$ *is found by noting* $e^{2\pi i/5}$ *is the primitive* 5*-th root of unity and* $-1 = e^{i\pi}$ *hence*

$$(-1)^{1/5} = \{e^{i\pi/5}, e^{i\pi/5}\omega, e^{i\pi/5}\omega^2, e^{i\pi/5}\omega^3, e^{i\pi/5}\omega^4\}.$$

Add a few fractions and use the 2π*-periodicity of the imaginary exponential to see:*

$$(-1)^{1/5} = \{e^{i\pi/5}, e^{3\pi i/5}, e^{5\pi i/5}, e^{7\pi i/5}, e^{9\pi i/5}\} = \{e^{i\pi/5}, e^{3\pi i/5}, -1, e^{-3\pi i/5}, e^{-\pi i/5}\}.$$

We can use the example above to factor $p(z) = z^5 + 1$. Notice $p(z) = 0$ implies $z \in (-1)^{1/5}$. Thus, the zeros of p are precisely the fifth roots of -1. This observation and the factor theorem yield:

$$p(z) = (z+1)(z - e^{i\pi/5})(z - e^{-i\pi/5})(z - e^{3i\pi/5})(z - e^{-3i\pi/5}).$$

If you start thinking about the pattern here (it helps to draw a picture which shows how the roots of unity are balanced below and above the x-axis) you can see that the conjugate pair factors for $p(z)$ are connected to that pattern. Furthermore, if you keep digging for patterns in factoring polynomials these appear again whenever it is possible. In particular, if $n \in 1 + 2\mathbb{Z}$ then -1 is a root of unity and all other roots are arranged in conjugate pairs.

The words below are a translation of the words written by Galois the night before he died in a duel at the age of 21:

> Go to the roots of these calculations! Group the operations. Classify them according to their complexities rather than their appearances! This, I believe, is the mission of future mathematicians. This is the road on which I am embarking in this work.

Galois' theory is still interesting. You can read about it in many places. For example, see Chapter 14 of Dummit and Foote's *Abstract Algebra*.

Chapter 2

Functions of a complex variable

A **function** from $f : S \to T$ is a single-valued assignment of $f(s) \in T$ for each $s \in S$. This clear definition of function was not clear until the middle of the nineteenth century. It is true that the term originates with Leibniz in 1692 to (roughly) describe magnitudes which depended on the point in question. Then Euler saw fit to call any analytic expression built from variables and some constants a *function*. In other words, Euler essentially defined a function by its formula. However, later, Euler did discuss an idea of an arbitrary function in his study of variational calculus. The clarity to state the modern definition apparently goes to Dirichlet. In 1837 he wrote:

> It is certainly not necessary that the law of dependence of $f(x)$ on x be the same throughout the interval; in fact one need not even think of the dependence as given by explicit mathematical operations.

See [R91] pages 37-38 for more detailed references.

2.1 The Square and Square Root Functions

The title of this section is quite suspicious given our discussion of the n-th roots of unity. We learned that $z^{1/2}$ is not a function because it is double-valued. Therefore, to create a function based on $z^{1/2}$ we must find a method to select one of the values. Gamelin spends several paragraphs to describe how $w = z^2$ maps half of the z-plane onto all of the w-plane except the negative real axis. In particular, he explains how $\{z \in \mathbb{C} \mid \mathbf{Re}(z) > 0\}$ maps to the slit-plane defined below:

Definition 2.1.1. *The (negative)* **slit plane** *is defined as* $\mathbb{C}^- = \mathbb{C} - (-\infty, 0]$. *Explicitly,*

$$\mathbb{C}^- = \mathbb{C} - \{z \in \mathbb{C} \mid \mathbf{Re}(z) \leq 0,\ \mathbf{Im}(z) = 0\}.$$

We also define the **positive slit plane**

$$\mathbb{C}^+ = \mathbb{C} - \{z \in \mathbb{C} \mid \mathbf{Re}(z) \geq 0,\ \mathbf{Im}(z) = 0\}.$$

Generically, when a ray is removed from $\mathbb{C}$ the resulting object is called a slit-plane. We mostly find use of $\mathbb{C}^-$ since it is tied to the principle argument function. Let us introduce some notation to sort out what is said in this section. Mostly we need function notation and the concept of a **restriction**.

Definition 2.1.2. *Let $S \subseteq \mathbb{C}$ and $f : S \to \mathbb{C}$ a function. If $U \subseteq S$ then we define the* **restriction of f to U** *to be the function $f|_U : U \to \mathbb{C}$ where $f|_U(z) = f(z)$ for all $z \in U$.*

Often a function is not injective on its domain, but, if we make a suitable restriction of the domain then an inverse function exists. In calculus I call this a **local inverse** of the function. In the context of complex analysis, the process of restricting the domain such that the range of the restriction does not multiply cover $\mathbb{C}$ is known as making a **branch cut**. The reason for that terminology is manifest in the pictures on page 17 of Gamelin. In what follows I show how a different branch of the square root may be selected.

Example 2.1.3. *Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^2$. Suppose we wish to make a branch cut of $z^{1/2}$ along $[0, \infty)$. This would mean we wish to delete the postive real axis from the range of the square function. Let us denote $\mathbb{C}^+ = \mathbb{C} - [0, \infty)$. The deletion of $[0, \infty)$ means we need to eliminate z which map to the positive real axis. This suggests we limit the argument of z such that $Arg(z^2) \neq 0$. In particular, let us define $U = \{z \in \mathbb{C} \mid \mathbf{Im}(z) > 0\}$. This is the upper half plane. Notice if $z \in U$ then $Arg(z) \in (0, \pi)$. That is, $z \in U$ implies $z = |z|e^{i\theta}$ for $0 < \theta < \pi$. Note:*

$$f|_U(z) = z^2 = \left(|z|e^{i\theta}\right)^2 = |z|^2 e^{2i\theta}$$

Observe $0 < 2\theta < 2\pi$ hence $Arg(z^2) \in (-\pi, 0) \cup (0, \pi]$. To summarize, if $z \in U$ and $w = z^2$ then $w \in \mathbb{C}^+$. Furthermore, we can provide a nice formula for $f_3 = (f|_U)^{-1} : \mathbb{C}^+ \to U$. For $\rho e^{i\phi} \in \mathbb{C}^+$ where $0 < \phi < 2\pi$,

$$f_3(\rho e^{i\phi}) = \sqrt{\rho} e^{i\phi/2}.$$

We could also use the lower half-plane to map to $\mathbb{C}^+$. Let $V = \{z \in \mathbb{C} \mid -\pi < Arg(z) < 0\}$ and notice for $z \in V$ we have $z^2 = |z|^2 e^{2i\theta}$. Thus, once again the standard angle of $w = z^2$ takes on all angles except $\theta = 0$. This is awkwardly captured in terms of the principal argument as $Arg(w) \in (-\pi, 0) \cup (0, \pi]$. Define $f_4 = (f|_V)^{-1} : \mathbb{C}^+ \to V$ for $\rho e^{i\phi} \in \mathbb{C}^+$ where $0 < \phi < 2\pi$ by

$$f_4(\rho e^{i\phi}) = -\sqrt{\rho} e^{i\phi/2}.$$

Together, the ranges of f_3 and f_4 cover almost the whole z-plane. You can envision how to draw pictures for f_3 and f_4 which are analogus to those given for the principal branch and its negative.

It is customary to use the notation $\sqrt{w}$ for the principal branch. Likewise, for other root functions the same convention is made:

Definition 2.1.4. *The* **principal branch** *of the n-th root is defined by:*

$$\sqrt[n]{w} = \sqrt[n]{|w|} e^{i\frac{Arg(w)}{n}}$$

for each $w \in \mathbb{C}^\times$.

Notice that $(\sqrt[n]{w})^n = \left(\sqrt[n]{|w|} e^{i\frac{Arg(w)}{n}}\right)^n = |w| e^{iArg(w)} = w$. Therefore, $f(z) = z^n$ has a local inverse function given by the principal branch. The range of the principal branch function gives the domain on which the principal branch serves as an inverse function. Since $-\pi < Arg(w) < \pi$ for $w \in \mathbb{C}^-$ it follows that $-\pi/n < Arg(w)/n < \pi/n$. Thus, the principal branch serves as the inverse function of $f(z) = z^n$ for $z \in \mathbb{C}^\times$ with $-\pi/n < Arg(z) < \pi/n$. In general, it will take n-branches to cover the z-plane. We can see those arising from rotating the sector centered about zero by the primitive n-th root. Notice this agrees nicely with what Gamelin shows for $n = 2$ in the text as the primitive root of unity in the case of $n = 2$ is just -1 and we obtain the second branch by merely multiplying by -1. This is still true for non-principal branches as I introduce below.

Honestly, to treat this problem in more generality it is useful to introduce other choices for "Arg". I'll introduce the notation here so we have it later if we need it[1].

Definition 2.1.5. *The* **α-Argument** *for $\alpha \in \mathbb{R}$ is denoted $Arg_\alpha : \mathbb{C}^\times \to [\alpha, \alpha + 2\pi)$. In particular, for each $z \in \mathbb{C}^\times$ we define $Arg_\alpha(z) \in arg(z)$ such that $z \in (\alpha, \alpha + 2\pi)$.*

Unfortunately, $Arg_{-\pi} \neq Arg$ since $Arg_{-\pi}$ has $-\pi$ in its range whereas Arg has range without $-\pi$. To be clear, Range(Arg) $= (-\pi, \pi]$ whereas Range($Arg_{-\pi}$) $= [-\pi, \pi)$. In retrospect, we could use $Arg_0 : \mathbb{C}^\times \to [0, 2\pi)$ to construct the branch-cut f_3 from Example 2.1.3:

$$f_3(w) = \sqrt{|w|}e^{iArg_0(w)/2} \qquad \text{for } w \in \mathbb{C}^+ = \mathbb{C} - [0, \infty).$$

We can use the modified argument function above to give branch-cuts for the n-th root function which delete the ray at standard angle α. These correspond to local inverse functions for $f(z) = z^n$ restricted to $\{z \in \mathbb{C}^\times \mid arg(z) = (\alpha/n, (\alpha + 2\pi)/n) + 2\pi\mathbb{Z}\}$.

Riemann Surfaces: if we look at all the branches of the n-root then it turns out we can sew them together along the branches to form the Riemann surface $\mathcal{R}$. Imagine replacing the w-plane $\mathbb{C}$ with n-copies of the appropriate slit plane attached to each other along the branch-cuts. This separates the values of $f(z) = z^n$ hence $f : \mathbb{C} \to \mathcal{R}$ is invertible. The idea of replacing the codomain of $\mathbb{C}$ with a Riemann surface constructed by weaving together different branches of the function is a challenging topic in general. I suspect this article by Teleman on Riemann surfaces is a good place to start.

2.2 The Exponential Function

In this section we extend our transitional definition for the exponential to complex values. What follows is simply the combination of the real and imaginary exponential functions:

Definition 2.2.1. *The* **complex exponential function** *is defined by $z \mapsto e^z$ where for each $z \in \mathbb{C}$ we define $e^z = e^{\mathbf{Re}(z)}e^{\mathbf{Im}(z)}$. In particular, if $x, y \in \mathbb{R}$ and $z = x + iy$,*

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \left(\cos(y) + i\sin(y)\right).$$

When convenient, we also use the notation $e^z = \exp(z)$ to make the argument of the exponential more readable. [2]. Consider, as $|e^{iy}| = \sqrt{e^{iy}e^{-iy}} = \sqrt{e^0} = 1$ we find

$$|e^z| = |e^x e^{iy}| = |e^x||e^{iy}| = |e^x| = e^x.$$

The magnitude of the complex exponential is unbounded as $x \to \infty$ whereas the magnitude approaches zero as $x \to -\infty$. If $z = x + iy$ then $arg(e^{x+iy}) = \{y + 2\pi k \mid k \in \mathbb{Z}\}$. Since $e^{x+iy} = e^x e^{iy}$ it is clear that e^x does not change the direction of e^{x+iy}; $arg(e^{x+iy}) = arg(e^{iy})$.

Observe domain(e^z) $= \mathbb{C}$ however range(e^z) $= \mathbb{C}^\times$ as we know $e^{iy} \neq 0$ for all $y \in \mathbb{R}$. Furthermore, the complex exponential is not injective precisely because the imaginary exponential is not injective. If two complex exponentials agree then their arguments need not be equal. In fact:

$$e^z = e^w \qquad \Leftrightarrow \qquad z - w \in 2\pi i\mathbb{Z}.$$

[1] this is due to §26 of Brown and Churchill you can borrow from me if you wish

[2] Notice, we have not given a careful definition of e^x here for $x \in \mathbb{R}$. We assume, for now, the reader has some base knowledge from calculus which makes the exponential function at least partly rigorous. Later in this our study we find a definition for the exponential which supercedes the one given here and provides a rigorous underpinning for all these fun facts

Moreover, $e^z = 1$ iff $z = 2\pi ik$ for some $k \in \mathbb{Z}$. The complex exponential function is a $2\pi i$-periodic function; $e^{z+2\pi i} = e^z$. We also have

$$e^{z+w} = e^z e^w \qquad \& \qquad (e^z)^{-1} = 1/e^z = e^{-z}.$$

The proof of the addition rule above follows from the usual laws of exponents for the real exponential function as well as the addition rules for cosine and sine which give the addition rule for imaginary exponentials. Of course, $e^z e^{-z} = e^{z-z} = e^0 = 1$ shows $1/e^z = e^{-z}$ but it is also fun to work it out from our previous formula for the reciprocal $1/z = \bar{z}/|z|^2$. We showed $|e^{x+iy}| = e^x$ hence:

$$\frac{1}{e^z} = \frac{e^x e^{-iy}}{(e^x)^2} = e^{-x}e^{-iy} = e^{-(x+iy)} = e^{-z}.$$

As is often the case, the use of x, y notation clutters the argument.

To understand the geometry of $z \mapsto e^z$ we study how the exponential maps the z-plane to the $w = u + iv$-plane where $w = e^z$. Often we look at how lines or circles transform. In this case, lines work well. I'll break into cases to help organize the thought:

1. A **vertical line** in the $z = x + iy$-plane has equation $x = x_o$ whereas y is free to range over $\mathbb{R}$. Consider, $e^{x_o+iy} = e^{x_o}e^{iy}$. As y-varies we trace out a circle of radius e^{x_o} in the $w = u + iv$-plane. In particular, it has equation $u^2 + v^2 = (e^{x_o})^2$.

2. A **horizontal line** in the $z = x + iy$-plane has equation $y = y_o$ whereas x is free to range over $\mathbb{R}$. Consider, $e^{x+iy_o} = e^x e^{iy_o}$. As x-varies we trace out a ray at standard angle y_o in the w-plane.

If you put these together, we see a little rectangle $[a, b] \times [c, d]$ in the z-plane transforms to a little sector in the w-plane with $|w| \in [e^a, e^b]$ and $Arg(w) \in [c, d]$ (assuming $[c, d] \subseteq (-\pi, \pi]$ otherwise we'd have to deal with some alternate argument function). See Figure 5.3 at this website.

2.3 The Logarithm Function

As we discussed in the previous section, the exponential function is not injective. In particular, $e^z = e^{z+2\pi i}$ hence as we study $z \mapsto w = e^z$ we find each horizontal strip $\mathbb{R} \times (y_o, y + 2\pi)$ maps to $\mathbb{C}^\times - \{w \in \mathbb{C} \mid arg(w) \cap \{y_o\}$. In other words, we map horizontal strips of height 2π to the slit-plane where the slit is at standard angle y_o. To cover $\mathbb{C}^-$ we map the horizontal strip $\mathbb{R} \times (-\pi, \pi)$ to $\mathbb{C}^-$. This gives us the **principal logarithm**

Definition 2.3.1. *The* **principal logarithm** *is defined by $Log(z) = \ln(|z|) + iArg(z)$ for each $z \in \mathbb{C}^\times$. In particular, for $z = |z|e^{i\theta}$ with $-\pi < \theta \leq \pi$ we define:*

$$Log(x + iy) = \ln|z| + i\theta.$$

We can also simplify the formula by the power property of the real logarithm to

$$Log(x + iy) = \frac{1}{2}\ln(x^2 + y^2) + iArg(x + iy).$$

Notice: we use "ln" for the **real** logarithm function. In contrast, we reserve the notations "*log*" and "*Log*" for complex arguments. Please do not write $\ln(1 + i)$ as in our formalism that is just nonsense. There is a multiply-valued function of which this is just one branch. In particular:

Definition 2.3.2. *The* **logarithm** *is defined by $log(z) = \ln(|z|) + iarg(z)$ for each $z \in \mathbb{C}^{\times}$. In particular, for $z = x + iy \neq 0$*

$$log(x+iy) = \left\{ \ln\sqrt{x^2+y^2} + i[Arg(x+iy) + 2\pi k] \mid k \in \mathbb{Z} \right\}.$$

Example 2.3.3. *To calculate $Log(1+i)$ we change to polar form $1+i = \sqrt{2}e^{i\pi/4}$. Thus*

$$Log(1+i) = \ln\sqrt{2} + i\pi/4.$$

Note $arg(1+i) = \pi/4 + 2\pi\mathbb{Z}$ hence

$$log(1+i) = \ln\sqrt{2} + i\pi/4 + 2\pi i\mathbb{Z}.$$

There are many values of the logarithm of $1+i$. For example, $\ln\sqrt{2} + 9i\pi/4$ and $\ln\sqrt{2} - 7i\pi/4$ are also a logarithms of $1+i$. These are the beginnings of the two tails[3] which Gamelin illustrates on page 22.

We could use Arg_α as given in Definition 2.1.5 to define other branches of the logarithm. In particular, a reasonable notation would be:

$$Log_\alpha(z) = \ln|z| + iArg_\alpha(z).$$

The set of values in $log(z)$ is formed from the union of all possible values for $Log_\alpha(z)$ as we vary α over $\mathbb{R}$. Notice, for $z \in \mathbb{C}^- = \mathbb{C} - (-\infty, 0]$ we have $Arg_{-\pi}(z) = Arg(z)$ hence the principal logarithm Log and Log_α are the same function on $\mathbb{C}^-$. However, $Log(-1) = i\pi$ whereas $Log_{-\pi}(-1) = -i\pi$.

Finally, let us examine how the logarithm does provide an inverse for the exponential. If we restrict to a particular branch then the calculation is simple. For example, the principal branch, let $z \in \mathbb{R} \times (-\pi, \pi)$ and consider

$$e^{Log(z)} = e^{\ln|z| + iArg(z)} = e^{\ln|z|}e^{iArg(z)} = |z|e^{iArg(z)} = z.$$

Conversely, for $z \in \mathbb{C}^-$,

$$Log(e^z) = \ln|e^z| + iArg(e^z) = \ln(e^{\mathbf{Re}(z)}) + i\mathbf{Im}(z) = \mathbf{Re}(z) + i\mathbf{Im}(z) = z.$$

The discussion for the multiply valued logarithm requires a bit more care. Let $z \in \mathbb{C}^{\times}$, by definition,

$$log(z) = \{\ln|z| + i(Arg(z) + 2\pi k) \mid k \in \mathbb{Z}\}.$$

Let $w \in \log(z)$ and consider,

$$\begin{aligned} e^w &= \exp\left(\ln|z| + i(Arg(z) + 2\pi k)\right) \\ &= \exp\left(\ln|z| + i(Arg(z))\right) \\ &= \exp(\ln|z|)\exp(i(Arg(z)) \\ &= |z|e^{iArg(z)} \\ &= z. \end{aligned}$$

[3] I can't help but wonder, is there a math with more tails

It follows that $e^{\log(z)} = \{z\}$. Sometimes, you see this written as $e^{\log(z)} = z$. if the author is not committed to viewing $\log(z)$ as a set of values. I prefer to use set notation as it is very tempting to use function-theoretic thinking for multiply-valued expressions. For example, a dangerous calculation:

$$1 = -i^2 = -ii = -(-1)^{1/2}(-1)^{1/2} = -((-1)(-1))^{1/2} = -(1)^{1/2} = -1.$$

Wait. This is troubling if we fail to appreciate that $1^{1/2} = \{1, -1\}$. What appears as equality for multiply-valued functions is better understood in terms of inclusion in a set. I will try to be explicit about sets when I use them, but, beware, Gamelin does not share my passion for pedantics.

The trouble arises when we ignore the fact there are multiple values for a complex power function and we try to assume it ought to behave as an honest, single-valued, function.

2.4 Power Functions

Definition 2.4.1. *Let $z, \alpha \in \mathbb{C}$ with $z \neq 0$. Define z^α to be the set of values $z^\alpha = exp(\alpha log(z))$.*

In particular,

$$z^\alpha = \{\exp(\alpha[Log(z) + 2\pi i k]) \mid k \in \mathbb{Z}\}.$$

However,

$$\exp(\alpha[Log(z) + 2\pi i k]) = \exp(\alpha[Log(z))\exp(2\alpha\pi i k).$$

We have already studied the case $\alpha = 1/n$. In that case $\exp(2\alpha\pi i k) = \exp(2\alpha\pi i/n)$ are the n-roots of unity. In the case $\alpha \in \mathbb{Z}$ the phase factor $\exp(2\alpha\pi i k) = 1$ and $z \mapsto z^\alpha$ is **single-valued** with domain $\mathbb{C}$. Generally, the complex power function is not single-valued unless we make some restriction on the domain.

Example 2.4.2. *Observe that $log(3) = \ln(3) + 2\pi i\mathbb{Z}$ hence:*

$$3^i = e^{i\,log(3)} = e^{i(\ln(3)+2\pi i\mathbb{Z})} = e^{i\ln(3)}e^{-2\pi\mathbb{Z}}.$$

In other words,

$$\begin{aligned} 3^i &= [\cos(\ln(3)) + i\sin(\ln(3))]e^{-2\pi\mathbb{Z}} \\ &= \{[\cos(\ln(3)) + i\sin(\ln(3))]e^{-2\pi k} \mid k \in \mathbb{Z}\}. \end{aligned}$$

In this example, the values fall along the ray at $\theta = \ln(3)$. As $k \to \infty$ the values approach the origin whereas as $k \to -\infty$ the go off to infinity. I suppose we could think of it as two tails, one stretched to ∞ and the other bunched at 0.

On page 25 Gamelin shows a similar result for i^i. However, as was known to Euler [R91] (p. 162), there is a **real** value of i^i. In a letter to Goldbach in 1746, Euler wrote:

> Recently I have found that the expression $(\sqrt{-1})^{\sqrt{-1}}$ has a real value, which in decimal fraction form = 0.2078795763; this seems remarkable to me.

On pages 160-165 of [R91] a nice discussion of the general concept of a logarithm is given. The problem of multiple values is dealt directly with considerable rigor.

2.5 Trigonometric and Hyperbolic Functions

If you've taken calculus with me then you already know that for $\theta \in \mathbb{R}$ the formulas:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \& \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

are of tremendous utility in the derivation of trigonometric identities. They also set the stage for our definitions of sine and cosine on $\mathbb{C}$:

Definition 2.5.1. *Let $z \in \mathbb{C}$. We define:*

$$\cos z = \frac{1}{2}\big(e^{iz} + e^{-iz}\big) \qquad \& \qquad \sin z = \frac{1}{2i}\big(e^{iz} - e^{-iz}\big)$$

All your favorite algebraic identities from real trigonometry hold here, unless, you are a fan of $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$. Those are not true for the complex sine and cosine. In particular, note:

$$e^{i(x+iy)} = e^{ix}e^{-y} \qquad \& \qquad e^{-i(x+iy)} = e^{-ix}e^{y}$$

Hence,

$$\cos(x+iy) = \frac{1}{2}\big(e^{ix}e^{-y} + e^{-ix}e^{y}\big) \qquad \& \qquad \sin(x+iy) = \frac{1}{2i}\big(e^{ix}e^{-y} - e^{-ix}e^{y}\big)$$

Clearly as $|y| \to \infty$ the moduli of sine and cosine diverge. I present explicit formulas for the moduli of sine and cosine later in terms of the hyperbolic functions.

I usually introduce hyperbolic cosine and sine as the even and odd parts of the exponential function:

$$e^x = \underbrace{\frac{1}{2}\big(e^x + e^{-x}\big)}_{\cosh(x)} + \underbrace{\frac{1}{2}\big(e^x - e^{-x}\big)}_{\sinh(x)}.$$

Once again, the complex hyperbolic functions are merely defined by replacing the real variable x with the complex variable z:

Definition 2.5.2. *Let $z \in \mathbb{C}$. We define:*

$$\cosh z = \frac{1}{2}\big(e^{z} + e^{-z}\big) \qquad \& \qquad \sinh z = \frac{1}{2}\big(e^{z} - e^{-z}\big).$$

The hyperbolic trigonometric functions and the circular trigonometric functions are linked by the following simple identities:

$$\cosh(iz) = \cos(z) \qquad \& \qquad \sinh(iz) = i\sin(z)$$

and

$$\cos(iz) = \cosh(z) \qquad \& \qquad \sin(iz) = i\sinh(z).$$

Return once more to cosine and use the adding angle formula (which holds in the complex domain as the reader is invited to verify)

$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

and
$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$
In view of these identities, we calculate the modulus of sine and cosine directly,
$$|\cos(x+iy)|^2 = \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y)$$
$$|\sin(x+iy)|^2 = \sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y).$$
However, $\cosh^2 y - \sinh^2 y = 1$ hence
$$\begin{aligned}\cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y) &= \cos^2(x)[1+\sinh^2(y)] + \sin^2(x)\sinh^2(y) \\ &= \cos^2(x) + [\cos^2(x) + \sin^2(x)]\sinh^2(y) \\ &= \cos^2(x) + \sinh^2(y).\end{aligned}$$
A similar calculation holds for $|\sin(x+iy)|^2$ and we obtain:
$$|\cos(x+iy)|^2 = \cos^2(x) + \sinh^2(y) \qquad \& \qquad |\sin(x+iy)|^2 = \sin^2(x) + \sinh^2(y).$$
Notice, for $y \in \mathbb{R}$, $\sinh(y) = 0$ iff $y = 0$. Therefore, the only way the moduli of sine and cosine can be zero is if $y = 0$. It follows that only zeros of sine and cosine are precisely those with which we are already familar on $\mathbb{R}$. In particular,
$$\sin(\pi\mathbb{Z}) = \{0\} \qquad \& \qquad \cos\left(\frac{2\mathbb{Z}+1}{2}\pi\right) = \{0\}.$$
There are pages and pages of interesting identities to derive for the functions introduced here. However, I resist. In part because they make nice homework/test questions for the students. But, also, in part because a slick result we derive later on forces identities on $\mathbb{R}$ of a particular type to necessarily extend to $\mathbb{C}$.

Definition 2.5.3. *Tangent and hyperbolic tangent are defined in the natural manner:*
$$\tan z = \frac{\sin z}{\cos z} \qquad \& \qquad \tanh z = \frac{\sinh z}{\cosh z}.$$
The domains of tangent and hyperbolic tangent are simply $\mathbb{C}$ with the zeros of the denominator function deleted. In the case of tangent, $\text{domain}(\tan z) = \mathbb{C} - \left(\frac{2\mathbb{Z}+1}{2}\right)\pi$.

Inverse Trigonometric Functions: consider $f(z) = \sin z$ then as $\sin(z + 2\pi k) = \sin(z)$ for all $k \in \mathbb{Z}$ we see that the inverse of sine is multiply-valued. If we wish to pick one of those values we should study how to solve $w = \sin z$ for z. Note:
$$2iw = e^{iz} - e^{-iz}$$
multiply by e^{iz} to obtain:
$$2iwe^{iz} = (e^{iz})^2 - 1.$$
Now, substitute $\eta = e^{iz}$ to obtain:
$$2iw\eta = \eta^2 - 1 \qquad \Rightarrow \qquad 0 = \eta^2 - 2iw\eta - 1.$$
Completing the square yields,
$$0 = (\eta - iw)^2 + w^2 - 1 \qquad \Rightarrow \qquad (\eta - iw)^2 = 1 - w^2.$$

Consequently, $\eta - iw \in (1-w^2)^{1/2}$ which in terms of the principal root implies $\eta = iw \pm \sqrt{1-w^2}$. But, $\eta = e^{iz}$ so we find:

$$e^{iz} = iw \pm \sqrt{1-w^2}.$$

There are many solutions to the equation above which are by custom included in the multiply-valued inverse sine mapping below:

$$z = \sin^{-1}(w) = -i\, log(iw \pm \sqrt{1-w^2}).$$

Usually in an application where the above expression was found the context would guide us to choose a particular logarithm. For example, for appropriate w we could study $z = -iLog\left(iw + \sqrt{1-w^2}\right)$ and find $\sin z = w$ for z so-defined.

Once again, the problem of defining an inverse sine **function** requires we reduce the domain of sine to a set which is small enough that sine is injective. The problem of ambiguity in defining an inverse sine function was already present in the context of the real sine function. It is our custom that $\text{range}\sin^{-1} = [-\pi/2, \pi/2]$, but this is just one of an infinitely many choices. Notice the sine function is injective going from any peak to valley of the sine graph. We could just as well have defined $\text{range}\sin^{-1} = [\pi/2, 3\pi/2]$ then inverse sine would be the honest inverse of sine restricted to $[\pi/2, 3\pi/2]$. Why not? To quote my littlest brother when he was little: *cause be why.*

Chapter 3

Numbers

The idea of a *number* is probably more general than you realize. The sort of numbers we introduce in this chapter are still relatively simple compared to other popular abstract number systems. For example, I have nothing here to say about pyadic, hyperreal or surreal numbers. Not to mention number fields of finite characteristic. All of the aforementioned topics have a vast literature which was largely authored in the past century. Instead, our concept of number in this work is more closely aligned with the structure we have already seen in $\mathbb{C}$. Are numbers form what is known as a *real associative algebra of finite dimension*. Such algebras used to be known as *linear algebras*, but that term is not in modern usage. The history of real associative algebras includes complex numbers and famously Hamilton's quaternions. However, the algebras I share in these notes are less known to the mathematics community of this century. One of my goals is to understand what we are missing by largely ignoring such number systems in our standard curriculum.

This Chapter is an attempt at simplifying my presentation in Section 4 of *Introduction to $\mathcal{A}$-Calculus*, https://arxiv.org/abs/1708.04135. In particular, I simply assume as a point set $\mathcal{A} = \mathbb{R}^n$ in this Chapter and use only the standard basis. For those students who have had linear algebra the presentation in my article follows the basis-dependence of the concepts introduced here.

3.1 Real Associative Algebra

A vector space paired with a multiplication forms an *algebra*. We assume the vector space is $\mathbb{R}^n$ for the sake of keeping the presentation elementary as possible here.

Definition 3.1.1. *Let $\mathcal{A} = \mathbb{R}^n$ paired with a function $\star : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is called* **multiplication**. *In particular, the multiplication map satisfies the properties below:*

(i.) bilinear: *$(cx + y) \star z = c(x \star z) + y \star z$ and $x \star (cy + z) = c(x \star y) + x \star z$ for all $x, y, z \in \mathcal{A}$ and $c \in \mathbb{R}$,*

(ii.) associative: *for which $x \star (y \star z) = (x \star y) \star z$ for all $x, y, z \in \mathcal{A}$ and,*

(iii.) unital: *there exists $\mathbb{1} \in \mathcal{A}$ for which $\mathbb{1} \star x = x$ and $x \star \mathbb{1} = x$.*

We say $x \in \mathcal{A}$ is an $\mathcal{A}$**-number**. *If $x \star y = y \star x$ for all $x, y \in \mathcal{A}$ then $\mathcal{A}$ is* **commutative**.

When there is no ambiguity we use $1 = \mathbb{1}$ and we replace $\star$ with juxtaposition; $xy = x \star y$. We assume $\mathcal{A}$ is an associative algebra of finite dimension over $\mathbb{R}$ throughout the remainder of this paper. In the commutative case there is no need to distinguish between *left* and *right* properties.

However, we allow the possibility that $\mathcal{A}$ be **noncommutative** at this point in our development.

Given an algebra $\mathcal{A}$ we can trade it for a set of equivalent matrices denoted $\text{M}_{\mathcal{A}}$ or left-multiplication maps which we denote by $\mathcal{R}_{\mathcal{A}}$. Let us describe how this trade is made[1].

Definition 3.1.2. *If $L_\alpha : \mathcal{A} \to \mathcal{A}$ is defined by $L_\alpha(x) = \alpha \star x$ for all $x \in \mathcal{A}$ then L_α is a* **left-multiplication map**. *We let $\mathcal{R}_{\mathcal{A}}$ denote the* **regular representation** *of $\mathcal{A}$ which is defined to be the set of all left-multiplication maps on $\mathcal{A}$.*

A mapping $T : \mathcal{A} \to \mathcal{A}$ is said to be **right-$\mathcal{A}$-linear** if $T(x \star y) = T(x) \star y$ for all $x, y \in \mathcal{A}$. It turns out this is just another way of looking at left-multiplication maps[2].

Proposition 3.1.3. *A map $T : \mathcal{A} \to \mathcal{A}$ is right-$\mathcal{A}$-linear if and only if $T \in \mathcal{R}_{\mathcal{A}}$.*

Proof: suppose $T : \mathcal{A} \to \mathcal{A}$ is right-$\mathcal{A}$-linear. Notice $x = 1 \star x$ thus for all $x \in \mathcal{A}$ hence $T(x) = T(1 \star x) = T(1) \star x = L_{T(1)}(x)$ and we find $T = L_{T(1)}$ which means T is the left-multiplication map by $T(1)$. Conversely, suppose $T \in \mathcal{R}_{\mathcal{A}}$ then there exists $\alpha \in \mathcal{A}$ such that $T = L_\alpha$. Let $x, y \in \mathcal{A}$ and observe by associativity of $\mathcal{A}$ we find

$$T(x \star y) = L_\alpha(x \star y) = \alpha \star (x \star y) = (\alpha \star x) \star y = L_\alpha(x) \star y = T(x) \star y.$$

Thus T is right-$\mathcal{A}$-linear and this concludes our proof. $\square$

Now, let us suppose $L_\alpha \in \mathcal{R}_{\mathcal{A}}$ then it is a simple exercise to show L_α is a linear transformation on $\mathcal{A} = \mathbb{R}^n$. Suppose $x, y \in \mathcal{A}$ and $c \in \mathbb{R}$ then

$$L_\alpha(cx + y) = \alpha \star (cx + y) = c\alpha \star x + \alpha \star y = cL_\alpha(x) + L_\alpha(y)$$

We learn in *Linear Algebra* that a linear transformation on $\mathbb{R}^n$ can be represented by *matrix multiplication* via the *standard matrix.* In particular if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then $[T] \in \mathbb{R}^{n \times n}$ denotes the **standard matrix** of T and $T(x) = [T]x$ for all $x \in \mathbb{R}^n$. Here $[T]x$ is the multiplication of the square $n \times n$ matrix $[T]$ with the $n \times 1$ column vector x. Let me give a brief example for those who have not taken linear algebra. This example also serves to illustrate the conventions I use for typsetting column vectors.

Example 3.1.4. *The map $T(x, y) = (x + 2y, 3x + 4y)$ is linear and*

$$T(x,y) = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad [T] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

The standard matrix is also found by evaluating T on the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$:

$$[T] = [T(e_1)|T(e_2)] = [T(1,0)|T(0,1)] = [(1,3)|(2,4)] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

[1] I should mention another way to define a regular representation is to consider the **structure constants** C_{ijk} defined implicitly by $e_i \star j_j = \sum_{i,j,k} C_{ijk} e_k$. The matrix R_i defined by $(R_i)_{kj} = C_{ijk}$ is in the first fundamental representation, S_i defined by $(S_j)_{ik} = C_{ijk}$ is in the second fundamental representation and finally $(Q_k)_{ij} = C_{ijk}$ defines a paraisotropic matrix of the algebra. If you enjoy this way of thinking then I would encourage you to read the literature of hypercomplex analysis written by Ward and Wagner. See [ward1940, ward1952, wagner1948].

[2] The property of right-$\mathcal{A}$-linearity will be important to future Chapters

In general, if $T:\mathbb{R}^n\to\mathbb{R}^n$ is a linear transformation then the **standard matrix** of T is the $n\times n$ matrix $[T]=[T(e_1)|T(e_2)|\cdots|T(e_n)]$ where[3] $(e_i)_j=\delta_{ij}$ is the standard basis for $\mathbb{R}^n$. When we calculate the standard matrix of a left-multiplication map the resulting matrix is part of the *matrix representation of* $\mathcal{A}$.

Definition 3.1.5. *The* **matrix representation of** $\mathcal{A}$ *is the set of all standard matrices for left-multiplication maps of* $\mathcal{A}$*;* $M_{\mathcal{A}}=\{[T]\mid T\in\mathcal{R}_{\mathcal{A}}\}$. *Equivalently,*

$$M_{\mathcal{A}}=\{[\alpha\star e_1|\alpha\star e_2|\cdots|\alpha\star e_n]\mid\alpha\in\mathcal{A}\}.$$

We also denote $\mathbf{M}(\alpha)=[\alpha\star e_1|\alpha\star e_2|\cdots|\alpha\star e_n]$ *and say* $\mathbf{M}(\alpha)$ **represents** α.

Observe that when $e_1=1\in\mathcal{A}$ the matrix representation is very simple to understand:

$$\mathbf{M}(\alpha)=[\alpha|\alpha\star e_2|\cdots|\alpha\star e_n],$$

the first column of such a matrix representation fixes the remaining columns. In contrast, the identity of the algebra is not e_1 (or any other e_j) in Examples 3.1.18 and 3.1.25. The next example is a bit silly, but I think it is worth including:

Example 3.1.6. *The* **real numbers** *with their usual addition and multiplication is an associative algebra over* $\mathbb{R}$. *If* $a\in\mathbb{R}$ *then* $[a]\in M_{\mathbb{R}}=\mathbb{R}^{1\times 1}$ *is its left regular representation. Usually we will not distinguish between* a *and* $[a]$.

Model II of Section 1.2 is seen once more:

Example 3.1.7. *We denote* **complex numbers** *by*

$$a+ib=\begin{bmatrix}a\\b\end{bmatrix}=a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}=ae_1+be_2$$

so our usual convention is $1=e_1$ *and* $i=e_2$. *We calculate the representation of* $a+ib$ *as follows:*

$$\mathbf{M}(a+ib)=[a+ib|(a+ib)i]=[a+ib|-b+ia]=\begin{bmatrix}a&-b\\b&a\end{bmatrix}.$$

If you know the formula for the 2×2 inverse matrix then it is interesting to note: for $a+ib\neq 0$,

$$(\mathbf{M}(a+ib))^{-1}=\begin{bmatrix}a&-b\\b&a\end{bmatrix}^{-1}=\frac{1}{a^2+b^2}\begin{bmatrix}a&b\\-b&a\end{bmatrix}=\mathbf{M}\left(\frac{a-ib}{a^2+b^2}\right)=\mathbf{M}\left((a+ib)^{-1}\right). \tag{3.1}$$

That is, the multiplicative inverse of the representation of a complex number is the representation of the multiplicative inverse of the number. Moreover, notice the formula makes sense whenever $a^2+b^2\neq 0$. In particular, only $a=b=0$ is a problem. Only zero fails to have a multiplicative inverse for $\mathbb{C}$. In fact, the complex numbers form a **field** since they have the structure of a commutative algebra **and** have the pleasant feature that every nonzero number has a multiplicative inverse. The definition which follows gives us some less clumsy language to continue this discussion:

Definition 3.1.8. *We say* $x\in\mathcal{A}$ *is a* **unit** *if there exists* $y\in\mathcal{A}$ *for which* $x\star y=y\star x=\mathbb{1}$. *The set of all units is known as the* **group of units** *and we denote this by* $\mathbf{U}(\mathcal{A})$. *We say* $a\in\mathcal{A}$ *is a* **zero-divisor** *if* $a\neq 0$ *and there exists* $b\neq 0$ *for which* $a\star b=0$ *or* $b\star a=0$. *We also denote* $\mathbf{zd}(\mathcal{A})=\{x\in\mathcal{A}\mid x=0$ *or* x *is a zero-divisor*$\}$

[3] The notation δ_{ij} is called the **Kronecker delta**, it is defined to be 1 if $i=j$ and 0 if $i\neq j$.

Notice the following result puts rather strict restrictions on the possible geometries for $\mathbf{zd}(\mathcal{A})$.

Theorem 3.1.9. *The set $\mathbf{zd}(\mathcal{A})$ is fixed under negation.*

Proof: suppose $x \in \mathbf{zd}(\mathcal{A})$ then there exists $y \in \mathcal{A}$ for which $x \star y = 0$. Thus $-x \star y = 0$ and we find $-x \in \mathbf{zd}(\mathcal{A})$. $\square$

In a field there are no zero-divisors; every nonzero element in a field is a unit. Equation 3.1 shows there is some connection between units of $\mathbb{C}$ and units of $\mathbf{M}_{\mathbb{C}}$. In fact, we can say much more about the interconnection of $\mathcal{A}$ and $\mathbf{M}_{\mathcal{A}}$. I think the proof of the following makes good homework:

Theorem 3.1.10. *Let $\mathcal{A} = \mathbb{R}^n$ be a unital associative algebra and define the matrix representation of $\mathcal{A}$ by $\mathbf{M}(\alpha) = [L_\alpha]$ for each $\alpha \in \mathcal{A}$ then for $x, y, z \in \mathcal{A}$ and $c \in \mathbb{R}$,*

(i.) $\mathbf{M}(1) = I$ *(here I denotes the identity matrix),*

(ii.) $\mathbf{M}(cx + y) = c\mathbf{M}(x) + \mathbf{M}(y)$,

(iii.) $\mathbf{M}(x \star y) = \mathbf{M}(x)\mathbf{M}(y)$.

In the language of abstract algebra, the result above means that $\mathbf{M} : \mathcal{A} \to \mathbf{M}_{\mathcal{A}}$ is an **isomorphism** of algebras. Isomorphisms allow transport of structure. Consider, if $x \star y = 1$ then by the above theorem,

$$\mathbf{M}(x \star y) = \mathbf{M}(x)\mathbf{M}(y) = \mathbf{M}(1) = I \quad \Rightarrow \quad \boxed{\mathbf{M}(x)^{-1} = \mathbf{M}(x^{-1})}. \tag{3.2}$$

Equation 3.1 is simply an example of this general feature of the representation map $\mathbf{M}$. This theorem requires a bit of matrix theory. In particular, we learn in linear algebra that a matrix is a unit if and only if it has nonzero determinant. It follows we can determine the zero divisors of the algebra by finding which matrices in $\mathbf{M}_{\mathcal{A}}$ have zero determinant.

Theorem 3.1.11. *Let $\mathcal{A} = \mathbb{R}^n$ be a unital associative algebra with matrix representation $\mathbf{M}_{\mathcal{A}}$ where $\mathbf{M}(\alpha) = [L_\alpha]$ and $\mathbf{M}^{-1}([L_\alpha]) = \alpha$ for each $\alpha \in \mathcal{A}$. Then,*

(i.) $\mathbf{U}(\mathcal{A}) = \mathbf{M}^{-1}\left(\{A \in \mathbf{M}_{\mathcal{A}} \mid det(A) \neq 0\}\right)$,

(ii.) $\mathbf{zd}(\mathcal{A}) = \mathbf{M}^{-1}\left(\{A \in \mathbf{M}_{\mathcal{A}} \mid det(A) = 0\}\right)$.

Example 3.1.12. *The* **hyperbolic** *numbers are given by $\mathcal{H} = \mathbb{R} \oplus j\mathbb{R}$ where $j^2 = 1$. If $a + jb, c + jd \in \mathcal{H}$ then*

$$(a + jb)(c + jd) = ac + adj + jbc + j^2bd = ac + bd + j(ad + bc).$$

Here we have $e_1 = 1$ and $e_2 = j$ thus calculate

$$\mathbf{M}(a + bj) = [a + bj|(a + bj)j] = [a + bj|b + ja] = \left[\begin{array}{cc} a & b \\ b & a \end{array}\right].$$

Calculate $det(\mathbf{M}(a + bj)) = det\left[\begin{array}{cc} a & b \\ b & a \end{array}\right] = a^2 - b^2$ thus $\mathbf{zd}(\mathcal{H}) = \{a + bj \mid a^2 = b^2\}$ whereas $\mathbf{U}(\mathcal{H}) = \{a + bj \mid a^2 \neq b^2\}$. Using Equation 3.2we find the inverse of a unit in $\mathcal{H}$ is simply

$$\frac{1}{a + bj} = \frac{a - bj}{a^2 - b^2}. \tag{3.3}$$

Ignoring Equation 3.2 for a moment, another way we could derive the above formula is by direct algebra in the hyperbolic numbers. Notice the identity $(a + bj)(a - bj) = a^2 - b^2$ given $a^2 - b^2 \neq 0$

allows us to normalize to obtain $(a+bj)\left(\frac{a-bj}{a^2-b^2}\right) = 1$. *Notice* $\mathcal{H}$ *is not a field. There are many nonzero hyperbolic numbers which fail to have multiplicative inverses. Geometrically, if* $z = x + jy$ *is a point in the hyperbolic plane then we find the zero divisors along the lines* $y = \pm x$. *Fascinating things happen with these zero-divisors, they are usually tied to the more novel features of hyperbolic analysis.*

One last algebraic idea before I share my big list of examples. Sometimes two algebras are really the same algebra given different notation.

Definition 3.1.13. *Let* $\mathcal{A}$ *be an algebra with multiplication* $\star$ *and* $\mathcal{B}$ *be an algebra with multiplication* $\diamond$ *then* $\mathcal{A}$ *is* **isomorphic** *to* $\mathcal{B}$ *if there exists a linear bijection* $\Psi : \mathcal{A} \to \mathcal{B}$ *for which* $\Psi(x \star y) = \Psi(x) \diamond \Psi(y)$ *for all* $x, y \in \mathcal{A}$.

Example 3.1.14. *This discussion continues Example 3.1.12. Let* $\mathcal{B} = \mathbb{R} \times \mathbb{R}$ *with* $(a, b)(c, d) = (ac, bd)$ *for all* $(a, b), (c, d) \in \mathcal{B}$. *We can show that*

$$\Psi(a,b) = a\left(\frac{1+j}{2}\right) + b\left(\frac{1-j}{2}\right) \qquad \& \qquad \Psi^{-1}(x+jy) = (x+y, x-y) \tag{3.4}$$

provide an isomorphism of $\mathcal{H}$ *and* $\mathbb{R} \times \mathbb{R}$. *We can use this isomorphism to transfer problems from* $\mathcal{H}$ *to* $\mathcal{B}$ *and vice-versa. For example, to solve* $z^2 + Bz + C = 0$ *in the hyperbolic numbers we note*

$$z^2 + Bz + C = 0 \quad \Rightarrow \quad \Psi^{-1}(z)^2 + \Psi^{-1}(B)\Psi^{-1}(z) + \Psi^{-1}(C) = 0 \tag{3.5}$$

Setting $\Psi^{-1}(B) = (b_1, b_2)$ *and* $\Psi^{-1}(C) = (c_1, c_2)$ *and* $\Psi^{-1}(z) = (x, y)$ *we arrive at*

$$(x,y)^2 + (b_1,b_2)(x,y) + (c_1,c_2) = 0 \tag{3.6}$$

which reduces to

$$(x^2 + b_1x + c_1, y^2 + b_2y + c_2) = (0,0). \tag{3.7}$$

Of course, these are just quadratic equations in $\mathbb{R}$ *so we can solve them and transfer back the result to the general solution of* $z^2 + Bz + C = 0$ *in* $\mathcal{H}$. *Given this correspondence, we deduce there are either zero, two or four solutions to the quadratic hyperbolic equation.*

3.1.1 Examples

To explain the structure of complex numbers it suffices to say $i^2 = -1$ and then just add and multiply $a + bi, c + di$ as usual. Of course, we can be more explicit in our construction if the audience knows about field extensions or group algebras, but, as a starting point it is convenient to provide definitions of algebras which are accessible to every level of student.

Example 3.1.15. *The* **dual numbers** *are given by* $\mathcal{N} = \mathbb{R} \oplus \epsilon\mathbb{R}$ *where* $\varepsilon^2 = 0$. *If* $a + \varepsilon b, c + \varepsilon d \in \mathcal{N}$ *then*

$$(a+\varepsilon b)(c+\varepsilon d) = ac + ad\varepsilon + bc\varepsilon + \varepsilon^2 bd = ac + (ad+bc)\varepsilon. \tag{3.8}$$

Observe $\mathbf{M}(a+b\varepsilon) = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} \in M_{\mathcal{N}}$ *and* $\mathbf{zd}(\mathcal{N}) = \{a + b\epsilon \mid a^2 = 0\} = \varepsilon\mathbb{R}$. *The units in the dual numbers are of the form* $a + b\varepsilon$ *where* $a \neq 0$. *Note* $(a+b\varepsilon)(a-b\varepsilon) = a^2$ *hence* $\frac{1}{a+b\varepsilon} = \frac{a-b\varepsilon}{a^2}$ *provided* $a \neq 0$.

For higher dimensional algebras the multiplicative inverse of a general element can be calculated by computing the inverse of the element's regular representation.

Example 3.1.16. *The n-**th order dual numbers** are given by $\mathcal{N}_n = \mathbb{R} \oplus \varepsilon\mathbb{R} \oplus \cdots \oplus \varepsilon^{n-1}\mathbb{R}$ where $\varepsilon^n = 0$ and $\varepsilon^k \neq 0$ for $1 \leq k \leq n-1$. The regular representation is formed by lower triangular matrices of a particular type:*

$$\mathbf{M}(a_1 + a_2\varepsilon + \cdots + a_n\varepsilon^{n-1}) = \begin{bmatrix} a_1 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_1 & 0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{bmatrix} \in M_{\mathcal{N}_n} \tag{3.9}$$

Notice $a_1 + a_2\varepsilon + \cdots + a_n\varepsilon^{n-1} \in \mathbf{zd}(\mathcal{N}_n)$ only if $a_1 \neq 0$.

Example 3.1.17. *Let $\mathcal{H}_3 = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R}$ where $j^3 = 1$. The matrix representatives of these numbers have an interesting shape; note: $A \in M_{\mathcal{A}}$ implies $A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$. We note an isomorphism $\mathcal{H}_3 \approx \mathbb{R} \times \mathbb{C}$ is given by mapping j to $(1, \omega)$ where ω is a third root of unity.*

Example 3.1.18. *Let $\mathcal{A} = \mathbb{R} \times \mathcal{H}$ where $\mathbb{1} = (1, 1 + 0j)$. Let $\beta = \{(1,0), (0,1), (0,j)\}$ gives block-diagonal $A \in M_{\mathcal{A}}(\beta)$;*

$$\mathbf{M}_\beta((a, b + cj)) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{bmatrix}. \tag{3.10}$$

This algebra is isomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with $(a_1, a_2, a_3) \star (b_1, b_2, b_3) = (a_1b_1, a_2b_2, a_3b_3)$.

Example 3.1.19. *Let the 4-hyperbolic numbers be defined by $\mathcal{H}_4 = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R} \oplus j^3\mathbb{R}$ where $j^4 = 1$. Observe,*

$$\mathbf{M}(a + bj + cj^2 + dj^3) = \begin{bmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{bmatrix}. \tag{3.11}$$

This algebra is naturally isomorphic to $\mathbb{C} \oplus \mathcal{H}$ which is clearly isomorphic to $\mathbb{C} \times \mathbb{R} \times \mathbb{R}$.

In close analogy to the n-hyperbolic numbers $\mathcal{H}_n$ discussed above we discuss the n-complicated numbers $\mathcal{C}_n$ in what follows. In the case $n = 2$ the complicated numbers are just $\mathbb{C}$ so we begin with $n = 3$.

Example 3.1.20. *Let $\mathcal{C}_3 = \mathbb{R} \oplus k\mathbb{R} \oplus k^2\mathbb{R}$ where $k^3 = -1$. Observe*

$$\mathbf{M}(a + bk + ck^2) = \begin{bmatrix} a & -c & -b \\ b & a & -c \\ c & b & a \end{bmatrix}.$$

Example 3.1.21. *Let $\mathcal{C}_4 = \mathbb{R} \oplus k\mathbb{R} \oplus k^2\mathbb{R} \oplus k^3\mathbb{R}$ where $k^4 = -1$. Observe*

$$\mathbf{M}(a + bk + ck^2) = \begin{bmatrix} a & -d & -c & -b \\ b & a & -d & -c \\ c & b & a & -d \\ d & c & b & a \end{bmatrix}.$$

Example 3.1.22. *Let $\mathcal{A} = \mathcal{H} \times \mathcal{H}$ where $\mathbb{1} = (1 + 0j, 1 + 0j)$. This means $(1, 1)$ is naturally represented by the identity matrix. Set $\beta = \{(1, 0), (j, 0), (0, 1), (0, j)\}$ and observe*

$$\mathbf{M}_\beta((a + bj, c + dj)) = \left[\begin{array}{cc|cc} a & b & 0 & 0 \\ b & a & 0 & 0 \\ \hline 0 & 0 & c & d \\ 0 & 0 & d & c \end{array}\right]. \tag{3.12}$$

This algebra is isomorphic to $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ with the Hadamard product $(a_1, a_2, a_3, a_4)*(b_1, b_2, b_3, b_4) = (a_1b_1, a_2b_2, a_3b_3, a_4b_4)$.

Example 3.1.23. *Let $\mathcal{A} = \mathbb{C} \times \mathbb{C}$ where $\mathbb{1} = (1 + 0i, 1 + 0i)$. Here we study the problem of two complex variables. In this algebra $(1+0i, 1+0i)$ corresponds to the identity and hence $(1, 1)$ is naturally represented by the identity matrix. In total we have once more a block-diagonal representation:*

$$A \in M_{\mathcal{A}} \text{ implies } A = \left[\begin{array}{cc|cc} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ \hline 0 & 0 & c & -d \\ 0 & 0 & d & c \end{array}\right] \text{ and this matrix represents } (a + bi, c + di).$$

Example 3.1.24. *Let $\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$ where $i^2 = j^2 = k^2 = -1$ and $ij = k$. These are Hamilton's famed* **quaternions**. *We can show $ij = -ji$ hence these are not commutative. With respect to the natural basis $e_1 = 1, e_2 = i, e_3 = j, e_4 = k$ we find the matrix representative of $a + ib + cj + dk$ is as follows:*

$$A = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \in M_{\mathbb{H}}. \tag{3.13}$$

Example 3.1.25. *Let $\mathcal{A} = \mathbb{R}^4$ with the multiplication $\star$ induced from the multiplication of 2×2 matrices. This is a noncommutative algebra. In particular, this multiplication is induced in the natural manner:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} t & x \\ y & z \end{bmatrix} = \begin{bmatrix} at + by & ax + bz \\ ct + dy & cx + dz \end{bmatrix}. \tag{3.14}$$

It follows that $(a, b, c, d) \star (t, x, y, z) = (at + by, ax + bz, ct + dy, cx + dz)$. We can read from this multiplication that the representative of $(a, b, c, d) \in \mathcal{A}$ is given by

$$\mathbf{M}(a, b, c, d) = \left[\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & a & 0 & b \\ \hline c & 0 & d & 0 \\ 0 & c & 0 & d \end{array}\right] = \left[\begin{array}{c|c} aI & bI \\ \hline cI & dI \end{array}\right] \in M_{\mathcal{A}}. \tag{3.15}$$

Given how we invented the multplication defining $\mathcal{A}$ we expect the multiplicative identity for $\mathcal{A}$ is given by $(1, 0, 0, 1)$ since this corresponds to the identity matrix in our construction.

A group is a set paired with a multiplication for which every element is a unit. I don't expect you know much about what a group is in this course, but I think it's important to mention the following as a method of construction:

Example 3.1.26. *Let G be a finite multiplicative group; $G = \{g_1, \dots, g_n\}$ then we define*

$$\mathcal{A}_G = g_1\mathbb{R} \oplus \cdots \oplus g_n\mathbb{R} \tag{3.16}$$

with natural multiplication inherited from G. For example, for all $a, b, c, d \in \mathbb{R}$,

$$(ag_1 + bg_2)(cg_3 + dg_4) = acg_1g_3 + adg_1g_4 + bcg_2g_3 + bdg_2g_4. \tag{3.17}$$

The group algebra allows us to multiply $\mathbb{R}$-linear combinations of group elements by extending the group multiplication linearly. By construction, $\{g_1, \dots, g_n\}$ serves as a basis for $\mathcal{A}_G$. As G is a group we know for each $i, j \in \{1, \dots, n\}$ there exists $k \in \{1, \dots, n\}$ for which $g_ig_j = g_k$. If we define structure constants C_{ijk} by $g_ig_j = \sum_l C_{ijl}g_l$ then $g_ig_j = g_k$ implies $C_{ijl} = \delta_{kl}$.

Example 3.1.27. *The cyclic group of order n in multiplicative notation has the form $G = \{e, g, g^2, \dots, g^{n-1}\}$. The group algebra $\mathcal{A}_G = e\mathbb{R} \oplus g\mathbb{R} \oplus \cdots \oplus g^{n-1}\mathbb{R}$. We usually call this algebra the n-hyperbolic numbers.*

3.2 Real and Unreal Parts

We should generalize the terminology of *real* and *imaginary part* to algebras. It is convenient to use the terminology of the **dot-product** to create a formula in that which follows. We remind the reader that $x \bullet y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ for all $x, y \in \mathbb{R}^n$. Note the usual length $\|x\| = \sqrt{x \bullet x}$. Moreover, any vector $x \neq 0$ can be expressed at $x = \|x\|\hat{x}$ where $\|\hat{x}\| = 1$ is a **unit-vector**. To calculate the component of y in the direction of x we simply calculate $y \bullet \hat{x}$. Moreover, the vector-projection of y onto x is given by $\text{Proj}_x(y) = (y \bullet \hat{x})\hat{x} = \left(\frac{y \bullet x}{x \bullet x}\right) x$. Hopefully we saw these ideas for at least $\mathbb{R}^3$ in our multivariate calculus course. History aside, we use the idea of projection to select the real part of a number in $\mathcal{A}$ in what follows:

Definition 3.2.1. *Let $\mathcal{A} = \mathbb{R}^n$ be an associative unital algebra with unity $\mathbb{1}$. If $z \in \mathcal{A}$ then the* **real part** *of z is $\mathbf{Re}(z)$ which is defined by*

$$\mathbf{Re}(z) = \left(\frac{z \bullet \mathbb{1}}{\mathbb{1} \bullet \mathbb{1}}\right)\mathbb{1}.$$

The **unreal part** *of z is $\mathbf{Ur}(z)$ is likewise defined by $\mathbf{Ur}(z) = z - \mathbf{Re}(z)$.*

Notice $z = \mathbf{Re}(z) + \mathbf{Ur}(z)$ for any $z \in \mathcal{A}$. We should discuss the precise connection of this new terminology for $\mathbb{C}$. Consider

$$\mathbf{Ur}(x + iy) = iy = i\mathbf{Im}(x).$$

It is amusing that the imaginary part of a complex number $z = x + iy$ is by definition the real number y.

Example 3.2.2. *Let $\zeta = x + jy + j^2z \in \mathcal{H}_3$ then $\mathbf{Re}(\zeta) = x$ and $\mathbf{Ur}(\zeta) = jy + j^2z$. However, to be clear, if we use notation $\zeta = (x, y, z)$ then $\mathbf{Re}(\zeta) = (x, 0, 0)$ whereas $\mathbf{Ur}(\zeta) = (0, y, z)$.*

Example 3.2.3. *For the direct product algebra $\mathbb{R} \times \mathbb{R}$ the unity is $(1, 1)$ since $(x, y)(1, 1) = (x, y)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Notice $(1, 1) \bullet (1, 1) = 2$ hence*

$$\mathbf{Re}(x, y) = \frac{x + y}{2}(1, 1) = \left(\frac{x + y}{2}, \frac{x + y}{2}\right) \qquad \& \qquad \mathbf{Ur}(x, y) = \left(\frac{x - y}{2}, \frac{y - x}{2}\right).$$

By our construction, $\mathbf{Re}(z)$ and $\mathbf{Ur}(z)$ are **orthogonal** (meaning $\mathbf{Re}(z) \bullet \mathbf{Ur}(z) = 0$) and thus

$$\|z\|^2 = \|\mathbf{Re}(z)\|^2 + \|\mathbf{Ur}(z)\|^2.$$

Indeed, $\|\mathbf{Re}(z)\| \leq \|z\|$ and $\|\mathbf{Ur}(z)\| \leq \|z\|$ is clear from the equation above.

Chapter 4

Topology and Analysis

Our goal in this chapter is to give a basic introduction to topology and analysis for $\mathbb{R}^n$. We wish to describe carefully terms such as *open*, *closed*,*boundary*,*open ball*,*closed ball*, *deleted-open ball*, *interior*, *compact*, *connected*, *path-connected*, *continuous*, *limit point* and ultimately *limit*. All of these terms are defined in terms which ultimately rely on the concept of a metric, or distance function, on $\mathbb{R}^n$ which itself is induced from the Euclidean norm in our context[1].

Unique to these notes, we also discuss how the norm interacts with the multiplication of the algebra. This is not emphasized in the usual course since the norm on $\mathbb{C}$ is *multiplicative*. We explain here that for most algebras we can only expect the norm is *submultiplicative*. Eventually, submultiplicative norms do damage to the theory of power series over an algebra, but that we discuss much later in this course. For the present purpose the submultiplicativity of the norm on an algebra will not impact the results we study about limits.

What is analysis? I sometimes glibly say something along the lines of: *algebra is about equations whereas analysis is about inequality.* Of course, any one-liner cannot hope to capture the entirety of a field of mathematics. We can certainly say both analysis and algebra are about structure. In algebra, the type of structure tends to be about generalizing rules of arithmetic. In analysis, the purpose of the structure is to generalize the process of making imprecise statements precisely known. For example, we might say a function is continuous if its values don't jump for nearby inputs. But, the structure of the limit replaces this fuzzy idea with a precise framework to judge continuity. Remmert remarks on page 38 of [R91] that the idea of continuity pre-rigorization is not strictly in one-one correspondence to our modern usage of the term. Sometimes, the term continuous implied the function had much more structure than mere continuity. The process of clarifying definitions is not always immediate. Cauchy's imprecise language captured the idea of the limit, but, it is generally agreed that it was not until the work of Weierstrauss (and his introduction of the dreaded $\varepsilon\delta$-notation which haunts students to the present day) that the limit was precisely defined.

[1]this is not a necessary restriction, one can show any norm on a finite dimensional vector space induces the same topology, in other words the calculation of limits does not care if you set it up with the Euclidean or Taxi-cab norm

4.1 Submultiplicative Property of Norm

Clearly there are many algebras we can consider. In order to make analysis over an algebra we need a concept of **length** of a number. For $\mathbb{C}$ we defined $|x+iy| = \sqrt{x^2+y^2}$ and rather beautifully it happened that $|zw| = |z||w|$ for all $z, w \in \mathbb{C}$. For an algebra this is not usually possible! Let us define the norm for $\mathcal{A} = \mathbb{R}^n$ by the usual **Euclidean norm**

$$\|x\| = \|(x_1, x_2, \dots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

This norm assigns a length of 1 to each standard basis element and has the excellent property that

$$|x_j| \leq \|x\|$$

for any $j = 1, 2, \dots, n$; in other words, the length of a number in $\mathcal{A} = \mathbb{R}^n$ is longer than the length of any component of the number.

I have known there exists a constant for which $\|x \star y\| \leq M\|x\|\|y\|$ for several years. See Theorem 5.1 in *Introduction to $\mathcal{A}$-Calculus* https://arxiv.org/abs/1708.04135. Khang Nguyen refined that Theorem further and proved the following is reasonable in [CK18].

Definition 4.1.1. *For $\mathcal{A} = \mathbb{R}^n$ with multiplication $\star$ we say the* **submultiplicative constant** *of $\mathcal{A}$ is the smallest constant $m_{\mathcal{A}} \in \mathbb{R}$ such that $\|x \star y\| \leq m_{\mathcal{A}}\|x\|\|y\|$ for all $x, y \in \mathcal{A}$. Furthermore, we say $x, y \in \mathcal{A}$ for which $\|x \star y\| = m_{\mathcal{A}}\|x\|\|y\|$ are known as* **sharpening numbers***.*

At the present time I only know precise values for $m_{\mathcal{A}}$ for the n-hyperbolic numbers, the n-complicated numbers and also dual numbers of small order. These results are an ongoing joint work with Khang Nguyen [KC18]. In particular, we have shown:

(1.) The n-hyperbolic numbers $\mathcal{H}_n$ we find $m_{\mathcal{H}_n} = \sqrt{n}$. I have known for a few years that for hyperbolic numbers of the form $z = x + jy$ with $j^2 = 1$ it is known $\|zw\| \leq \sqrt{2}\|z\|\|w\|$. The result for $n \geq 3$ is thanks to Khang in the Spring 2018 Semester. It is interesting to note the inequality $\|xy\| \leq \sqrt{n}\|x\|\|y\|$ is sharp for $x = y = 1 + j + j^2 + \cdots + j^{n-1}$ since $\|x\| = \sqrt{n}$ and as

$$x^2 = (1 + j + j^2 + \cdots + j^{n-1})^2 = n(1 + j + j^2 + \cdots + j^{n-1}) = nx \tag{4.1}$$

Thus $\|xy\| = n\|x\| = n\sqrt{n} = \sqrt{n}\|x\|\|y\|$. The same x has $x(x-n) = 0$ which shows x and $x - n$ are zero-divsors.

(2.) The n-complicated numbers $\mathcal{C}_n$ it depends on whether n is even or odd. If n is odd then $m_{\mathcal{C}_n} = \sqrt{n}$. However, for n even, we find $m_{\mathcal{C}_n} = \sqrt{n/2}$ and of course $n = 2$ is a very special case; $\|zw\| = \|z\|\|w\|$ for all $z, w \in \mathcal{C}_2 = \mathbb{C}$ which means every complex number is a sharpening number[2]. For $n > 2$ the norm is not multiplicative. In fact, $\|xy\| = m_{\mathcal{C}_n}\|x\|\|y\|$ only for select x, y. In particular, if n is odd then we found sharpening number $x = 1 - k + k^2 - \cdots + k^{n-1} \in \mathcal{C}_n$ has $x^2 = nx$ thus $\|x^2\| = \sqrt{n}\|x\|^2$. In contrast, for even $n \neq 2$ we found sharpening number $x = \sum_{j=0}^{n-1} \cos\left(\frac{j\pi}{n}\right) k^j$ which is a nonzero zero-divisor with $x^2 = \frac{n}{2}x$ and $\|x\| = \sqrt{\frac{n}{2}}$. Thus $\|x^2\| = \sqrt{\frac{n}{2}}\|x\|^2$.

(3.) The n-dual numbers are denoted by $\Delta_n = \mathbb{R}^n$ with $\varepsilon^n = 0$ and typical element $z = x_1 + x_2\varepsilon + \cdots + x_n\varepsilon^{n-1}$ then

[2] be warned, this terminology is special to these notes to my knowledge, so if you tell someone from outside this course this term you'll need to explain it most likely

(a) for $n = 2$, $m_{\Delta_2} = \frac{2}{\sqrt{3}}$ with sharpening number $x = \sqrt{2} + \varepsilon$.

(b) for $n = 3$, $m_{\Delta_3} = \frac{4}{3}$ with sharpening number $x = 2 + 2\varepsilon + \varepsilon^2$.

(c) $n = 4$: $m_{\Delta_4} = \sqrt{\frac{2(1103+33\sqrt{33})}{1153}} \approx 1.49736$. We note equality in $\|xy\| \leq m_{\Delta_4}\|x\|\|y\|$ is attained for $x = y = \sqrt{5+\sqrt{\frac{11}{3}}} + \left(1+\sqrt{\frac{11}{3}}\right)\varepsilon + \sqrt{3+\sqrt{\frac{11}{3}}}\varepsilon^2 + \varepsilon^3 \approx 2.62961 + 2.91485\varepsilon + 2.21695\varepsilon^2 + \varepsilon^3$.

(d) $n = 5$: the closed form expression is too lengthy for us to display here, however an approximation of the constant is $m_{\Delta_5} \approx 1.64748$. Equality in $\|xy\| \leq m_{\Delta_5}\|x\|\|y\|$ is attained when $x = y \approx 1 - 1.14862\varepsilon + 1.03046\varepsilon^2 - 0.700308\varepsilon^3 + 0.304849\varepsilon^4$.

(e) $n = 6$: the submultiplicative constant near $m_2(\Delta_6) \approx 1.78611$ and equality in $\|xy\| \leq m_{\Delta_6}\|x\|\|y\|$ is attained at $x = y \approx 1 - 1.16152\varepsilon + 1.12963\varepsilon^2 - 0.915093\varepsilon^3 + 0.589126\varepsilon^4 - 0.253601\varepsilon^5$.

It was interesting that the sharpening numbers for the dual numbers were not zero divisors in contrast to the n-hyperbolic or n-complicated numbers. In any event, I share these fun facts for your amusement primarily. We really just need to know $m_{\mathcal{A}}$ exists for which $\|x \star y\| \leq m_{\mathcal{A}}\|x\|\|y\|$ for all $x, y \in \mathcal{A}$. That inequality alone suffices for most of our work in the near future. That said, when we study power series over an algebra we will likely discuss the deeper results in [CK18]. Let us summarize the known properties of the norm for $\mathcal{A}$.

Theorem 4.1.2. *If $\mathcal{A} = \mathbb{R}^n$ and $\|\cdot\|$ denotes the Euclidean norm then*

(i.) $\|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality)*

(ii.) $\|cx\| = |c|\|x\|$ *(homogeneity)*

(iii.) $\|x \star y\| \leq m_{\mathcal{A}}\|x\|\|y\|$ *(submultiplicativity)*

for all $x, y \in \mathcal{A}$ and $c \in \mathbb{R}$ where $|c| = \sqrt{c^2}$.

I should mention we need the extension of (i.) for finite sums:

$$\|x_1 + x_2 + \cdots + x_k\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_k\|.$$

Also, beware that $||a/b|| \neq ||a||/||b||$ in general[3]. However, we can say something productive:

Theorem 4.1.3. *Suppose $m_{\mathcal{A}} > 0$ is a real constant such that $||x \star y|| \leq m_{\mathcal{A}}||x||||y||$ for all $x, y \in \mathcal{A}$. If $b \in \mathbf{U}(\mathcal{A})$ and $a \in \mathcal{A}$ then*

$$\frac{||a||}{||b||} \leq m_{\mathcal{A}}\left|\left|\frac{a}{b}\right|\right|. \tag{4.2}$$

Proof: we know there exists $m_{\mathcal{A}} > 0$ for which $||x \star y|| \leq m_{\mathcal{A}}||x||||y||$ for all $x, y \in \mathcal{A}$. Consider, if $b \in \mathbf{U}(\mathcal{A})$ and $a \in \mathcal{A}$ then $a/b \in \mathcal{A}$ and $a = b \star (a/b)$ thus
$||a|| = ||b \star (a/b)|| \leq m_{\mathcal{A}}\,||b||\,\left|\left|\frac{a}{b}\right|\right|$ and we deduce $\frac{||a||}{||b||} \leq m_{\mathcal{A}}\left|\left|\frac{a}{b}\right|\right|$. $\square$

[3] we assume $\mathcal{A}$ is commutative and thus denote both $a \star b^{-1}$ and $b^{-1} \star a$ by a/b.

4.2 Topology

At this point we turn to the geometric topology of $\mathbb{R}^n$. Fundamental to almost everything is the idea of a **disk** ($n = 2$) or **ball**. Here the term ball is used in a rather abstract fashion, it includes line-segments, disks, actual three dimensional balls and a host of higher-dimensional balls which I personally am unable to directly visualize. Let $\mathcal{A} = \mathbb{R}^n$ where $\|x\| = \sqrt{x \bullet x}$ (Euclidean norm) throughout what follows.

Definition 4.2.1. *An* **open ball** *of radius ε centered at $z_o \in \mathcal{A}$ is the subset all numbers which are less than an ε distance from z_o, we denote this open ball by*

$$B_\varepsilon(z_o) = \{z \in \mathcal{A} \mid \|z - z_o\| < \varepsilon\}.$$

The **deleted-ball** *with radius ε centered at z_o is likewise defined*

$$B_\varepsilon^o(z_o) = \{z \in \mathcal{A} \mid 0 < \|z - z_o\| < \varepsilon\}.$$

The **closed ball** *of radius ε centered at $z_o \in \mathcal{A}$ is defined by*

$$\overline{B}_\varepsilon(z_o) = \{z \in \mathcal{A} \mid \|z - z_o\| \leq \varepsilon\}.$$

We use balls to define topological concepts in $\mathcal{A}$.

Definition 4.2.2. *Let $S \subseteq \mathcal{A}$. We say $y \in S$ is an* **interior point** *of S iff there exists some open ball centered at y which is completely contained in S. If each point in S is an interior point then we say S is an* **open** *set.*

Roughly, an open set is one with fuzzy edges. A closed set has solid edges. Furthermore, an open set is the same as its interior and closed set is the same as its closure.

Definition 4.2.3. *We say $y \in \mathcal{A}$ is a* **limit point** *of S iff every open ball centered at y contains points in $S - \{y\}$. We say $y \in \mathcal{A}$ is a* **boundary point** *of S iff every open ball centered at y contains points not in S and other points which are in $S - \{y\}$. We say $y \in S$ is an* **isolated point** *or* **exterior point** *of S if there exist open ball about y which do not contain other points in S. The set of all interior points of S is called the* **interior of** S*. Likewise the set of all boundary points for S is called the*[4] **boundary** *of S and is denoted ∂S. The* **closure** *of S is defined to be $\overline{S} = S \cup \{y \in \mathbb{C} \mid y \text{ a limit point of } S\}$.*

To avoid certain pathological cases we often insist that the set considered is a **domain** or a **region**. These are technical terms in this context and we should be careful not to confuse them with their previous uses in mathematical discussion.

Definition 4.2.4. *If $a, b \in \mathcal{A}$ then we define the* **directed line segment from** a **to** b *by*

$$[a, b] = \{a + t(b - a) \mid t \in [0, 1]\}$$

This notation is pretty slick as it agrees with interval notation on $\mathbb{R}$ when we think about them as line segments along the real axis of $\mathcal{A}$. However, certain things I might have called crazy in precalculus now become totally sane. For example, $[4, 3]$ has a precise meaning. I think, to be fair, if you teach precalculus and someone tells you that $[4, 3]$ meant the same set of points, but they prefer to look at them Manga-style then you have to give them credit.

[4]technically this is the topological boundary which we can distinguish from the manifold boundary. For example, (a, b) has topological boundary $\{a, b\}$ whereas the manifold boundary is empty. In a manifold with boundary, if it has a nonempty boundary then the boundary is necessarily a subset of the manifold. I digress mightily here.

Definition 4.2.5. *A subset U of $\mathcal{A}$ is called* **star shaped** *with* **star center $\mathbf{z_o}$** *if there exists z_o such that each $z \in U$ has $[z_o, z] \subseteq U$.*

A given set may have many star centers[5]. For example, $\mathbb{C}^-$ is star shaped and the only star centers are found on $[0, \infty)$. Likewise, $\mathbb{C}^+$ is star shaped with possible star centers found on $(-\infty, 0]$.

Definition 4.2.6. *A* **polygonal path** *γ from a to b in $\mathcal{A}$ is the union of finitely many line segments which are placed end to end; $\gamma = [a, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-2}, z_{n-1}] \cup [z_{n-1}, b]$.*

Gamelin calls a polygonal path in $\mathbb{C}$ a **broken line segment**.

Definition 4.2.7. *A set $S \subseteq \mathcal{A}$ is* **connected** *iff there exists a polygonal path contained in S between any two points in S. That is for all $a, b \in S$ there exists a polygonal path γ from a to b such that $\gamma \subseteq S$*

Technically, the definition above defines **path-connected** for $\mathcal{A}$. Fortunately, even for general topologies, path-connected implies connected. More generally, in topology a set is connected if it has no separation. A set U is separated if there exist open sets U_1, U_2 such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = U$. This characterization of connectedness plays an important role in certain proofs of complex analysis so it's wise to mention it here despite it's apparent weirdness. Actually, the definition I just shared for connectedness is not nearly as weird as the fact that in general connected does not imply path-connected. See this blurb by Professor Keith Conrad if you want some gory detail on why connected does not always imply path connected. For $\mathbb{R}^n$ open and connected implies path connected so we can interchange the ideas of path-connected and connected for open sets.

Definition 4.2.8. *An open connected set is called a* **domain**. *We say R is a* **region** *if $R = D \cup S$ where D is a domain D and $S \subseteq \partial D$.*

The concept of a domain is most commonly found in the remainder of our study. You should take note of its meaning as it will not be emphasized every time it is used later.

Definition 4.2.9. *A subset $U \subseteq \mathcal{A}$ is* **bounded** *if there exists $M > 0$ and $z_o \in U$ for which $U \subseteq B_\delta(z_o)$. If $U \subseteq \mathcal{A}$ is both closed and bounded then we say U is* **compact**.

I should mention the definition of compact given here is not a primary definition, when you study topology or real analysis you will learn a more fundamental characterization of compactness. We may combine terms in reasonable ways. For example, a domain which is also star shaped is called a **star shaped domain**. A region which is also compact is a **compact region**.

The theorem which follows is interesting because it connects an algebraic condition $\nabla h = 0$ with a topological trait of connectedness. Recall that $h : \mathbb{R}^n \to \mathbb{R}$ is **continuously differentiable** if each of the partial derivatives of h is continuous. We need this condition to avoid pathological issues which arise from merely assuming the partial derivatives exist. In the real case, the existence of the partial derivatives does not imply their continuity. We'll see something different for $\mathbb{C}$ as we study complex differentiability.

[5]if a person knew something about this activity called basketball there must be team-specific jokes to make here

Theorem 4.2.10. *If $h : D \to \mathbb{R}$ is a continuously differentiable function on a domain $D \subseteq \mathcal{A}$ such that $\nabla h = \left\langle \frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \dots, \frac{\partial h}{\partial x_n} \right\rangle = 0$ at each point on D then h is constant.*

Proof: Let $p, q \in D$. As D is connected there exists a polygonal path γ from p to q. Let $p_1, p_2, \dots, p_n$ be the points at which the line segments comprising γ are joined. In particular, γ_1 is a path from p to p_1 and we parametrize the path such that $\text{dom}(\gamma_1) = [0, 1]$. By the chain rule,

$$\frac{d}{dt}(h(\gamma_1(t))) = \nabla h(\gamma_1(t))) \bullet \frac{d\gamma_1(t))}{dt}$$

however, $\gamma_1(t) \in D$ for each t hence $\nabla h(\gamma_1(t))) = 0$. Consequently,

$$\frac{d}{dt}(h(\gamma_1(t))) = 0$$

It follows from calculus $h \circ \gamma_1$ is constant on $[0, 1]$ thus $h(\gamma_1(0)) = h(\gamma_1(1))$ hence $h(p) = h(p_1)$. But, we can repeat this argument to show $h(p_2) = h(p_3)$ and so forth and we arrive at:

$$h(p) = h(p_1) = h(p_2) = \cdots = h(p_n) = h(q).$$

But, p, q were arbitrary thus h is constant on D. $\square$

We have much more to say about real differential calculus in later parts of these notes. I included this here because of its topological content.

4.3 Limits and Continuity

What follows is the natural extension of the $(\varepsilon\delta)$-definition to our current context. Once more we assume $\mathcal{A} = \mathbb{R}^n$ throughout what follows.

Definition 4.3.1. *(**Limit and Continuity of function on** $\mathcal{A}$) Let $f : U \subseteq \mathcal{A} \to \mathcal{A}$ be a function and z_o a limit point of U. Also, suppose $L \in \mathcal{A}$. We say $\lim\limits_{z \to z_o} f(z) = L$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $z \in \mathcal{A}$ with $0 < \|z - z_o\| < \delta$ implies $\|f(z) - L\| < \varepsilon$. We say f is continuous at z_o if $\lim_{z \to z_o} f(z) = f(z_o)$.*

We also write $f(z) \to L$ as $z \to z_o$ when the limit exists. Limit laws generalize to our context as you might expect:

Theorem 4.3.2. *Suppose $\lim\limits_{z \to z_o} f(z), \lim\limits_{z \to z_o} g(z) \in \mathcal{A}$ and $c \in \mathcal{A}$ then*

(a.) $\lim\limits_{z \to z_o} [f(z) + g(z)] = \lim\limits_{z \to z_o} f(z) + \lim\limits_{z \to z_o} g(z)$

(b.) $\lim\limits_{z \to z_o} [f(z) \star g(z)] = \lim\limits_{z \to z_o} f(z) \star \lim\limits_{z \to z_o} g(z)$

(c.) $\lim\limits_{z \to z_o} [c \star f(z)] = c \star \lim\limits_{z \to z_o} f(z)$

(d.) $\lim\limits_{z \to z_o} \left[\dfrac{f(z)}{g(z)}\right] = \dfrac{\lim_{z \to z_o} f(z)}{\lim_{z \to z_o} g(z)}$

where in the last property we assume $\lim\limits_{z \to z_o} g(z) \in U(\mathcal{A})$.

Proof: parts (a.) and (c.) can be proved in nearly the same fashion as was done in first semester calculus. Time permitting I'll show in lecture what I mean by that. In contrast, (b.) and (d.) require a different approach for $\mathcal{A}$ due to the possibility of zero-divisors and the submultiplicativity of the norm. Let me sketch the argument here for (b.):

1. Notice $f = (f_1, f_2, \dots, f_n)$ where $f_j : U \subseteq \mathcal{A} \to \mathbb{R}$ are the component functions of f for $j = 1, 2, \dots, n$. The vector limit theorem of advanced calculus says $\lim f = L = (L_1, \dots, L_n)$ if and only if $\lim f_j = L_j$ for $j = 1, 2, \dots, n$. In other words, we can attack a vector limit one component a time if we so choose.

2. The product function defined pointwise by $(f \star g)(z) = f(z) \star g(z)$ can be written as a sum of component functions of f and g weighted against the structure constants of $\mathcal{A}$; $fg = \sum_{i,j,k} C_{ijk} f_i g_j e_k$ hence $(f \star g)_k = \sum_{i,j,k} C_{ijk} f_i g_j$. Then, as the limits of f_i, g_j are known since we assume the limits of f, g are given we are able to calculate:

$$\begin{aligned}
\lim_{z \to z_o} (f \star g)_k &= \lim_{z \to z_o} \sum_{i,j,k} C_{ijk} f_i g_j \\
&= \sum_{i,j,k} C_{ijk} \lim_{z \to z_o} f_i g_j \qquad \text{(by parts (a.) and (c.) which I might prove in class)} \\
&= \sum_{i,j,k} C_{ijk} \left(\lim_{z \to z_o} f_i\right)\left(\lim_{z \to z_o} g_j\right) \qquad \text{(need Lemma I for this step)} \\
&= \sum_{i,j,k} C_{ijk} \left(\lim_{z \to z_o} f\right)_i \left(\lim_{z \to z_o} g\right)_j \qquad \text{(vector limit theorem)} \\
&= \left(\lim_{z \to z_o} f \star \lim_{z \to z_o} g\right)_k
\end{aligned}$$

Since k is arbitrary this proves $(b.)$ if we can prove the Lemma to follow:

Lemma I: if $f_i, g_j : U \subseteq \mathcal{A} \to \mathbb{R}$ where z_o is a limit point of U and $\lim_{z \to z_o} f_i = L_i$ and $\lim_{z \to z_o} g_j = M_j$ then $\lim_{z \to z_o}(f_i g_j) = L_i M_j$.

I think proving this Lemma this will make a good homework once I give a bit more guidance.

To prove (d.) notice $\frac{f}{g} = f \star g^{-1}$ by assumption and to remove digression about left and right factors let us assume $\mathcal{A}$ is commutative. Furthermore, we need the following

Lemma II: if $\lim_{z \to z_o} g = G \in U(\mathcal{A})$ then $\lim_{z \to z_o} g^{-1} = G^{-1}$

In fact, the above Lemma is a particular case of a much more general theorem about limits of composite functions. Assuming Lemma II and part (b.) we have:

$$\lim_{z \to z_o} \left[\frac{f(z)}{g(z)}\right] = \lim_{z \to z_o} \left(f \star g^{-1}\right) = \left(\lim_{z \to z_o} f\right) \star \left(\lim_{z \to z_o} g^{-1}\right) = \lim_{z \to z_o} f \star \left(\frac{1}{\lim_{z \to z_o} g}\right) = \frac{\lim_{z \to z_o} f}{\lim_{z \to z_o} g}. \ \square$$

The composite limit rule takes Lemma II as a special case where the outside function is the reciprocal function $f(z) = \frac{1}{z}$ and the inside function is just $g(z)$ hence $(f \circ g)(z) = \frac{1}{g(z)}$.

Theorem 4.3.3. *Suppose $f : U \subseteq \mathcal{A} \to \mathcal{A}$ and $g \subseteq V \subseteq \mathcal{A} \to \mathcal{A}$ are functions such that z_o is a limit point of V and $\lim_{z \to z_o}(g(z)) = w_o \in U$ such that f is continuous at w_o then*

$$\lim_{z \to z_o} (f(g(z))) = f(\lim_{z \to z_o} g(z)).$$

If time permits I will prove this in class. It is a theorem which is true in much greater generality than that which we consider here. Let me repeat the definition of continuity once more:

Definition 4.3.4. *If $f : dom(f) \subseteq \mathcal{A} \to \mathcal{A}$ is a function $z_o \in dom(f)$ such that $\lim\limits_{z \to z_o} f(z) = f(z_o)$ then f is* **continuous at $\mathbf{z}_o$**. *Iif f is continuous at each point in $U \subseteq dom(f)$ then we say f is* **continuous on U**. *When f is continuous on $dom(f)$ we say f is* **continuous**. *The set of all continuous functions on $U \subseteq \mathcal{A}$ is denoted $C^0(U)$.*

The definition above gives continuity at a point, continuity on a set and finally continuitiy of the function itself. In view of Theorem 4.3.2 we may immediately conclude that if f, g are continuous then $f + g, f \star g, cf$ and f/g are continuous provided $g(z) \in U(\mathcal{A})$ for each $z \in dom(g)$. The conclusion holds at a point, on a common subset of the domains of f, g and finally on the domains of the new functions $f + g, f \star g, c \star f, f/g$.

Example 4.3.5. *Let $f(z) = z$ for all $z \in \mathcal{A}$. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$ then observe $0 < \|z - z_o\| < \delta$ implies $\|f(z) - f(z_o)\| = \|z - z_o\| < \varepsilon$. Consequently, $\lim_{z \to z_o}(z) = z_o$ for any $z_o \in \mathcal{A}$. In other words, the function $f = Id_{\mathcal{A}}$ is continuous on $\mathcal{A}$.*

Example 4.3.6. *Let $f(z) = z^2 = z \star z$ then using a limit law and Example 4.3.6 we find:*

$$\lim_{z \to z_o} z \star z = \lim_{z \to z_o} z \star \lim_{z \to z_o} z = z_o \star z_o = z_o^2.$$

Thus the square function $z \mapsto z^2$ is continuous on $\mathcal{A}$.

As a general rule, if the formula for a function *makes sense* then the function is continuous at such a point. We can continue down the path of the above examples and show any polynomial function on $\mathcal{A}$ is continuous. In addition $z \mapsto \mathbf{Re}z$ and $z \mapsto \mathbf{Ur}z$ as well as $z \mapsto \|z\|$ are all continuous on $\mathcal{A}$. We are also interested in limits of expressions which are not continuous at the limit point, that is the principle obession of the next Chapter. We define derivatives by just such a limit.

Chapter 5

Real Differentiation

This chapter contains a condensed introduction to the theory of real differentiation. I think the treatment I give in Advanced Calculus[1] is better since it embraces and uses normed linear spaces and some abstract linear algebra. That said, about half the audience of this course has no such background so I will focus our attention here on functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. The linear algebra needed is simply basic facts about linear transformations, matrix multiplication and the use of the standard basis for calculation. In short, the same corner of matrix theory which are already encountered in the previous chapter.

In particular, for $F : \mathbb{R}^n \to \mathbb{R}^m$ we define the *differential* at p for F to be the linear transformation $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ which best approximates the change in F near p. We quantify *best* by insisting dF_p satisfy the *Frechet quotient*. In contrast to first semester calculus, this definition only *implicitly* defines the differential. Since dF_p is a linear transformation its action can be expressed in terms of matrix multiplication by the standard matrix; $dF_p(h) = J_F(p)h$ where $J_F(p)$ is the *Jacobian matrix* of F. The Jacobian matrix is the standard matrix of the differential; $[dF_p] = J_F(p)$. The partial derivative with respect to the j-th Cartesian coordinate is given by $dF_p(e_j) = \partial_j F(p)$ thus $J_F = [\partial_1 F| \cdots |\partial_n F]$.

It turns out $p \mapsto dF_p$ is continuous[2] provided all the partial derivatives of F are continuous near p[3]. Furthermore, we will show that continuous differentiability of partial derivative functions at p implies differentiability of a function at p. The theorem that continuously differentiable implies differentiable is one of the cornerstone theorems of Advanced Calculus. I will not go over it in class, but I include the proof here for the sake of completeness.

Finally, I share a few basic theorems of general calculus on $\mathbb{R}^n$. Linearity, chain and a rather general product rule justify much of the differential calculus you ever saw in previous coursework.

[1] I believe this course is offered again in Fall 2019 if you are interested

[2] in a sense which we'd rather not explain here

[3] which is easy enough to understand in terms of limits we've already discussed

5.1 Frechet and Partial Differentiability

The definition below says that $\triangle F = F(a+h) - F(a) \cong dF_a(h)$ when h is close to zero. I should mention, going forward in this course, when I say a function is real differentiable I mean that it is real differentiable in the Frechet sense defined below:

Definition 5.1.1. *Suppose that $U \subseteq \mathbb{R}^n$ is open and $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a function the we say that F is* **differentiable** *at $a \in U$ iff there exists a linear mapping $dF_a : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{h \to 0} \left[\frac{F(a+h) - F(a) - dF_a(h)}{\|h\|} \right] = 0.$$

In such a case we call the linear mapping dF_a the **differential at** *a.*

Partial derivatives are defined in the usual fashion. If $F : dom(F) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ we define, for such points $a \in dom(F)$ as the limit exists,

$$\frac{\partial F}{\partial x_i}(a) = \lim_{h \to 0} \frac{F(a+he_i) - F(a)}{h}.$$

Here $e_i \in \mathbb{R}^n$ has all components 0 except for the i-th component which is 1. The connection of the differential to partial derivatives is a bit subtle. On the one hand, if the differential exists then partial derivatives exist and allow a nice formula for the differential.

Theorem 5.1.2. *If $F : dom(F) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at p then the partial derivatives $\frac{\partial F}{\partial x_i}$ for $i = 1, 2, \dots, n$ all exist at p and $dF_p(h) = J_F(p)h$ where, suppressing p, $J_F = \left[\frac{\partial F}{\partial x_1} \middle| \frac{\partial F}{\partial x_2} \middle| \cdots \middle| \frac{\partial F}{\partial x_n}\right]$.*

In contrast, it is possible for J_F to exist at p and yet dF_p fails to exist! We explore some of the nefarious ways this may occur and then offer a remedy in the following subsection. But first let me share some explicit examples of the Jacobian matrix.

You may recall the notation from calculus III at this point, omitting the a-dependence,

$$\boxed{\nabla F_j = grad(F_j) = [\ \partial_1 F_j,\ \partial_2 F_j,\ \cdots,\ \partial_n F_j\ ]^T}$$

So if the derivative exists we can write it in terms of a stack of gradient vectors of the component functions: (I used a transpose to write the stack side-ways),

$$F' = \left[\nabla F_1 | \nabla F_2 | \cdots | \nabla F_m\right]^T$$

Finally, just to collect everything together,

$$\boxed{F' = \begin{bmatrix} \partial_1 F_1 & \partial_2 F_1 & \cdots & \partial_n F_1 \\ \partial_1 F_2 & \partial_2 F_2 & \cdots & \partial_n F_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 F_m & \partial_2 F_m & \cdots & \partial_n F_m \end{bmatrix} = \left[\partial_1 F \mid \partial_2 F \mid \cdots \mid \partial_n F\ \right] = \begin{bmatrix} (\nabla F_1)^T \\ \hline (\nabla F_2)^T \\ \hline \vdots \\ \hline (\nabla F_m)^T \end{bmatrix}}$$

5.1.1 Examples of Jacobian Matrices

I have two goals in mind in the first five or so examples. First, show how we can use an algebra formula to define a mapping on $\mathbb{R}^n$. Second, to calculate the Jacobian of such functions. After this first batch of somewhat preachy examples, I move on to a select collection of more random functions which illustrate other cases for domain and codomain dimension.

Example 5.1.3. *Suppose $f(z) = z^2$ defines a mapping on $\mathbb{R}^2 = \mathbb{C}$. If $z = x + iy$ then $z^2 = x^2 - y^2 + 2xyi$. Real notation for f reads $f(x, y) = (x^2 - y^2, 2xy)$ thus*

$$J_f = [\partial_x f | \partial_y f] = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

Interesting, the Jacobian matrix of $f(z) = z^2$ is the matrix representation of $2z = 2x+2yi$. Curious.

Example 5.1.4. *Let $f(z) = z^2$ where $z = x + jy$ and $j^2 = 1$ for $\mathcal{H}$. Observe $z^2 = x^2 + y^2 + 2xyj$ hence $f(x, y) = (x^2 + y^2, 2xy)$ and*

$$J_f = [\partial_x f | \partial_y f] = \begin{bmatrix} 2x & 2y \\ 2y & 2x \end{bmatrix}$$

Once more, the Jacobian matrix of $f(z) = z^2$ in the hyperbolic numbers is the matrix representation of $2z = 2x + 2yj$. Curious.

Example 5.1.5. *Let $f(z) = e^z$ where $z = x + iy \in \mathbb{C}$. Recall $e^z = e^x(\cos y + i \sin y)$ hence $f(x, y) = (e^x \cos y, e^x \sin y)$ and*

$$J_f = [\partial_x f | \partial_y f] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

Apparently the Jacobian matrix of the complex exponential function is the matrix representation of e^z. Perhaps we start to see a pattern. See if you can guess the Jacobian matrix without direct calculation in the next example.

Example 5.1.6. *Let $f(z) = \frac{1}{z}$ where $z = x + jy \in \mathcal{H}$. Recall $\frac{1}{z} = \frac{1}{x+jy} = \frac{x-jy}{x^2-y^2}$ hence*

$$f(x, y) = \left(\frac{x}{x^2 - y^2}, \frac{-y}{x^2 - y^2} \right)$$

thus calculate (the last equality requires some steps)

$$\frac{\partial f}{\partial x} = \left(\frac{x^2 - y^2 - 2x^2}{(x^2 - y^2)^2}, \frac{-2xy}{(x^2 - y^2)^2} \right) = -\left(\frac{x^2 + y^2}{(x^2 - y^2)^2}, \frac{2xy}{(x^2 - y^2)^2} \right) = \frac{-1}{z^2}$$

and

$$\frac{\partial f}{\partial y} = \left(\frac{-2xy}{(x^2 - y^2)^2}, \frac{-(x^2 - y^2) - 2y^2}{(x^2 - y^2)^2} \right) = -\left(\frac{2xy}{(x^2 - y^2)^2}, \frac{x^2 + y^2}{(x^2 - y^2)^2} \right) = j\left(\frac{-1}{z^2} \right)$$

Thus, we find the expected result, the derivative of the $1/z$ function is $-1/z^2$ in the following sense:

$$J_f = [\partial_x f | \partial_y f] = -\begin{bmatrix} \frac{x^2+y^2}{(x^2-y^2)^2} & \frac{2xy}{(x^2-y^2)^2} \\ \frac{2xy}{(x^2-y^2)^2} & \frac{x^2+y^2}{(x^2-y^2)^2} \end{bmatrix} = \left[\frac{-1}{z^2} \middle| j\left(\frac{-1}{z^2} \right) \right] = \mathbf{M}(-1/z^2).$$

Example 5.1.7. *Let $f(z) = 3z + \bar{z}$ where $z = x + iy$ and $\bar{z} = x - iy$ in $\mathbb{C}$. In real notation, $f(x, y) = (4x, 2y)$ thus $J_f = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$. You can check, J_f is not in the matrix representation of $\mathbb{C}$.*

Example 5.1.8. *Let $f(t) = (t, t^2, t^3)$ then $f'(t) = (1, 2t, 3t^2)$. In this case we have*

$$f'(t) = [df_t] = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}$$

Example 5.1.9. *Let $f(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$ be a mapping from $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. I'll denote the coordinates in the domain by $(x_1, x_2, x_3, y_1, y_2, y_3)$ thus $f(\vec{x}, \vec{y}) = x_1y_1 + x_2y_2 + x_3y_3$. Calculate,*

$$[df_{(\vec{x},\vec{y})}] = \nabla f(\vec{x}, \vec{y})^T = [y_1, y_2, y_3, x_1, x_2, x_3]$$

Example 5.1.10. *Suppose $F(x, y) = (x^2 + y^2, xy, x + y)$ we find the Jacobian is a 3×2 matrix:*

$$J_F = \left[\frac{\partial F}{\partial x} \middle| \frac{\partial F}{\partial y} \right] = \begin{bmatrix} 2x & 2y \\ y & x \\ 1 & 1 \end{bmatrix}.$$

Example 5.1.11. *When other variables are used we still follow the same pattern. Suppose that $F(r, \theta) = (r\cos\theta, r\sin\theta)$. We calculate,*

$$J_F = [\partial_r F | \partial_\theta F] = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Example 5.1.12. *Let $f(x, y, z) = (x + y, y + z, x + z, xyz)$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ yz & xz & xy \end{bmatrix}$$

Example 5.1.13. *Let $f(x, y, z) = xyz$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \end{bmatrix}$$

Example 5.1.14. *Let $f(x, y, z) = (xyz, 1 - x - y)$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \\ -1 & -1 & 0 \end{bmatrix}$$

5.1.2 Continuous Differentiability

We have noted that differentiablility on some set U implies all sorts of nice formulas in terms of the partial derivatives. Curiously the converse is not quite so simple. It is possible for the partial derivatives to exist on some set and yet the mapping may fail to be differentiable. We need an extra topological condition on the partial derivatives if we are to avoid certain pathological[4] examples.

[4]"pathological" as in, "your clothes are so pathological, where'd you get them?"

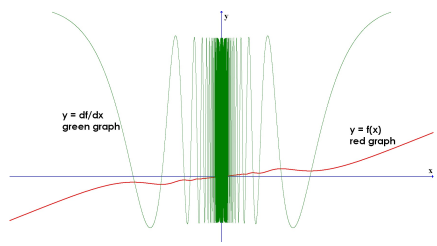
Example 5.1.15. *I found this example in Hubbard's advanced calculus text(see Ex. 1.9.4, pg. 123). It is a source of endless odd examples, notation and bizarre quotes. Let* $f(x) = 0$ *and*

$$f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}$$

for all $x \neq 0$*. I can be shown that the derivative* $f'(0) = 1/2$*. Moreover, we can show that* $f'(x)$ *exists for all* $x \neq 0$*, we can calculate:*

$$f'(x) = \frac{1}{2} + 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

Notice that $dom(f') = \mathbb{R}$*. Note then that the tangent line at* $(0,0)$ *is* $y = x/2$*.*



You might be tempted to say then that this function is increasing at a rate of 1/2 for x *near zero. But this claim would be false since you can see that* $f'(x)$ *oscillates wildly without end near zero. We have a tangent line at* $(0,0)$ *with positive slope for a function which is not increasing at* $(0,0)$ *(recall that increasing is a concept we must define in a open interval to be careful). This sort of thing cannot happen if the derivative is continuous near the point in question.*

The one-dimensional case is really quite special, even though we had discontinuity of the derivative we still had a well-defined tangent line to the point. However, many interesting theorems in calculus of one-variable require the function to be continuously differentiable near the point of interest. For example, to apply the 2nd-derivative test we need to find a point where the first derivative is zero and the second derivative exists. We cannot hope to compute $f''(x_o)$ unless f' is continuous at x_o. The next example is *sick*.

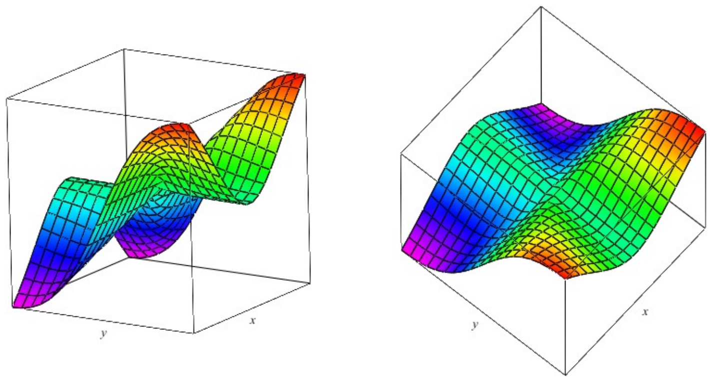
Example 5.1.16. *Let us define* $f(0,0) = 0$ *and*

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}$$

for all $(x,y) \neq (0,0)$ *in* $\mathbb{R}^2$*. It can be shown that* f *is continuous at* $(0,0)$*. Moreover, since* $f(x,0) = f(0,y) = 0$ *for all* x *and all* y *it follows that* f *vanishes identically along the coordinate axis. Thus the rate of change in the* e_1 *or* e_2 *directions is zero. We can calculate that*

$$\frac{\partial f}{\partial x} = \frac{2xy^3}{(x^2+y^2)^2} \qquad \textit{and} \qquad \frac{\partial f}{\partial y} = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}$$

If you examine the plot of $z = f(x,y)$ *you can see why the tangent plane does not exist at* $(0,0,0)$*.*



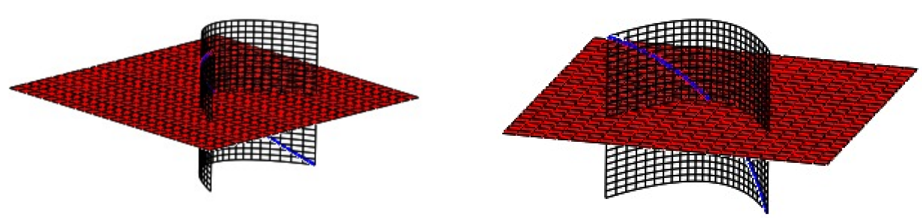
Notice the sides of the box in the picture are parallel to the x and y axes so the path considered below would fall on a diagonal slice of these boxes[5]*. Consider the path to the origin $t \mapsto (t,t)$ gives $f_x(t,t) = 2t^4/(t^2+t^2)^2 = 1/2$ hence $f_x(x,y) \to 1/2$ along the path $t \mapsto (t,t)$, but $f_x(0,0) = 0$ hence the partial derivative f_x is not continuous at $(0,0)$. In this example, the discontinuity of the partial derivatives makes the tangent plane fail to exist.*

One might be tempted to suppose that if a function is continuous at a given point and if all the possible directional derivatives exist then differentiability should follow. It turns out this is not sufficient since continuity of the function does not imply some continuity along the partial derivatives. For example:

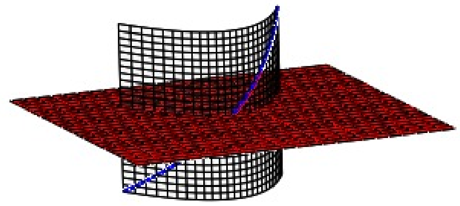
Example 5.1.17. *Let us define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = 0$ for $y \neq x^2$ and $f(x,x^2) = x$. I invite the reader to verify that this function is continuous at the origin. Moreover, consider the directional derivatives at $(0,0)$. We calculate, if $v = \langle a,b\rangle$*

$$D_v f(0,0) = \lim_{h\to 0} \frac{f(0+hv)-f(0)}{h} = \lim_{h\to 0} \frac{f(ah,bh)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

To see why $f(ah,bh) = 0$, consider the intersection of $\vec{r}(h) = (ha,hb)$ and $y = x^2$ the intersection is found at $hb = (ha)^2$ hence, noting $h = 0$ is not of interest in the limit, $b = ha^2$. If $a = 0$ then clearly (ah,bh) only falls on $y = x^2$ at $(0,0)$. If $a \neq 0$ then the solution $h = b/a^2$ gives $f(ha,hb) = ha$ a nontrivial value. However, as $h \to 0$ we eventually reach values close enough to $(0,0)$ that $f(ah,bh) = 0$. Hence we find **all** *directional derivatives exist and are zero at $(0,0)$. Let's examine the graph of this example to see how this happened. The pictures below graph the xy-plane as red and the nontrivial values of f as a blue curve. The union of these forms the graph $z = f(x,y)$.*



[5]the argument to follow stands alone, you don't need to understand the picture to understand the math here, but it's nice if you do



Clearly, f is continuous at $(0,0)$ as I invited you to prove. Moreover, clearly $z = f(x,y)$ cannot be well-approximated by a tangent plane at $(0,0,0)$. If we capture the xy-plane then we lose the blue curve of the graph. On the other hand, if we use a tilted plane then we lose the xy-plane part of the graph.

The moral of the story in the last two examples is simply that derivatives at a point, or even all directional derivatives at a point do not necessarily tell you much about the function near the point. This much is clear: something else is required if the differential is to have meaning which extends beyond one point in a nice way. Therefore, we consider the following:

It would seem the trouble has something to do with discontinuity in the derivative[6].

Definition 5.1.18.

A mapping $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is **continuously differentiable** at $a \in U$ iff the partial derivative mappings D_jF exist on an open set containing a and are continuous at a.

The import of the theorem below is that we can build the tangent plane from the Jacobian matrix provided the partial derivatives exist near the point of tangency and are continuous at the point of tangency. This is a very nice result because the concept of the linear mapping is quite abstract but partial differentiation of a given mapping is often easy. The proof that follows here is found in many texts, in particular see C.H. Edwards *Advanced Calculus of Several Variables* on pages 72-73.

Theorem 5.1.19.

If $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable at a then F is differentiable at a

Proof: We give a proof for $m = 1$ since the result then extends to $m > 1$ by the vector limit theorem. Consider $a+h$ sufficiently close to a that all the partial derivatives of F exist. Furthermore, consider going from a to $a + h$ by traversing a hyper-parallel-piped travelling n-perpendicular paths:

$$\underbrace{a}_{p_o} \to \underbrace{a + h_1e_1}_{p_1} \to \underbrace{a + h_1e_1 + h_2e_2}_{p_2} \to \cdots \underbrace{a + h_1e_1 + \cdots + h_ne_n}_{p_n} = a + h.$$

Let us denote $p_j = a + b_j$ where clearly b_j ranges from $b_o = 0$ to $b_n = h$ and $b_j = \sum_{i=1}^{j} h_ie_i$. Notice that the difference between p_j and p_{j-1} is given by:

$$p_j - p_{j-1} = a + \sum_{i=1}^{j} h_ie_i - a - \sum_{i=1}^{j-1} h_ie_i = h_je_j$$

[6]see commentary near Equations 2.18 and 2.19 in my Advanced Calculus notes for why the term continuously differentiable is quite natural

Consider then the following identity,

$$F(a+h) - F(a) = F(p_n) - F(p_{n-1}) + F(p_{n-1}) - F(p_{n-2}) + \cdots + F(p_1) - F(p_o)$$

This is to say the change in F from $p_o = a$ to $p_n = a + h$ can be expressed as a sum of the changes along the n-steps. Furthermore, if we consider the difference $F(p_j) - F(p_{j-1})$ you can see that only the j-th component of the argument of F changes. Since the j-th partial derivative exists on the interval for h_j considered by construction we can apply the mean value theorem to locate c_j such that:

$$h_j \partial_j F(p_{j-1} + c_j e_j) = F(p_j) - F(p_{j-1})$$

Therefore, using the mean value theorem for each interval, we select $c_1, \dots c_n$ with:

$$F(a+h) - F(a) = \sum_{j=1}^{n} h_j \partial_j F(p_{j-1} + c_j e_j)$$

It follows we should propose L to satisfy the definition of Frechet differentation as follows:

$$L(h) = \sum_{j=1}^{n} h_j \partial_j F(a)$$

It is clear that L is linear (in fact, perhaps you recognize this as $L(h) = (\nabla F)(a) \bullet h$). Let us prepare to study the Frechet quotient,

$$\begin{aligned} F(a+h) - F(a) - L(h) &= \sum_{j=1}^{n} h_j \partial_j F(p_{j-1} + c_j e_j) - \sum_{j=1}^{n} h_j \partial_j F(a) \\ &= \sum_{j=1}^{n} h_j \underbrace{\left[\partial_j F(p_{j-1} + c_j e_j) - \partial_j F(a)\right]}_{g_j(h)} \end{aligned}$$

Observe that $p_{j-1} + c_j e_j \to a$ as $h \to 0$. Thus, $g_j(h) \to 0$ by the continuity of the partial derivatives at $x = a$. Finally, consider the Frechet quotient:

$$\lim_{h \to 0} \frac{F(a+h) - F(a) - L(h)}{||h||} = \lim_{h \to 0} \frac{\sum_j h_j g_j(h)}{||h||} = \lim_{h \to 0} \sum_j \frac{h_j}{||h||} g_j(h)$$

Notice $|h_j| \leq ||h||$ hence $\left|\frac{h_j}{||h||}\right| \leq 1$ and

$$0 \leq \left|\frac{h_j}{||h||} g_j(h)\right| \leq |g_j(h)|$$

Apply the squeeze theorem to deduce each term in the sum $\star$ limits to zero. Consquently, $L(h)$ satisfies the Frechet quotient and we have shown that F is differentiable at $x = a$ and the differential is expressed in terms of partial derivatives as expected; $dF_x(h) = \sum_{j=1}^{n} h_j \partial_j F(a)$ $\square$.

5.2 Connections with Our Past and Future

The discussion below connects the difference quotient definition for the derivative with the Frechet quotient introduced above.

Example 5.2.1. *Suppose $f : dom(f) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at x. It follows that there exists a linear function $df_x : \mathbb{R} \rightarrow \mathbb{R}$ such that $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)-df_x(h)}{|h|} = 0$. Note that*

$$\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)-df_x(h)}{|h|} = 0 \quad \Leftrightarrow \quad \lim_{h \rightarrow 0^{\pm}} \frac{f(x+h)-f(x)-df_x(h)}{|h|} = 0.$$

In the left limit $h \rightarrow 0^-$ we have $h < 0$ hence $|h| = -h$. On the other hand, in the right limit $h \rightarrow 0^+$ we have $h > 0$ hence $|h| = h$. Thus, differentiability suggests that $\lim_{h \rightarrow 0^{\pm}} \frac{f(x+h)-f(x)-df_x(h)}{\pm h} = 0$. But we can pull the minus out of the left limit to obtain $\lim_{h \rightarrow 0^-} \frac{f(x+h)-f(x)-df_x(h)}{h} = 0$. Therefore, after an algebra step, we find:

$$\lim_{h \rightarrow 0} \left[\frac{f(x+h)-f(x)}{h} - \frac{df_x(h)}{h} \right] = 0.$$

Linearity of $df_x : \mathbb{R} \rightarrow \mathbb{R}$ implies there exists $m \in \mathbb{R}^{1 \times 1} = \mathbb{R}$ such that $df_x(h) = mh$. Observe that

$$\lim_{h \rightarrow 0} \frac{df_x(h)}{h} = \lim_{h \rightarrow 0} \frac{mh}{h} = m.$$

It is a simple exercise to show that if $\lim(A - B) = 0$ and $\lim(B)$ exists then $\lim(A)$ exists and $\lim(A) = \lim(B)$. Identify $A = \frac{f(x+h)-f(x)}{h}$ and $B = \frac{df_x(h)}{h}$. Therefore,

$$m = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}.$$

Consequently, we find the 1×1 matrix m of the differential is precisely $f'(x)$ as we defined it via a difference quotient in first semester calculus. In summary, we find $\boxed{df_x(h) = f'(x)h}$.

I should mention, for a path $\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^n$ there are arguments largely mirroring the example above and we can prove[7]:

$$d\vec{r}_t(h) = \vec{r}\,'(t)h$$

where $\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$ is the derivative of a path you previously studied in multivariate calculus. For a less familar application, a path in $\mathbb{C}$ could be denoted $z(t) = x(t) + iy(t)$ and then we have

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

and the differential $dz_t(h) = \frac{dz}{dt}(t)h$. In any event, I mostly mention this so you understand the theoretical results I share later in this chapter apply meaningfully and naturally to paths in $\mathbb{C}$ or $\mathcal{A} = \mathbb{R}^n$. We use many paths later in this course.

Remark 5.2.2.

Incidentally, I should mention that df_x is the differential of f at the point x. The differential of f would be the mapping $x \mapsto df_x$. Technically, the differential df is a function from $\mathbb{R}$ to the set of linear transformations on $\mathbb{R}$. You can contrast this view with that of first semester calculus. There we say the mapping $x \mapsto f'(x)$ defines the derivative f' as a function from $\mathbb{R}$ to $\mathbb{R}$. This simplification in perspective is only possible because calculus in one-dimension is so special. More on this later. This distinction is especially important to understand if you begin to look at questions of higher derivatives.

[7]see Example 2.1.9 etc. in my Advanced Calculus 2017 notes if you wish the gory details

5.3 Rules for Real Derivatives

Linearity and the chain rule naturally generalize for Frechet derivatives on normed linear spaces[8]. It is helpful for me to introduce some additional notation to analyze the convergence of the Frechet quotient: supposing that $F : dom(F) \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a we set:

$$\eta_F(h) = F(a+h) - F(a) - dF_a(h) \tag{5.1}$$

hence the **Frechet quotient** can be written as:

$$\frac{\eta_F(h)}{\|h\|} = \frac{F(a+h) - F(a) - dF_a(h)}{\|h\|}. \tag{5.2}$$

Thus differentiability of F at a requires $\frac{\eta_F(h)}{\|h\|} \to 0$ as $h \to 0$. For $h \neq 0$ and $\|h\| < 1$ we have:

$$0 \leq \|\eta_F(h)\| < \frac{\|\eta_F(h)\|}{\|h\|}. \tag{5.3}$$

Thus $\|\eta_F(h)\| \to 0$ as $h \to 0$ by the squeeze theorem. Consequently,

$$\lim_{h \to 0} \eta_F(h) = 0. \tag{5.4}$$

Therefore, $\eta_F : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $h = 0$ since $\eta_F(0) = F(a) - F(a) - dF_a(0) = 0$ (I remind the reader that the linear transformation dF_a must map zero to zero). Continuity of η_F at $h = 0$ allows us to use theorems for continuous functions on η_F.

Theorem 5.3.1. Linearity of the Frechet derivatives: *If $F : dom(F) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $G : dom(G) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are differentiable at a and $c \in \mathbb{R}$ then $cF + G$ is differentiable at a and*

$$d(cF + G)_a = cdF_a + dG_a$$

Proof: Let $\eta_F(h) = F(a+h) - F(a) - dF_a(h)$ and $\eta_G(h) = G(a+h) - G(a) - dG_a(h)$ for all $h \in V$. Assume F and G differentiable at a hence $\lim_{h \to 0} \frac{\eta_F(h)}{\|h\|} = 0$ and $\lim_{h \to 0} \frac{\eta_G(h)}{\|h\|} = 0$. Moreover, $dF_a, dG_a : \mathbb{R}^n \to \mathbb{R}^m$ are linear hence $cdF_a + dG_a : \mathbb{R}^n \to \mathbb{R}^m$ is linear. Hence calculate,

$$\begin{aligned} \eta_{cF+G}(h) &= (cF + G)(a+h) - (cF + G)(a) - (cdF_a + dG_a)(h) \\ &= c\,(F(a+h) - F(a) - dF_a(h)) + G(a+h) - G(a) - dG_a(h) \\ &= c\eta_F(h) + \eta_G(h) \end{aligned} \tag{5.5}$$

Therefore, by the usual limit laws,

$$\lim_{h \to 0} \frac{\eta_{cF+G}(h)}{\|h\|} = \lim_{h \to 0} \frac{c\eta_F(h) + \eta_G(h)}{\|h\|} = c \lim_{h \to 0} \left(\frac{\eta_F(h)}{\|h\|} \right) + \lim_{h \to 0} \frac{\eta_G(h)}{\|h\|} = 0. \quad \square$$

Setting $c = 1$ or $G = 0$ we obtain important special cases:

$$d(F + G)_a = dF_a + dG_a \qquad \& \qquad d(cF)_a = cdF_a. \tag{5.6}$$

The chain rule is also a general rule of calculus on a normed linear space. I found these notes by J. C. M. Grajales on page 40 have a proof of the chain rule which appears complete. That said, the chain rule is much easier to remember than to prove:

[8]but, we only study $\mathbb{R}^n$ here. Also, rules for radicals is something entirely different

Theorem 5.3.2. Chain Rule: *Let V_1, V_2, V_3 be open subsets of $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^k$ respective. Suppose $G : V_1 \to V_2$ is differentiable at a and $F : V_2 \to V_3$ is differentiable at $G(a)$ then $F \circ G$ is differentiable at a and*

$$d(F \circ G)_a = dF_{G(a)} \circ dG_a.$$

When I first wrote notes for advanced calculus I realized I was writing the same argument over and over. The result below is a result. This argument simultaneously covers derivatives of scalar multiplications, matrix multiplications, dot and cross products. More importantly for us, it equally well applies to the multiplication of an algebra $\mathcal{A}$. For those who have not had linear algebra, as you read the Theorem and proof below you can just think of a finite dimensional real normed linear space as $\mathbb{R}^n$ and each basis is just the standard basis.

Theorem 5.3.3. One-size fits all product rule: *Let W_1, W_2, W_3, V be finite dimensional real normed linear spaces and suppose $U \subseteq V$ is open. Let $\beta = \{r_1, \dots, r_n\}$ be a basis for V with coordinates $x_1, \dots, x_n$. Let $\gamma_1 = \{w_1, \dots, w_{m_1}\}$ be the basis for W_1. Let $\gamma_2 = \{v_1, \dots, v_{m_2}\}$ be the basis for W_2. Let $\gamma_3 = \{\varepsilon_1, \dots, \varepsilon_{m_3}\}$ be the basis for W_3. Assume there exists a product $\star : W_1 \times W_2 \to W_3$ such that*

$$(cx + y) \star z = c(x \star z) + y \star z \qquad \& \qquad x \star (cz + w) = c(x \star z) + x \star w$$

for all $c \in \mathbb{R}$ and $x, y \in W_1$ and $z, w \in W_2$. Then, if $F : U \to W_1$ and $G : U \to W_2$ are continuously differentiable at $a \in U$ then $F \star G$ is continuously differentiable at $a \in U$ where $(F \star G)(a) = F(a) \star G(a)$. Moreover, denoting $\partial/\partial x_j$ by ∂_j we have

$$\partial_j(F \star G)(a) = (\partial_j F)(a) \star G(a) + F(a) \star (\partial_j G)(a).$$

Hence, for each $h \in V$, $d(F \star G)_a(h) = dF_a(h) \star G(a) + F(a) \star dG_a(h)$.

Proof: assume the notation given in the Theorem and define constants $c_{ijk} \in \mathbb{R}$ such that:

$$v_i \star w_j = \sum_{k=1}^{m_3} c_{ijk}\varepsilon_k. \tag{5.7}$$

These constants characterize the nature of the multiplication $\star$. Interestingly, they have little to do with the proof, essentially the play the role of bystanders. Assuming $F : U \to W_1$ and $G : U \to W_2$ are continuously differentiable at a means their component functions $F_1, \dots, F_{m_1} : U \to \mathbb{R}$ with respect to γ_1 and $G_1, \dots, G_{m_2} : U \to \mathbb{R}$ with respect to γ_2 are continuous at a. The component functions of $F \star G$ are naturally related to those of F and G as follows:

$$\begin{aligned} F \star G &= \left(\sum_{i=1}^{m_1} F_i v_i\right) \star \left(\sum_{j=1}^{m_2} G_j w_j\right) \qquad (5.8) \\ &= \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} F_i G_j \left(v_i \star w_j\right) \\ &= \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} F_i G_j \left(\sum_{k=1}^{m_3} c_{ijk}\varepsilon_k\right) \\ &= \sum_{k=1}^{m_3}\left(\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} F_i G_j c_{ijk}\right)\varepsilon_k \end{aligned}$$

Thus $F \star G$ has component function $\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} F_iG_jc_{ijk}$. Observe this is the sum of products of continuously differentiable functions at a which is once again continuously differentiable a. Thus $F \star G$ is continuously differentiable at a as it has component functions whose partial derivative functions are continous at a. This becomes explicitly clear if we calculate the partial derivative of $F \star G$ with respect to x_l for points near a,

$$\begin{aligned}
\partial_l(F \star G) &= \sum_{k=1}^{m_3} \partial_l \left(\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} F_iG_jc_{ijk} \right) \varepsilon_k \qquad : \partial_l \text{ done componentwise} \\
&= \sum_{k=1}^{m_3} \left(\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} c_{ijk}\partial_l(F_iG_j) \right) \varepsilon_k \qquad : \text{linearity of } \partial_l \\
&= \sum_{k=1}^{m_3} \left(\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} c_{ijk}\left[(\partial_l F_i)G_j + F_i\partial_l G_j\right] \right) \varepsilon_k \qquad : \text{ordinary product rule} \\
&= \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\sum_{k=1}^{m_3} c_{ijk}(\partial_l F_i)G_j\varepsilon_k + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\sum_{k=1}^{m_3} c_{ijk}F_i(\partial_l G_j)\varepsilon_k \\
&= (\partial_l F) \star G + F \star (\partial_l G).
\end{aligned} \tag{5.9}$$

where I used the calculation of Equation 5.8 in reverse in order to make the final step. The calculation makes it explicitly clear that the partial derivatives of $F \star G$ are sums and products of continuous functions hence $F \star G$ is continuously differentiable as claimed. Finally, we can construct the differential from partial derivatives: for $h = \sum_{l=1}^{n} h_l r_l$ calculate:

$$\begin{aligned}
d(F \star G)_a(h) &= \sum_{l=1}^{n} h_l \partial_l(F \star G)(a) \\
&= \sum_{l=1}^{n} h_l \left[(\partial_l F)(a) \star G(a) + F(a) \star (\partial_l G)(a)\right] \\
&= \left[\sum_{l=1}^{n} h_l(\partial_l F)(a)\right] \star G(a) + F(a) \star \left[\sum_{l=1}^{n} h_l(\partial_l G)(a)\right]. \\
&= dF_a(h) \star G(a) + F(a) \star dG_a(h).
\end{aligned} \tag{5.10}$$

This completes the proof. $\square$

Let's unwrap a few common cases of this general product rule. I'll continue to use the W_1, W_2, W_3 and V notation to connect directly to Theorem 5.3.3.

(1.) Set $W_1 = W_2 = W_3 = \mathbb{R}$ and $V = \mathbb{R}$ to produce the usual first semester calculus product rule:
$$\frac{d}{dt}(fg) = \frac{df}{dt}g + f\frac{dg}{dt}.$$
Of course, this was the heart of the proof.

(2.) Set $W_1 = W_2 = W_3 = \mathbb{R}$ and $V = \mathbb{R}^n$ to produce the usual product rule for real-valued functions of several variables:
$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i}g + f\frac{\partial g}{\partial x_i}.$$

(3.) Set $W_1 = \mathbb{R}$ and $W_2 = W_3$ and $V = \mathbb{R}^n$ to produce the usual product rule for a scalar function multiplied on a vector-valued function:

$$\frac{\partial}{\partial x_i}(f\vec{v}) = \frac{\partial f}{\partial x_i}\vec{v} + f\frac{\partial \vec{v}}{\partial x_i}.$$

(4.) Set $W_1 = W_2 = \mathbb{R}^n$ and $W_3 = \mathbb{R}$ and $V = \mathbb{R}$ to produce the product rule for dot-products of paths:

$$\frac{d}{dt}(\vec{v} \bullet \vec{w}) = \frac{d\vec{v}}{dt} \bullet \vec{w} + \vec{v} \bullet \frac{d\vec{w}}{dt}.$$

(5.) Set $W_1 = W_2 = \mathbb{R}^3$ and $W_3 = \mathbb{R}^3$ and $V = \mathbb{R}$ to produce the product rule for cross-products of paths:

$$\frac{d}{dt}(\vec{v} \times \vec{w}) = \frac{d\vec{v}}{dt} \times \vec{w} + \vec{v} \times \frac{d\vec{w}}{dt}.$$

(6.) Set $W_1 = W_2 = W_3 = \mathbb{R}^{\ n\times n}$ and $V = \mathbb{R}$ to produce the product rule for matrix-valued functions of a real variable: $t \mapsto A(t)$, $t \mapsto B(t)$,

$$\frac{d}{dt}(AB) = \frac{dA}{dt}B + A\frac{dB}{dt}.$$

(7.) Set $W_1 = W_2 = W_3 = \mathbb{C}$ and $V = \mathbb{C}$ with $z = x + iy$ we find for $f_1 = u_1 + iv_1$ and $f_2 = u_2 + iv_2$

$$\frac{\partial}{\partial x}(f_1f_2) = \frac{\partial f_1}{\partial x}f_2 + f_1\frac{\partial f_2}{\partial x} \qquad \& \qquad \frac{\partial}{\partial y}(f_1f_2) = \frac{\partial f_1}{\partial y}f_2 + f_1\frac{\partial f_2}{\partial y}.$$

(8.) Set $W_1 = W_2 = W_3 = \mathcal{A}$ and $V = \mathcal{A}$ with multiplication $\star$ and typical variable $\zeta = x_1e_1 + x_2e_2 + \cdots + x_ne_n$ we find for $f = f_1e_1 + f_2e_2 + \cdots + f_ne_n$ and $g = g_1e_1 + g_2e_2 + \cdots + g_ne_n$

$$\frac{\partial}{\partial x_j}(f \star g) = \frac{\partial f}{\partial x_j} \star g + f \star \frac{\partial g}{\partial x_j}.$$

I simply wish to impress on you that these product rules are **all** simply the standard product rule married to the algebraic structure of the given product. So long as the product has the needed linearity properties, there will be a corresponding product rule for functions.

5.4 Higher Derivatives

Comment: you can safely skip this section, I merely include it for the curious. Something strange and wonderful happens for calculus over an algebra which makes this background essentially unecessary.

Given normed linear spaces V, W and $U \subseteq V$ open and a differentiable map $F : U \to W$ we find a linear transformation $dF_a : V \to W$ for each $a \in U$. Therefore, we can define the map $f' : U \to \mathcal{L}(V; W)$ by the natural map $a \mapsto dF_a$. That is, $f'(a) = df_a$. Furthermore, since $\mathcal{L}(V; W)$ is itself a normed linear space we may study derivatives of f'. In particular, if $df'_a : V \to \mathcal{L}(V; \mathcal{L}(V; W))$ is linear for each $a \in U$ and satisfies the needed Frechet quotient then we may likewise define

$f'' : U \to \mathcal{L}(V; \mathcal{L}(V; W))$ by $f''(a) = (f')'(a) = (df')_a \in \mathcal{L}(V; \mathcal{L}(V; W))$ for each $a \in U$. This all gets a bit meta, so, its helpful to make use of an isomorphism $\Psi : \mathcal{L}(V; \mathcal{L}(V; W)) \to \mathcal{L}(V, V; W)$ defined by:

$$\Psi(T)(x, y) = (T(x))(y) \tag{5.11}$$

for all $x, y \in V$ and $T \in \mathcal{L}(V, W)$. Typically the Ψ is not written. With this abuse of language, we have $f''(a) : V \times V \to W$ given by

$$f''(a)(h, k) = df'_a(h, k) = d(h \mapsto df_h)_a(k) \tag{5.12}$$

Thus, in stark contrast to first semester calculus, each added derivative brings out a new object. Using the isomorphism and its extension to higher derivatives, we find the n-th derivative of $f : V \to W$ is naturally understood as an n-linear map from V to W. What is beautiful is that we can capture this simply in terms of iterated partial derivatives provided a certain continuity is given. I'll attempt to explain this for the case of second derivatives this semester. For the sake of time, I'll let Zorich provide the many details I omit here. If I find time to prepare and Lecture for Advanced Calculus of Fall 2019, we may examine the proof that partial derivatives commute. Whether or not we have time for the proof, the fact that partial derivatives commute is a cornerstone of abstract calculus.

Chapter 6

Differentiation on an Algebra

In the usual course of complex analysis it is a simple matter to define differentiation with respect to a complex variable. In particular, the usual definition is that for $f : dom(f) \subseteq \mathbb{C} \to \mathbb{C}$ the **complex derivative** of f exists at z if and only if the following limit exists and

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

It is then a relatively straight-forward exercise to verify this complex differentiation satisfies all the usual features of real calculus. Unfortunately, in an algebra $\mathcal{A}$ the division by h is rather troubling. For $\mathbb{C}$ there is no problem since $h \neq 0$ suffices to give a well-defined meaning for $1/h$. In contrast, $\mathbf{zd}(\mathcal{A}) \neq 0$ and we cannot reasonably consider $\frac{1}{h}$ as $h \to 0$ without some qualification. In short, the difference quotient does not generalize to general differentiation over an algebra. Well, some researchers might disagree with me on that, they would just assert we limit[1] the limit to $h \in \mathbf{U}(\mathcal{A})$. I call this modification of the difference quotient the *deleted difference quotient* [2]

In this chapter we use the theory of real differentiation set forth in the previous chapter to frame the definition of **$\mathcal{A}$-differentiability** for a function $f : dom(f) \subseteq \mathcal{A} \to \mathcal{A}$. The definition we provide reproduces the usual definition for $f'(z)$ with the proper interpretation **and** it allows us to meaningfully develop a theory of differential calculus for algebras with zero-divisors. We prove the derivative with respect to an algebra variable has the usual linearity, product and chain rules. Our proofs make ample use of the corresponding results for real derivatives which were offered in the previous chapter.

In addition, we study the $\mathcal{A}$-Cauchy Riemann Equations ($\mathcal{A}$-CREqns). In particular, if a **continuously differentiable** $f : dom(f) \subseteq \mathcal{A} \to \mathcal{A}$ satisfies the $\mathcal{A}$-CREqns then f is $\mathcal{A}$-differentiable. Since the $\mathcal{A}$-CREqns are relatively simple PDEs this provides a nice tool to check differentiability with respect to $\mathcal{A}$. In the case $\mathcal{A} = \mathbb{C}$ these PDEs are simply called the **Cauchy Riemann Equations**.

Complex conjugation plays an important role in the usual complex analysis. You can express the Cauchy Riemann Equations as $\frac{\partial f}{\partial \bar{z}} = 0$. We show the $\mathcal{A}$-CREqns may likewise be expressed by the

[1] fine, pun intended

[2] I discuss the merits and failings of the deleted difference quotient viewpoint in Section 6.4. I include this for breadth, you can skip it in your first reading of the material. Perhaps you might return to it if you decide to study $\mathcal{A}$-Calculus more deeply.

condition that $n-1$ conjugate derivatives vanish for the n-dimensional algebra $\mathcal{A}$.

We also study further differential consequences of the $\mathcal{A}$-CREqns. Of course Laplace's equation is the most famous of these and we will likely spend a whole day to focus on Laplace's equation place in complex analysis. That said, it is convenient to introduce a general framework in which the $\mathcal{A}$-Laplace equations are found. We also detail how Taylor's Theorem beautifully generalizes to our context.

To the struggling student, please be patient with me, I soon will shift gears and focus almost entirely on $\mathbb{C}$. This Chapter finishes much of what I have to say about $\mathcal{A}$-Calculus until much later in the course.

6.1 $\mathcal{A}$-Differentiability

In what follows we assume $\mathcal{A} = \mathbb{R}^n$ has Euclidean norm $\|x\| = \sqrt{x \bullet x}$. The treatment of this material in [cookAcalculusI] allows $\mathcal{A}$ to be an abstract vector space of finite dimension and many results involving the choice of basis are given. Once again I emphasize, in this course we assume $\mathcal{A} = \mathbb{R}^n$ in the interest of making this more understandable for students lacking a linear algebra background. We now make use of Definition 4.3.1 where we introduced the limit on $\mathcal{A}$.

Definition 6.1.1. *Let $U \subseteq \mathcal{A}$ be an open set containing p. If $f : U \to \mathcal{A}$ is a function then we say f is* **$\mathcal{A}$-differentiable at** *p if there exists a linear function $d_pf \in \mathcal{R}_{\mathcal{A}}$ such that*

$$\lim_{h \to 0} \frac{f(p+h) - f(p) - d_pf(h)}{||h||} = 0. \tag{6.1}$$

Definition 3.1.2 says $\mathcal{R}_{\mathcal{A}}$ is the set of all left-multiplication maps of $\mathcal{A}$. As we have before discussed, $L_b(x) = b \star x$ defines the **left-multiplication map** on $\mathcal{A}$ by b. Proposition 3.1.3 argued that we could also characterize $\mathcal{R}_{\mathcal{A}}$ as the set of **right-$\mathcal{A}$-linear** maps on $\mathcal{A}$. Let's review the algebra once again: for each $b \in \mathcal{A}$ and any pair $x, y \in \mathcal{A}$,

$$L_b(x \star y) = b \star (x \star y) = (b \star x) \star y = L_b(x) \star y.$$

Thus each left-multiplication map allows us to pull $\mathcal{A}$-numbers out to the right of its argument (that is L_b is **right-$\mathcal{A}$-linear**). Let us agree that we may either characterize the fundamental representation as left-multiplications or as right-$\mathcal{A}$-linear maps in the remainder of this course[3].

I should mention, it is also clear that each map in $\mathcal{R}_{\mathcal{A}}$ is linear over $\mathbb{R}$ since: for $x, y \in \mathcal{A}$ and $c \in \mathbb{R}$

$$L_b(cx + y) = b \star (cx + y) = cb \star x + b \star y = cL_b(x) + L_b(y).$$

In summary, $d_pf \in \mathcal{R}_{\mathcal{A}}$ implies $d_pf : \mathcal{A} \to \mathcal{A}$ is $\mathbb{R}$-linear mapping on $\mathcal{A}$ and $d_pf(v \star w) = d_pf(v) \star w$ for all $v, w \in \mathcal{A}$.

Theorem 6.1.2. *If f is $\mathcal{A}$ differentiable at p then f is $\mathbb{R}$-differentiable at p.*

Proof: if f is $\mathcal{A}$-differentiable at p then we know $d_pf \in \mathcal{R}_{\mathcal{A}}$ satisfies the Frechet limit given in Equation 6.1. Moreover, d_pf is $\mathbb{R}$-linear. $\square$

There are $\mathbb{R}$-differentiable functions which are not $\mathcal{A}$-differentiable. The condition $d_pf \in \mathcal{R}_{\mathcal{A}}$ is not met by all functions on $\mathcal{A}$.

[3] in other words, we need not speak of Proposition 3.1.3 any longer, this issue is settled

Example 6.1.3. *Let $\mathcal{A}$ be an algebra of dimension $n \geq 2$ and $\{e_1, \dots, e_n\}$ an invertible basis with $e_1 = \mathbb{1}$ and coordinates $x_1, \dots, x_n$ then we have $\zeta = x_1 + \cdots + x_n e_n$ and $\overline{\zeta}_2 = x_1 - x_2 e_2 + \cdots + x_n e_n$. The function $f(\zeta) = \overline{\zeta}_2$ is everywhere real differentiable and nowhere $\mathcal{A}$-differentiable. These observations are most easily verified using the tools of Section 6.2 which culminate in Theorem 6.3.5.*

If f is $\mathcal{A}$-differentiable at each $p \in V$ then f is **$\mathcal{A}$-differentiable on** V and we write $f \in \mathrm{C}_{\mathcal{A}}(V)$. If there exists an open set containing p on which f is $\mathcal{A}$-differentiable then we say f is **$\mathcal{A}$-differentiable near** p and write $f \in \mathrm{C}_{\mathcal{A}}(p)$. There are several ways to characterize $\mathcal{A}$-differentiability at a point. These all follow from the isomorphism of $\mathcal{A}$ with $\mathcal{R}_{\mathcal{A}}$ or $\mathrm{M}_{\mathcal{A}}$.

Theorem 6.1.4. *Let $U \subseteq \mathcal{A}$ and $p \in U$. Let $f : U \to \mathcal{A}$ be a $\mathbb{R}$-differentiable function at p. The following are equivalent*

(i.) *$df_p(v \star w) = df_p(v) \star w$ for all $v, w \in \mathcal{A}$,*

(ii.) *there exists $\lambda \in \mathcal{A}$ for which $df_p(v) = \lambda \star v$ for each $v \in \mathcal{A}$,*

(iii.) *$[df_p] \in M_{\mathcal{A}}$.*

Proof: (i.) is right-$\mathcal{A}$-linearity and (ii.) says df_p is a left-multiplication by λ thus we have already shown the equivalence of (i.) and (ii.) in Proposition 3.1.3. Let us suppose f is $\mathbb{R}$-differentiable at p as in the statement of the Theorem. Furthermore, suppose $[df_p] \in \mathrm{M}_{\mathcal{A}}$ thus there exists $b \in \mathcal{A}$ for which $[L_b] = [df_p]$. But, $\mathcal{A} = \mathbb{R}^n$ and we know two linear transformation with the same standard matrix are the same transformation thus $L_b = df_p$ thus (ii.) follows with $b = \lambda$. Conversely, if (ii.) is true then by definition of $\mathrm{M}_{\mathcal{A}}$ we have $[df_p] \in \mathrm{M}_{\mathcal{A}}$. $\square$

The $\mathcal{A}$-number λ which appears above in (ii.) is known as the **derivative** of f at p. Notice: if $d_pf(v) = \lambda \star v$ for each $v \in \mathcal{A}$ then $(d_pf)(\mathbb{1}) = \lambda$. We should appreciate the importance of the isomorphism of $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{A}$ as it allows derivatives of functions on $\mathcal{A}$ to be viewed once more as functions on $\mathcal{A}$. This is a great simplification as derivatives of $\mathbb{R}$-differentiable maps are not usually objects of the same type. The following quote[4] is from Dieudonné in [Dmaster]

> ...on a one-dimensional vector space, there is a one-to-one correspondence between linear forms and numbers, and therefore the derivative at a point is defined as a number instead of a linear form.

Dieudonné says this to encourage students to place linear transformations at the center stage of their analysis. In contrast, we find the correspondence of right-$\mathcal{A}$-linear transformations, or later k-linear transformations on $\mathcal{A}$[5], with $\mathcal{A}$ itself allows us to perform calculations in $\mathcal{A}$-calculus in nearly the same fashion as introductory real calculus. In other words, since there is also a natural correspondence between $\mathcal{A}$-linear transformations and $\mathcal{A}$ we escape the sophistication which Dieudonné could not avoid.

Definition 6.1.5. *Let $U \subseteq \mathcal{A}$ be an open set and $f : U \to \mathcal{A}$ an $\mathcal{A}$-differentiable function on U then we define $f' : U \to \mathcal{A}$ by $f'(p) = (d_pf)(\mathbb{1})$ for each $p \in U$.*

Many theorems of calculus hold for $\mathcal{A}$-differentiable functions.

[4] from the introduction to Dieudonné's chapter on differentiation in Modern Analysis Chapter VIII

[5] in particular, see Theorem 6.5.2

Theorem 6.1.6. *If $f, g \in C_{\mathcal{A}}(p)$ and $c \in \mathcal{A}$ and we define $f+g$ by $(f+g)(x) = f(x)+g(x)$ and $(c \star f)(x) = c \star f(x)$. Then*

(i.) $f+g \in C_{\mathcal{A}}(p)$ *and* $(f+g)'(p) = f'(p)+g'(p)$,

(ii.) $c \star f \in C_{\mathcal{A}}(p)$ *and* $(c \star f)'(p) = c \star f'(p)$.

Proof: suppose $f, g \in \mathrm{C}_{\mathcal{A}}(p)$ and $c \in \mathcal{A}$. Notice f, g are $\mathbb{R}$-differentiable at p so we are free to use the general linearity of the differential as was demonstrated to prove Theorem 5.3.1. In particular, $f+g$ is $\mathbb{R}$-differentiable and $d(f+g)_p = df_p + dg_p$. Observe, for $x, y \in \mathcal{A}$,

$$d(f+g)_p(x \star y) = df_p(x \star y) + dg_p(x \star y) = df_p(x) \star y + dg_p(x) \star y$$

where we used right-$\mathcal{A}$-linearity of df_p and dg_p in the last step. Factor out the multiplication by y as to see:

$$d(f+g)_p(x \star y) = (df_p(x) + dg_p(x)) \star y = (df_p + g_p)(x) \star y$$

which provides $d(f+g)_p(x \star y) = (d(f+g)_p(x)) \star y$ thus $d(f+g)_p \in \mathcal{R}_{\mathcal{A}}$. Moreover

$$(f+g)'(p) = d(f+g)_p(\mathbb{1}) = df_p(\mathbb{1}) + dg_p(\mathbb{1}) = f'(p) + g'(p).$$

Thus (i.) is shown true. To prove (ii.) it is helpful to recall the general product rule proved in Theorem 5.3.3. In particular, we think about case (8.) presented after the theorem which implies $c \star f$ is $\mathbb{R}$-differentiable with

$$d(c \star f)_p = dc_p \star f(p) + c \star df_p.$$

I propose a homework for the reader; if $G(x) = c$ for all $x \in \mathcal{A}$ then $dG_p = 0$ for any $p \in \mathcal{A}$. In words, I invite the reader to prove the differential of a constant function is the zero transformation; $d(c_p) = 0$. With this trivial result in hand, $d(c \star f)_p = c \star df_p$. Thus observe

$$(d(c \star f)_p)(x \star y) = c \star (df_p(x \star y)) = c \star (df_p(x) \star y) = (c \star df_p(x)) \star y = (d(c \star f)_p(x)) \star y$$

where we used right-$\mathcal{A}$-linearity of df_p in the crucial middle step above. We find $d(c \star f)_p \in \mathcal{R}_{\mathcal{A}}$ hence $c \star f \in \mathrm{C}_{\mathcal{A}}(p)$ and $(c \star f)'(p) = d(c \star f)_p(\mathbb{1}) = c \star df_p(\mathbb{1}) = c \star f'(p)$ which concludes the proof of (ii.). $\square$

Proof of (ii.) without using uber-product rule: If $f \in \mathrm{C}_{\mathcal{A}}(p)$ then $d_p f \in \mathcal{R}_{\mathcal{A}}$ which means $d_p f(v \star w) = d_p f(v) \star w$. Let $c \in \mathcal{A}$ and define $g(p) = c \star f(p)$. Let $L(h) = c \star d_p f(h)$ for each $h \in \mathcal{A}$. If $v, w \in \mathcal{A}$ then

$$L(v \star w) = c \star d_p f(v \star w) = c \star d_p f(v) \star w = L(v) \star w. \tag{6.2}$$

thus $L \in \mathcal{R}_{\mathcal{A}}$. It remains to show g is differentiable with $d_p g = L$. If $h \neq 0$ let $\mathcal{F}_f, \mathcal{F}_g$ denote the Frechet quotients of f, g respective. Since f is differentiable at p means $\lim_{h \to 0} \mathcal{F}_f = 0$. Calculate:

$$\begin{aligned} \mathcal{F}_g = \frac{g(p+h) - g(p) - L(h)}{||h||} &= \frac{c \star f(p+h) - c \star f(p) - c \star d_p f(h)}{||h||} \\ &= c \star \frac{f(p+h) - f(p) - d_p f(h)}{||h||} \\ &= c \star \mathcal{F}_f. \end{aligned} \tag{6.3}$$

Thus, recalling Theorem 4.1.2 we find $||\mathcal{F}_g|| = ||c \star \mathcal{F}_f|| \leq m_{\mathcal{A}}||c||\,||\mathcal{F}_f||$. Since $||\mathcal{F}_f|| \to 0$ as $h \to 0$ it follows $\lim_{h\to 0} \mathcal{F}_g = 0$[6]. Hence, g is $\mathbb{R}$-differentiable with $d_p g = L$. Thus $c \star f \in \mathrm{C}_{\mathcal{A}}(p)$ with $d_p(c \star f) = c \star d_p f$. Note

$$d_p(c \star f)(\mathbb{1}) = c \star d_p f(\mathbb{1}) = c \star f'(p) \tag{6.4}$$

Consequently, $(c \star f)'(p) = c \star f'(p)$. $\square$

The product of two $\mathcal{A}$-differentiable functions is not necessarily $\mathcal{A}$-differentiable in the case that $\mathcal{A}$ is a noncommutative algebra. However, the product of two $\mathcal{A}$-differentiable functions is always $\mathbb{R}$-differentiable as is clear from Theorem 5.3.3.

Theorem 6.1.7. *Suppose f, g are $\mathcal{A}$-differentiable at p then*

$$d_p(f \star g)(v) = d_p f(v) \star g(p) + f(p) \star d_p g(v)$$

for each $v \in \mathcal{A}$. Furthermore, if $\mathcal{A}$ is commutative then $f \star g$ is $\mathcal{A}$-differentiable at p and

$$(f \star g)'(p) = f'(p) \star g(p) + f(p) \star g'(p).$$

Proof: as discussed in Case (8.) following Theorem 5.3.3 we have $f \star g$ is real-differentiable at p with $d_p(f \star g)(v) = d_p f(v) \star g(p) + f(p) \star d_p g(v)$. Next, suppose $\mathcal{A}$ is commutative. If $v, w \in \mathcal{A}$ then

$$\begin{aligned} d_p(f \star g)(v \star w) &= d_p f(v \star w) \star g(p) + f(p) \star d_p g(v \star w) \\ &= d_p f(v) \star w \star g(p) + f(p) \star d_p g(v) \star w \\ &= \big(d_p f(v) \star g(p) + f(p) \star d_p g(v)\big) \star w \\ &= d_p(f \star g)(v) \star w \end{aligned} \tag{6.5}$$

Thus $d_p(f \star g) \in \mathcal{R}_{\mathcal{A}}$ and we have shown $f \star g$ is $\mathcal{A}$-differentiable at p. Moreover,

$$\begin{aligned} (f \star g)'(p) &= d_p(f \star g)(\mathbb{1}) \\ &= d_p f(\mathbb{1}) \star g(p) + f(p) \star d_p g(\mathbb{1}) \\ &= f'(p) \star g(p) + f(p) \star g'(p). \;\; \square \end{aligned} \tag{6.6}$$

If $\mathcal{A}$ is not commutative then it is possible

$$d_p f(v) \star w \star g(p) \neq d_p f(v) \star g(p) \star w \tag{6.7}$$

If $f \star g$ is to be $\mathcal{A}$ differentiable at p then we need that

$$d_p f(v) \star w \star g(p) - d_p f(v) \star g(p) \star w = 0 \tag{6.8}$$

for all $v, w \in \mathcal{A}$. Equivalently,

$$d_p f(v) \star \big(w \star g(p) - g(p) \star w\big) = 0 \tag{6.9}$$

For example, if $f(p) = c$ is the constant function then $f \star g$ is differentiable as we already saw in Theorem 6.1.6 part (ii.). For a less trivial example, we could seek a function g for which

$$w \star g(p) - g(p) \star w = 0 \tag{6.10}$$

[6] alternatively, we could simply use the part (c.) of Theorem 4.3.2 to calculate directly that $\lim c \star \mathcal{F}_f = c \star \lim \mathcal{F}_f = c \star 0 = 0$.

for all $w \in \mathcal{A}$ and some p. The **center** of $\mathcal{A}$ is $\mathbf{Z}(\mathcal{A}) = \{x \in \mathcal{A} \mid x \star y = y \star x \text{ for all } y \in \mathcal{A}\}$. The center forms an ideal of $\mathcal{A}$. We say $\mathcal{A}$ is a simple algebra if it has no ideals except $\{0\}$ and $\mathcal{A}$. If $g(p) \in \mathbf{Z}(\mathcal{A})$ and $g(p) \neq 0$ then we find $\mathcal{A}$ is not a simple algebra. Another aspect to this result is the nonexistence of higher than first-order $\mathcal{A}$-polynomials. In particular, Rosenfeld shows in [Rosenfeld] that there are only linear $\mathcal{A}$-differentiable functions over simple associative or alternative algebras. Simple algebras aside, there are algebras with nontrivial centers which in turn support nontrivial functions which meet the criteria of Equation 6.10.

Example 6.1.8. *Let $\mathcal{A} = \mathbb{R}^6$ with the following noncommutative multiplication:*

$$(a,b,c,d,e,f) \star (x,y,z,u,v,w) = (ax, by, cz, au+dy, bv+ez, aw+dv+fz) \tag{6.11}$$

The regular representation of $\mathcal{A}$ has typical element

$$\mathbf{M}(a,b,c,d,e,f) = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & d & 0 & a & 0 & 0 \\ 0 & 0 & e & 0 & b & 0 \\ 0 & 0 & f & 0 & d & a \end{bmatrix} \tag{6.12}$$

Suppose $\mathcal{A}$ has variables $\zeta = (x_1, \dots, x_6)$ and define $f(\zeta) = (1,1,1,1,1,x_3^2)$ and define $g(\zeta) = (0,0,0,x_2,0,x_5)$. Calculate $(f \star g)(\zeta) = (0,0,0,x_2,0,x_5)$. We calculate,

$$\left[\frac{\partial f}{\partial x_i}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x_3 & 0 & 0 & 0 \end{bmatrix} \quad \& \quad \left[\frac{\partial g}{\partial x_i}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{6.13}$$

Observe f and g are $\mathcal{A}$-differentiable and $f \star g = g$ is likewise $\mathcal{A}$-differentiable. In contrast, $(g \star f)(\zeta) = (0,0,0,x_2,0,x_2+x_5)$ is not $\mathcal{A}$-differentiable as its Jacobian matrix is nonzero in the $(2,6)$-entry and hence is not[7] in $M_{\mathcal{A}}$. Recall Equation 6.9 showed we need d_pf to annihilate the center of the algebra in order that $f \star g$ be $\mathcal{A}$-differentiable at p. Likewise, to have $g \star f$ differentiable over $\mathcal{A}$ at p we need d_pg to annihilate the center of $\mathcal{A}$. This is the distinction between f and g in this example, only f has d_pf annihilating the center of $\mathcal{A}$.

Theorem 6.1.9. *Suppose $U, V \subseteq \mathcal{A}$ are open sets and $g : U \to V$ and $f : V \to \mathcal{A}$ are $\mathcal{A}$-differentiable functions. If $p \in U$ then*

$$(f \circ g)'(p) = f'(g(p)) \star g'(p).$$

Proof: if $f \in C_{\mathcal{A}}(U)$ and $g \in C_{\mathcal{A}}(V)$ then f and g are $\mathbb{R}$-differentiable on U, V respective. Moreover, by the usual real calculus of a normed linear space, if the composite $f \circ g$ is defined at

[7]The algebra in Example 6.1.8 is isomorphic to the algebra formed by upper triangular matrices in $\mathbb{R}^{3\times 3}$. The center of the upper triangular matrices is formed by the strictly upper triangular matrices. The element $A = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ annihilates everything in the center of the triangular matrices. This is the reason that $f \star g$ is differentiable whereas $g \star f$ is not; d_pf corresponds to A whereas d_pg does not annihilate the center of the algebra.

p we have an elegant chain rule in terms of differentials: $d(f \circ g)_p = df_{g(p)} \circ dg_p$. Let $v, w \in \mathcal{A}$ and consider

$$\begin{aligned} d(f \circ g)_p(v \star w) &= (df_{g(p)} \circ dg_p)(v \star w) && : \text{real chain rule} \\ &= df_{g(p)}(dg_p(v \star w)) && : \text{def. of composite} \\ &= df_{g(p)}(dg_p(v) \star w)) && : \text{as } g \in \mathrm{C}_{\mathcal{A}}(p) \\ &= df_{g(p)}(dg_p(v)) \star w && : \text{as } f \in \mathrm{C}_{\mathcal{A}}(g(p)) \\ &= d(f \circ g)_p(v) \star w && : \text{real chain rule} \end{aligned} \tag{6.14}$$

Thus $d(f \circ g)_p \in \mathcal{R}_{\mathcal{A}}$ which shows $f \circ g$ is $\mathcal{A}$-differentiable at p. Moreover, as $f \in \mathrm{C}_{\mathcal{A}}(g(p))$ implies $df_{g(p)}(w) = f'(g(p)) \star w$ and $g \in \mathrm{C}_{\mathcal{A}}(p)$ implies $dg_p(v) = g'(p) \star v$ we derive

$$d(f \circ g)_p(v) = df_{g(p)}(dg_p(v)) = f'(g(p)) \star dg_p(v) = f'(g(p)) \star g'(p) \star v \tag{6.15}$$

for each $v \in \mathcal{A}$. Therefore, $(f \circ g)'(p) = f'(g(p)) \star g'(p)$. $\square$

Example 6.1.10. Claim: *Let $f(\zeta) = \zeta^n$ for some $n \in \mathbb{N}$ then $f'(\zeta) = n\zeta^{n-1}$.*

We proceed by induction on n. If $n = 1$ then $f(\zeta) = \zeta$ which is to say $f = Id$ and hence $df_p = Id$ for each $p \in \mathcal{A}$. Moreover,

$$df_p(x \star y) = Id(x \star y) = x \star y = Id(x) \star y = df_p(x) \star y \tag{6.16}$$

which shows f is $\mathcal{A}$-differentiable on $\mathcal{A}$. We calculate $(df_p)(\mathbb{1}) = Id(\mathbb{1}) = \mathbb{1} = 1\zeta^0$ hence the claim is true for $n = 1$. Suppose the claim holds for some $n \in \mathbb{N}$. Define $f(\zeta) = \zeta^n$ and $g(\zeta) = \zeta$. We have $f'(\zeta) = n\zeta^{n-1}$ by the induction hypothesis and we already argued $g'(\zeta) = 1$. Thus, Theorem 6.1.7 applies to calculate $\zeta^{n+1} = f(\zeta) \star g(\zeta)$:

$$(f \star g)'(\zeta) = n\zeta^{n-1} \star \zeta + \zeta^n \star \mathbb{1} = (n+1)\zeta^{(n+1)-1} \tag{6.17}$$

thus the claim is true for $n + 1$ and we conclude $\frac{d}{d\zeta}(\zeta^n) = n\zeta^{n-1}$ for all $n \in \mathbb{N}$.

Admittedly, we just introduced a new notation; if f is an $\mathcal{A}$-differentiable function and ζ denotes an $\mathcal{A}$ variable then we write

$$f'(\zeta) = \frac{df}{d\zeta}(\zeta) = \frac{d}{d\zeta}(f(\zeta)) \qquad \& \qquad f' = \frac{df}{d\zeta} \tag{6.18}$$

If the $\mathcal{A}$-differentiability of $f : \mathcal{A} \to \mathcal{A}$ is not certain then we may still meaningfully calculate $\frac{\partial}{\partial \zeta}$ as a particular $\mathcal{A}$-linear combination of real partial derivatives. In other words, we are able to find an $\mathcal{A}$-generalization of Wirtinger's calculus. Details are given in the next section.

If $\mathcal{A} \approx \mathcal{B}$ then $\mathcal{A}$ and $\mathcal{B}$ differentiable functions are related through the isomorphism.

Theorem 6.1.11. *Let $\Psi : \mathcal{A} \to \mathcal{B}$ be an isomorphism of unital, associative finite dimensional algebras over $\mathbb{R}$. If f is $\mathcal{A}$ differentiable at p then $g = \Psi \circ f \circ \Psi^{-1}$ is $\mathcal{B}$-differentiable at $\Psi(p)$. Moreover, $g'(p) = (\Psi \circ f' \circ \Psi^{-1})(p)$.*

Proof: Let $(\mathcal{A}, \star)$ and $(\mathcal{B}, *)$ be finite dimensional isomorphic unital associative algebras via the isomorpism $\Psi : \mathcal{A} \to \mathcal{B}$. In particular, Ψ is a linear bijection and $\Psi(v \star w) = \Psi(v) * \Psi(w)$ for all $v, w \in \mathcal{A}$. Since Ψ and Ψ^{-1} are linear maps on normed linear spaces of finite dimension we know these are smooth real maps with $d\Psi_p = \Psi$ for each $p \in \mathcal{A}$ and $d\Psi_q^{-1} = \Psi^{-1}$ for each $q \in \mathcal{B}$. If f is $\mathcal{A}$ differentiable at p then $df_p \in \mathcal{R}_{\mathcal{A}}$. Define $g = \Psi \circ f \circ \Psi^{-1}$ and notice dg_q exists as g is formed from the composite of differentiable maps. The chain rule[8] provides,

$$dg = d\Psi \circ df \circ d\Psi^{-1} \quad \Rightarrow \quad dg = \Psi \circ df \circ \Psi^{-1} \tag{6.19}$$

as Ψ, Ψ^{-1} are linear maps. We seek to show dg_q is right-$\mathcal{B}$-linear at $q = \Psi(p)$. Calculate,

$$\begin{aligned} dg_q(v * w) &= \Psi(df_p(\Psi^{-1}(v * w))) \\ &= \Psi(df_p(\Psi^{-1}(v) \star \Psi^{-1}(w))) \\ &= \Psi(df_p(\Psi^{-1}(v)) \star \Psi^{-1}(w)) \\ &= \Psi(df_p(\Psi^{-1}(v))) * \Psi(\Psi^{-1}(w)) \\ &= dg_q(v) * w. \end{aligned} \tag{6.20}$$

Thus $dg_q \in \mathcal{L}_{\mathcal{B}}$ and we find g is $\mathcal{B}$-differentiable at $q = \Psi(p)$ as claimed. $\square$

Isomorphic algebras have algebra-differentiable functions which naturally correspond:

Corollary 6.1.12. *If $\Psi : \mathcal{A} \to \mathcal{B}$ be an isomorphism of unital, associative finite dimensional algebras over $\mathbb{R}$ and $U \subseteq \mathcal{A}$ an open set then each $f \in C_{\mathcal{A}}(U)$ can be written as a composite $f = \Psi^{-1} \circ g \circ \Psi$ for some $g \in C_{\mathcal{B}}(\Psi(U))$*

Proof: Observe $g = \Psi \circ f \circ \Psi^{-1}$ satisfies $f = \Psi^{-1} \circ g \circ \Psi$. Moreover, $\mathcal{B}$-differentiability of g at $\Psi(p) \in \Psi(U)$ is naturally derived from the given $\mathcal{A}$-differentiability of f at $\Psi(p) \in \Psi(U)$ via the result of Theorem 6.1.11. $\square$

6.2 $\mathcal{A}$-Cauchy Riemann Equations

The task of verifying $\mathcal{A}$-differentiability of a function on $\mathcal{A}$ is greatly simplified via the use of the $\mathcal{A}$-Cauchy Riemann Equations. Recall Theorem 5.1.19 provided that a continuously differentiable function was in fact a $\mathbb{R}$-differentiable function. It follows that $f : \mathcal{A} \to \mathcal{A}$ which is continuously differentiable at p and has $df_p \in \mathcal{R}_{\mathcal{A}}$ is $\mathcal{A}$-differentiable at p. Let us record this result for future reference:

Theorem 6.2.1. *Let $f : dom(f) \subseteq \mathcal{A} \to \mathcal{A}$ be a function which is continuously differentiable at p and $df_p \in \mathcal{R}_{\mathcal{A}}$ then $f \in C_{\mathcal{A}}(p)$. If f is continuously differentiable at each point on its domain[9] then f is $\mathcal{A}$-differentiable on its domain.*

We remind the reader that *continuously differentiability* of f at p means that all the component functions of f possess partial derivative functions which are continuous near p. The right-$\mathcal{A}$-linearity of df_p amounts to $n^2 - n$ equations which the partial derivatives of the entries in Jacobian matrix of f must satisfy. Let us give these a proper naming:

Definition 6.2.2. *Let $f = (u_1, u_2, \dots, u_n) : dom(f) \subseteq \mathcal{A} \to \mathcal{A}$ be a function with component functions $u_1, u_2, \dots, u_n$. We call the equations imposed by $df_p(x \star y) = df_p(x) \star y$ for all $x, y \in \mathcal{A}$ the $\mathcal{A}$* **Cauchy Riemann Equations***(ACREqns) for $\mathcal{A}$.*

[8]explicitly $dg_q = d\Psi_{f(\Psi^{-1}(q))} \circ df_{\Psi^{-1}(q)} \circ d\Psi_q^{-1}$
[9]ok, to be picky, I assume the domain is open given my previous definitions

Often the $\mathcal{A}$CREqns are studied at the level of component functions in the standard complex analysis course. Let us begin with the $\mathcal{A} = \mathbb{C}$ example.

Example 6.2.3. *Let $f = u + iv$ then $J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$. If $J_f \in \mathbf{M}_{\mathbb{C}}$ then $J_f = \mathbf{M}(a+ib)$ for some complex number $a + ib$. This means*

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

hence $a = u_x = v_y$ and $b = v_x = -u_y$. The equations $u_x = v_y$ and $v_x = -u_y$ are the standard **Cauchy Riemann** *equations. By Theorem 6.2.1, if $f = u + iv$ is continously differentiable at p and has $u_x = v_y$ and $v_x = -u_y$ at p then f is complex differentiable at p.*

Example 6.2.4. *Let $f = u + jv$ then $J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$. If $J_f \in \mathbf{M}_{\mathcal{H}}$ then $J_f = \mathbf{M}(a + jb)$ for some hyperbolic number $a + jb$. This means*

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

hence $a = u_x = v_y$ and $b = v_x = u_y$. The equations $u_x = v_y$ and $v_x = u_y$ are the **Hyperbolic Cauchy Riemann** *equations. By Theorem 6.2.1, if $f = u + jv$ is continously differentiable at p and has $u_x = v_y$ and $v_x = u_y$ at p then f is hyperbolic differentiable at p.*

We could continue with the pattern matching of the two examples above. That is how I used to think about it. But, there is a better way.

Theorem 6.2.5. *Suppose $\mathcal{A} = \mathbb{R}^n$ is an associative unital algebra with structure constants C_{ijk} defined by $e_i \star e_j = \sum_k C_{ijk} e_k$. If $f : \mathcal{A} \to \mathcal{A}$ is $\mathcal{A}$-differentiable at p then*

(i.) $\dfrac{\partial f}{\partial x_i} \star e_j = \displaystyle\sum_k C_{ijk} \dfrac{\partial f}{\partial x_k}$, *(we rarely use this formulation)*

(ii.) *given $e_1 = \mathbb{1}$ we find $\dfrac{\partial f}{\partial x_j} = \dfrac{\partial f}{\partial x_1} \star e_j$ for $j = 2, \dots, n$,*

(iii.) *if $\mathcal{A}$ is commutative then $\frac{\partial f}{\partial x_i} \star e_j = e_i \star \frac{\partial f}{\partial x_j}$ for all $i, j = 1, 2, \dots, n$.*

Proof: suppose f is $\mathcal{A}$-differentiable at p then $d_p f \in \mathcal{R}_{\mathcal{A}}$. The k-th partial derivatives with respect to the Cartesian coordinate x_k in $\mathcal{A}$ is given by $df_p(e_k) = \frac{\partial f}{\partial x_k}$. To derive (i.) suppose $e_i \star e_j = \sum_k C_{ij}^k e_k$ and calculate:

$$\frac{\partial f}{\partial x_i} \star v_j = df_p(e_i) \star e_j = df_p(e_i \star e_j) = df_p\left(\sum_k C_{ijk} e_k\right) = \sum_k C_{ijk} df_p(e_k) = \sum_k C_{ijk} \frac{\partial f}{\partial x_k}. \quad (6.21)$$

Likewise, (ii.) follows as

$$df_p(e_j) = df_p(\mathbb{1} \star e_j) = df_p(\mathbb{1}) \star e_j \quad \Rightarrow \quad \frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial x_1} \star e_j \quad (6.22)$$

for $j = 2, \dots, n$. Finally, in the case $\mathcal{A}$ is commutative we derive (iii.) as follows:

$$\frac{\partial f}{\partial x_i} \star e_j = df_p(e_i) \star e_j = df_p(e_i \star e_j) = df_p(e_j \star e_i) = df_p(e_j) \star e_i = \frac{\partial f}{\partial x_j} \star e_i. \tag{6.23}$$

Thus, once more using commutativity of $\mathcal{A}$, $\frac{\partial f}{\partial x_i} \star e_j = e_i \star \frac{\partial f}{\partial x_j}$. $\square$

The (i.), (ii.) or (iii.) equations above are known as *generalized Cauchy Riemann Equations* by many authors. We prefer to call them the $\mathcal{A}$-CR-equations as to be specific. Both (i.) and (ii.) are suitable sets of equations when $\mathcal{A}$ is noncommutative. If $\mathbb{1}$ is conveniently presented in a basis for $\mathcal{A}$ then (ii.) is the convenient description of $\mathcal{A}$-differentiable functions.

Example 6.2.6. *Consider the complicated numbers of the form $\zeta = x + ky + k^2 z$ where $k^3 = -1$ then the $\mathcal{A}$-CREqns can be expressed rather nicely as follows: for $f : dom(f) \subseteq \mathcal{C}_3 \to \mathcal{C}_3$ we have $\mathcal{C}_3$-CREqns: since $e_1 = 1$, $e_2 = k$ and $e_3 = k^2$,*

$$\frac{\partial f}{\partial y} = k\frac{\partial f}{\partial y} \qquad \& \qquad \frac{\partial f}{\partial z} = k^2\frac{\partial f}{\partial z}$$

Setting $f = u + kv + k^2 w$ the above vector equations translate into:

$$u_y + kv_y + k^2 w_y = k(u_x + kv_x + k^2 w_x) \qquad \& \qquad u_z + kv_z + k^2 w_z = k^2(u_x + kv_x + k^2 w_x)$$

from which we derive: the $n^2 - n = 9 - 3 = 6$ generalized Cauchy Riemann equations for the 3-complicated number system $\mathcal{C}_3$:

$$u_y = -w_x, \quad v_y = u_x, \quad w_y = v_x, \quad u_z = -v_x, \quad v_z = -w_x, \quad w_z = u_x.$$

Notice, if we use these equations it allows us to simplify the Jacobian matrix for f as follows:

$$J_f = [f_x|f_y|f_z] = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} = \begin{bmatrix} u_x & -w_x & -v_x \\ v_x & u_x & -w_x \\ w_x & v_x & u_x \end{bmatrix} = \mathbf{M}_{\mathcal{C}_3}(u_x + kv_x + k^2 w_x) = \mathbf{M}_{\mathcal{C}_3}\left(\frac{\partial f}{\partial x}\right).$$

6.3 Conjugation and the $\mathcal{A}$-Cauchy Riemann Equations

Our first goal in this section is to describe a Wirtinger calculus for $\mathcal{A}$. In particular, we define the partial derivatives of an algebra variable and its conjugate variables. We should caution, the term *conjugate* ought not be taken too literally. These conjugates generally do not form automorphisms of the algebra. Their utility is made manifest that the play much the same role as the usual conjugate in complex analysis. We follow the construction of Alvarez-Parrilla, Frías-Armenta, López-González and Yee-Romero directly for the definition below (this is Equation 4.3 of [pagr2012]):

Definition 6.3.1. *Suppose $\mathcal{A}$ has an invertible basis which begins with the multiplicative identity of the algebra. In particular, $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for $\mathcal{A}$ with $v_1 = \mathbb{1}$. If $\zeta = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$ then we define the* ***j*-th conjugate of** *ζ as follows:*

$$\overline{\zeta}_j = \zeta - 2x_j v_j = x_1 \mathbb{1} + \cdots + x_{j-1} v_{j-1} - x_j v_j + x_{j+1} v_{j+1} + \cdots + x_n v_n$$

for $j = 2, 3 \dots, n$.

In some sense, the variables $\zeta, \overline{\zeta}_2, \dots, \overline{\zeta}_n$ are simply an algebra notation for n real variables. Given a function of $x_1, \dots, x_n$ we are free to express the function in terms of the algebra variables $\zeta, \overline{\zeta}_2, \dots, \overline{\zeta}_n$.

Theorem 6.3.2. *Suppose $\mathcal{A}$ has invertible basis $\beta = \{\mathbb{1}, v_2, \dots, v_n\}$ and $\zeta = \sum_{i=1}^n x_i v_i$ and $\overline{\zeta}_j = \zeta - 2x_j v_j$ for $j = 2, \dots, n$ then using $\frac{1}{v_j}$ to denote v_j^{-1} and omit $\mathbb{1}$ we find:*

(i.) $x_j = \dfrac{1}{2v_j}(\zeta - \overline{\zeta}_j)$ *for $j = 2, \dots, n$.*

(ii.) $x_1 = \dfrac{1}{2}\left((3-n)\zeta + \displaystyle\sum_{j=2}^n \overline{\zeta}_j\right)$.

Proof: to obtain (i.) simply solve $\overline{\zeta}_j = \zeta - 2x_j v_j$ for x_j. Then to derive (ii.) we solve $\zeta = x_1\mathbb{1} + x_2 v_2 + \cdots + x_n v_n$ for $x_1\mathbb{1}$:

$$x_1\mathbb{1} = \zeta - \sum_{j=2}^n x_j v_j = \zeta - \frac{1}{2}\sum_{j=2}^n (\zeta - \overline{\zeta}_j) = \frac{1}{2}\left[(3-n)\zeta + \sum_{j=2}^n \overline{\zeta}_j\right] \tag{6.24}$$

finally, omit $\mathbb{1}$ to obtain (ii.) $\square$

It may be helpful to formally discover the usual results of the Wirtinger's [wirtinger] calculus for complex analysis. Consider $\mathcal{A} = \mathbb{C}$ where $z = x + iy$ and $\bar{z} = x - iy$ hence $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{i}{2}(\bar{z} - z)$. Hence, formally,

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \& \quad \frac{\partial}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{6.25}$$

Furthermore, $\partial_x = \partial_z + \partial_{\bar{z}}$ and $\partial_y = i(\partial_z - \partial_{\bar{z}})$. Observe, for a complex differentiable f we have $f_y = if_x$ hence $\partial_{\bar{z}} f = \frac{1}{2}(f_x + if_y) = \frac{1}{2}(f_x + i^2 f_x) = 0$. Whereas $\partial_z f = \frac{1}{2}(f_x - if_y) = f_x$. We find a complex differentiable function may depend on z however it should not have a nontrivial $\bar{z}$-dependence. Since z and $\bar{z}$ are obviously related by conjugation, the previous sentence has the potential to confuse! That said, these statements about z or $\bar{z}$-dependence ought to be understood as comments about a partial real dependence.

We seek similar formulas for $\mathcal{A}$. Hence, consider Theorem 6.3.2 shows that if $\mathcal{A}$ has invertible basis $\beta = \{\mathbb{1}, v_2, \dots, v_n\}$ and $\zeta = \sum_{i=1}^n x_i v_i$ and $\overline{\zeta}_j = \zeta - 2x_j v_j$ for $j = 2, \dots, n$ then

$$x_j = \frac{1}{2v_j}(\zeta - \overline{\zeta}_j) \quad \& \quad x_1 = \frac{1}{2}\left((3-n)\zeta + \sum_{j=2}^n \overline{\zeta}_j\right). \tag{6.26}$$

Formally, $\frac{\partial x_j}{\partial \zeta} = \frac{1}{2v_j}$ for $j = 2, \dots, n$ and $\frac{\partial x_1}{\partial \zeta} = \frac{3-n}{2}$ hence we suspect

$$\frac{\partial}{\partial \zeta} = \sum_{j=1}^n \frac{\partial x_j}{\partial \zeta}\frac{\partial}{\partial x_j} = \frac{1}{2}\left((3-n)\frac{\partial}{\partial x_1} + \frac{1}{v_2}\frac{\partial}{\partial x_2} + \cdots + \frac{1}{v_n}\frac{\partial}{\partial x_n}\right) \tag{6.27}$$

whereas $\frac{\partial x_j}{\partial \overline{\zeta}_k} = \frac{-1}{2v_j}\delta_{jk}$ and $\frac{\partial x_1}{\partial \overline{\zeta}_k} = \frac{1}{2}$ thus we speculate:

$$\frac{\partial}{\partial \overline{\zeta}_k} = \sum_{j=1}^n \frac{\partial x_j}{\partial \overline{\zeta}_k}\frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{1}{v_k}\frac{\partial}{\partial x_k}\right). \tag{6.28}$$

The calculations above convinced us to make the definition below (which is slightly different than the defintion offered in [pagr2012] where $\partial/\partial\zeta$ is defined differently in their Equation 4.7).

Definition 6.3.3. *Suppose $f : \mathcal{A} \to \mathcal{A}$ is $\mathbb{R}$-differentiable. Furthermore, suppose $\beta = \{\mathbb{1}, v_2, \dots, v_n\}$ is an invertible basis and $\zeta = x_1\mathbb{1} + x_2v_2 + \cdots + x_nv_n$. We define*

$$\frac{\partial}{\partial\zeta} = \frac{1}{2}\left((3-n)\frac{\partial}{\partial x_1} + \frac{1}{v_2}\frac{\partial}{\partial x_2} + \cdots + \frac{1}{v_n}\frac{\partial}{\partial x_n}\right) \qquad \& \qquad \frac{\partial}{\partial\bar{\zeta}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{1}{v_k}\frac{\partial}{\partial x_k}\right)$$

for $j = 2, 3, \dots, n$.

The merit of Definition 6.3.3 is seen in the theorem below[10]:

Theorem 6.3.4. *Given the notation of Definition 6.3.3,*

$$\frac{\partial\zeta}{\partial\zeta} = 1, \qquad \frac{\partial\bar{\zeta}_j}{\partial\zeta} = 0, \qquad \frac{\partial\bar{\zeta}_j}{\partial\bar{\zeta}_j} = 1, \qquad \frac{\partial\bar{\zeta}_j}{\partial\bar{\zeta}_k} = 0, \qquad \frac{\partial\zeta}{\partial\bar{\zeta}_j} = 0$$

for all $j = 2, \dots, n$ and $k \neq j$.

Proof: simple calculation. Consider:

$$\begin{aligned}\frac{\partial\zeta}{\partial\zeta} &= \frac{1}{2}\left((3-n)\frac{\partial}{\partial x_1} + \frac{1}{v_2}\frac{\partial}{\partial x_2} + \cdots + \frac{1}{v_n}\frac{\partial}{\partial x_n}\right)(x_1 + \cdots + x_nv_n) \\ &= \frac{(3-n)+n-1}{2} \\ &= 1.\end{aligned} \tag{6.29}$$

For $j = 2, \dots, n$ we calculate:

$$\begin{aligned}\frac{\partial\bar{\zeta}_j}{\partial\zeta} &= \frac{1}{2}\left((3-n)\frac{\partial}{\partial x_1} + \frac{1}{v_2}\frac{\partial}{\partial x_2} + \cdots + \frac{1}{v_n}\frac{\partial}{\partial x_n}\right)(x_1 + \cdots - x_jv_j + \cdots + x_nv_n) \\ &= \frac{3-n+n-3}{2} \\ &= 0,\end{aligned} \tag{6.30}$$

and

$$\frac{\partial\bar{\zeta}_j}{\partial\bar{\zeta}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{1}{v_j}\frac{\partial}{\partial x_j}\right)(x_1 + \cdots - x_jv_j + \cdots + x_nv_n) = \frac{1}{2}\left(1 + \frac{1}{v_j}v_j\right) = 1, \tag{6.31}$$

and for $k \neq j$,

$$\frac{\partial\bar{\zeta}_j}{\partial\bar{\zeta}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{1}{v_k}\frac{\partial}{\partial x_k}\right)(x_1 + \cdots - x_jv_j + \cdots + x_nv_n) = \frac{1}{2}\left(1 - \frac{1}{v_k}v_k\right) = 0. \tag{6.32}$$

and finally

$$\frac{\partial\zeta}{\partial\bar{\zeta}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{1}{v_j}\frac{\partial}{\partial x_j}\right)(x_1 + \cdots + x_jv_j + \cdots + x_nv_n) = \frac{1}{2}\left(1 - \frac{1}{v_j}v_j\right) = 0. \tag{6.33}$$

[10] compare with Equation 4.9 in [pagr2012]

In summary, the derivatives above show we may think of ζ and $\overline{\zeta}_j$ as independent variables. $\square$

We should connect the formal derivatives above with $\mathcal{A}$-differentiability of a function. It turns out we can recast the $\mathcal{A}$-Cauchy Riemann Equations in terms of the formal derivatives we thus far defined. In fact, $\mathcal{A}$-differentiability of a function can be thought of as saying the given function depends only on the variable of the algebra and not of any of its conjugates. More precisely:

Theorem 6.3.5. *Let $\beta = \{\mathbb{1}, v_2, \dots, v_n\}$ be an invertible basis for the commutative algebra $\mathcal{A}$. If $f : \mathcal{A} \to \mathcal{A}$ is $\mathcal{A}$-differentiable at p then $\dfrac{\partial f}{\partial \overline{\zeta}_j} = 0$ for $j = 2, \dots, n$.*

Proof: following Definition 6.3.3

$$\frac{\partial f}{\partial \overline{\zeta}_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - \frac{1}{v_k} \star \frac{\partial f}{\partial x_k}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - \frac{1}{v_k} \star \frac{\partial f}{\partial x_1} \star v_k\right) = 0 \tag{6.34}$$

as $\mathcal{A}$ is assumed commutative and $\frac{1}{v_k} \star v_k = 1$. $\square$

The usual additive and product rules hold for $\partial/\partial\zeta$ and $\partial/\partial\overline{\zeta}_j$.

Theorem 6.3.6. *Using the notation of Definition 6.3.3, if $f, g : \mathcal{A} \to \mathcal{A}$ are differentiable then*

$$\frac{\partial}{\partial \zeta}(f+g) = \frac{\partial f}{\partial \zeta} + \frac{\partial g}{\partial \zeta} \qquad \& \qquad \frac{\partial}{\partial \overline{\zeta}_j}(f+g) = \frac{\partial f}{\partial \zeta} + \frac{\partial g}{\partial \overline{\zeta}_j}$$

for $j = 2, \dots, n$. Likewise,

$$\frac{\partial}{\partial \zeta}(f \star g) = \frac{\partial f}{\partial \zeta} \star g + f \star \frac{\partial g}{\partial \zeta} \qquad \& \qquad \frac{\partial}{\partial \overline{\zeta}_j}(f \star g) = \frac{\partial f}{\partial \overline{\zeta}_j} \star g + f \star \frac{\partial g}{\partial \overline{\zeta}_j}$$

Proof: suppose f and g are real differentiable functions on $\mathcal{A}$,

$$\frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{1}{v_k}\frac{\partial}{\partial x_k}\right)(f+g) = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - \frac{1}{v_k}\frac{\partial f}{\partial x_k}\right) + \frac{1}{2}\left(\frac{\partial g}{\partial x_1} - \frac{1}{v_k}\frac{\partial g}{\partial x_k}\right) \tag{6.35}$$

thus $\frac{\partial}{\partial \overline{\zeta}_j}(f+g) = \frac{\partial f}{\partial \zeta} + \frac{\partial g}{\partial \overline{\zeta}_j}$. Using the structure constants C_{ijk} we express $f \star g = \sum_{ijk} C_{ijk} f_i g_j v_k$ and it follows $\partial_j(f \star g) = \partial_j f \star g + f \star \partial_j g$. Consequently,

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{1}{v_k}\frac{\partial}{\partial x_k}\right)(f \star g) &= \frac{1}{2}\left(\frac{\partial f}{\partial x_1} \star g + f \star \frac{\partial g}{\partial x_1} - \frac{1}{v_k}\left[\frac{\partial f}{\partial x_k} \star g + f \star \frac{\partial g}{\partial x_k}\right]\right) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x_1} + -\frac{1}{v_k}\frac{\partial f}{\partial x_k}\right) \star g + f \star \frac{1}{2}\left(\frac{\partial g}{\partial x_1} + -\frac{1}{v_k}\frac{\partial g}{\partial x_k}\right).\end{aligned} \tag{6.36}$$

Hence $\frac{\partial}{\partial \overline{\zeta}_j}(f \star g) = \frac{\partial f}{\partial \overline{\zeta}_j} \star g + f \star \frac{\partial g}{\partial \overline{\zeta}_j}$. The identities for $\partial/\partial\zeta$ follow from similar calculations. $\square$

The following example can be constructed in nearly every $\mathcal{A}$.

Example 6.3.7. *Suppose $dim(\mathcal{A}) \geq 2$. Let $f(\zeta) = \zeta\,\overline{\zeta}_2$ where $f : \mathcal{A} \to \mathcal{A}$ then*

$$\frac{\partial f}{\partial \zeta} = \frac{\partial \zeta}{\partial \zeta}\overline{\zeta}_2 + \zeta\frac{\partial \overline{\zeta}_2}{\partial \zeta} = \overline{\zeta}_2 \qquad \& \qquad \frac{\partial f}{\partial \overline{\zeta}_2} = \frac{\partial \zeta}{\partial \overline{\zeta}_2}\overline{\zeta}_2 + \zeta\frac{\partial \overline{\zeta}_2}{\partial \overline{\zeta}_2} = \zeta$$

This function is only $\mathcal{A}$-differentiable at the origin. In the usual complex analysis it is simply the square of the modulus; $f(z) = z\overline{z} = x^2 + y^2$ where $z = x + iy$ has $\overline{z}_2 = x - iy$.

Inverting Definition 6.3.3 for $\partial/\partial x_1, \dots, \partial/\partial x_n$ in terms of $\partial/\partial\zeta, \dots, \partial/\partial\bar{\zeta}_n$ yields:

Theorem 6.3.8. *Using the notation of Definition 6.3.3,*

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\bar{\zeta}_2} + \cdots + \frac{\partial}{\bar{\zeta}_n} \qquad \& \qquad \frac{\partial}{\partial x_k} = v_k\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\bar{\zeta}_2} + \cdots + \frac{\partial}{\bar{\zeta}_n} - 2\frac{\partial}{\bar{\zeta}_k}\right)$$

Proof: Begin with Definition 6.3.3 and note the identity for $\partial/\partial x_1$ follows immediately from summing $\partial/\partial\zeta$ with the $n-1$ conjugate derivatives $\frac{\partial}{\partial\bar{\zeta}_2}, \dots, \frac{\partial}{\partial\bar{\zeta}_n}$:

$$\frac{\partial}{\partial \zeta} + \frac{\partial}{\bar{\zeta}_2} + \cdots + \frac{\partial}{\bar{\zeta}_n} = \frac{3-n}{2}\frac{\partial}{\partial x_1} + \frac{n-1}{2}\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1}. \tag{6.37}$$

Following Definition 6.3.3 we substitute Equation 6.37 into the definition of $\frac{\partial}{\partial\bar{\zeta}_k}$ to obtain:

$$\frac{\partial}{\partial \bar{\zeta}_k} = \frac{1}{2}\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\bar{\zeta}_2} + \cdots + \frac{\partial}{\bar{\zeta}_n} - \frac{1}{v_k}\frac{\partial}{\partial x_k}\right) \tag{6.38}$$

It is now clear we can solve for $\frac{\partial}{\partial x_k}$ to obtain the desired result. $\square$

In principle we can take a given PDE in $x_1, \dots, x_n$ and convert it to an $\mathcal{A}$ differential equation in $\zeta, \bar{\zeta}_1, \dots, \bar{\zeta}_n$. If we assume a solution for which all the conjugate derivative vanish then the PDE simplifies to an ordinary $\mathcal{A}$-differential equation. For some PDEs the corresponding $\mathcal{A}$-ODE may be solvable using elementary calculus. See Example 6.6.1 for a demonstration.

Example 6.3.9. *Consider $\mathcal{A} = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R}$ where $j^3 = 1$. We consider the algebra variable $\zeta = x + jy + zj^2$ and conjugate variables*

$$\bar{\zeta}_2 = x - jy + j^2 z \qquad \& \qquad \bar{\zeta}_3 = x + jy - j^2 z \tag{6.39}$$

In our current notation $\{1, j, j^2\}$ forms an invertible basis with $v_2 = j$ and $v_3 = j^2$. Note $1/v_2 = j^2$ and $1/v_3 = j$. It follows we have derivatives

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left[j\frac{\partial}{\partial y} + j^2\frac{\partial}{\partial z}\right] \quad \& \quad \frac{\partial}{\partial \bar{\zeta}_2} = \frac{1}{2}\left[\frac{\partial}{\partial x} - j^2\frac{\partial}{\partial y}\right] \quad \& \quad \frac{\partial}{\partial \bar{\zeta}_3} = \frac{1}{2}\left[\frac{\partial}{\partial x} - j\frac{\partial}{\partial z}\right]. \tag{6.40}$$

Thus, by Theorem 6.3.8, or direct calculation, we find:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}_2} + \frac{\partial}{\partial \bar{\zeta}_3}, \quad \frac{\partial}{\partial y} = j\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \bar{\zeta}_2} + \frac{\partial}{\partial \bar{\zeta}_3}\right), \quad \frac{\partial}{\partial z} = j^2\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}_2} - \frac{\partial}{\partial \bar{\zeta}_3}\right) \tag{6.41}$$

From the formulas above we can derive the following differential identities:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\frac{\partial}{\partial z} &= 2\left(\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_2} + \frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_3} + \frac{\partial^2}{\partial \bar{\zeta}_2^2}\right) \\
\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial z}\frac{\partial}{\partial x} &= 2j^2\left(-2\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_2} + \frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_3} - \frac{\partial}{\partial \bar{\zeta}_2}\frac{\partial}{\partial \bar{\zeta}_3} + \frac{\partial^2}{\partial \bar{\zeta}_3^2}\right) \\
\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial x}\frac{\partial}{\partial y} &= 2j\left(-2\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_3} + \frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}_2} - \frac{\partial}{\partial \bar{\zeta}_2}\frac{\partial}{\partial \bar{\zeta}_3} + \frac{\partial^2}{\partial \bar{\zeta}_2^2}\right)
\end{aligned} \tag{6.42}$$

If $f = u + vj + j^2w$ is an $\mathcal{A}$-differentiable function then $\frac{\partial f}{\partial \bar{\zeta}_2} = 0$ and $\frac{\partial f}{\partial \bar{\zeta}_3} = 0$. Therefore, f is annihilated by the operators $\partial_x^2 - \partial_y \partial_z$, $\partial_y^2 - \partial_z \partial_x$ and $\partial_z^2 - \partial_x \partial_y$. It follows that the component functions of f must solve the corresponding PDEs:

$$\Phi_{xx} - \Phi_{yz} = 0, \quad \Phi_{yy} - \Phi_{zx} = 0, \quad \Phi_{zz} - \Phi_{xy} = 0.$$

These are known as the **generalized Laplace Equations** *for the 3-hyperbolic numbers.*

Generalized $\mathcal{A}$-Laplace Equations are differential consequences of the $\mathcal{A}$-CR equations. When system of PDEs happens to be the $\mathcal{A}$-Laplace equations we find any $\mathcal{A}$-differentiable function provides solutions to system. One may also wonder when a given system is consistent with the $\mathcal{A}$-Laplace equations. In the event a given system of PDEs was consistent then we may impose the $\mathcal{A}$-CR equations and their differential consequences on the given system of PDEs as to find a special subclass of $\mathcal{A}$-differentiable solutions. Computationally this section provides a roadmap for this procedure:

(1.) given a PDE in real independent variables $x_1, x_2, \dots, x_n$ choose an algebra $\mathcal{A}$ of dimension n to study in conjunction with the system.

(2.) convert the derivatives in the PDE with respect to $x_1, x_2, \dots, x_n$ to derivatives with respect to the algebra variables $\zeta, \overline{\zeta}_2, \dots, \overline{\zeta}_n$

(3.) impose that the derivatives with respect to $\overline{\zeta}_2, \dots, \overline{\zeta}_n$ vanish, study the resulting ordinary differential equation in ζ. If possible, solve the $\mathcal{A}$-ODE which results.

The possibility that the technique above may produce novel solutions to particular systems of PDEs is one of the major motivations of this work.

Example 6.3.10. *An algebra which has a basis which is natural, but not invertible is given by the direct product algebra $\mathcal{A} = \mathbb{R} \times \mathbb{R}$ where*

$$(a, b)(x, y) = (ax, by).$$

Notice $e_1 e_2 = (1, 0)(0, 1) = (0, 0)$ thus $e_1, e_2 \in \mathbf{zd}(\mathcal{A})$. In contrast, $(1, 1)(x, y) = (x, y)$ thus $(1, 1) = \mathbb{1}$. Let us set $v_1 = (1, 1)$ since it is certainly invertible. To complete the basis for $\mathcal{A}$ we need to find v_2 which is linearly independent with v_1. A nice choice is $v_2 = (1, -1)$ since $v_1 \bullet v_2 = 0$ it follows the angle between v_1, v_2 is $\pi/2$ radians and certainly v_1, v_2 are linearly independent. Furthermore, if $\zeta = y_1 v_1 + y_2 v_2$ then

$$\zeta = (x_1, x_2) = y_1(1, 1) + y_2(1, -1) = (y_1 + y_2, y_1 - y_2)$$

whereas,

$$\bar{\zeta} = y_1 v_1 - y_2 v_2 = (y_1 - y_2, y_1 + y_2).$$

Of course, $v_1 = e_1 + e_2$ and $v_2 = e_1 - e_2$ hence

$$\frac{\partial f}{\partial \zeta} = \frac{\partial f}{\partial y_1} = df(v_1) = df(e_1 + e_2) = df(e_1) + df(e_2) = f_x + f_y.$$

and since $v_2^{-1} = v_2$ as $v_2 v_2 = (1, -1)(1, -1) = (1, 1)$.

$$\frac{\partial f}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial f}{\partial y_1} - v_2^{-1}\frac{\partial f}{\partial y_2}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial y_1} - (1, -1)\frac{\partial f}{\partial y_2}\right).$$

But, $\frac{\partial f}{\partial y_1} = df(v_1) = df(e_1) + df(e_2) = f_x + f_y$ *and* $\frac{\partial f}{\partial y_2} = df(v_2) = df(e_1) - df(e_2) = f_x - f_y$. *Thus,*

$$\begin{aligned}
\frac{\partial f}{\partial \bar{\zeta}} &= \frac{1}{2}\left(f_x + f_y + (-1,1)(f_x - f_y)\right) \\
&= \frac{1}{2}\left((u_x + u_y, v_x + v_y) + (-1,1)(u_x - u_y, v_x - v_y)\right) \\
&= \frac{1}{2}\left((u_x + u_y, v_x + v_y) + (u_y - u_x, v_x - v_y)\right) \\
&= (u_y, v_x).
\end{aligned} \tag{6.43}$$

If f *is* $\mathcal{A}$*-differentiable then* $f = (u, v)$ *has* $u_y = 0$ *and* $v_x = 0$ *hence* $\frac{\partial f}{\partial \bar{\zeta}} = 0$ *as expected.*

6.4 Deleted difference quotients

There are several popular definitions of differentiability with respect to an algebra. Either we can follow the path of first semester calculus and use a difference quotient[11] or we can follow something involving a Frechet quotient[12]. To see a rather detailed exposition of how these are related in the particular context of bicomplex or multi-complex numbers see [price].

The general concept this section is an adaptation and generalization of the arguments given in [gadeaD1vsD2] for the context of the hyperbolic numbers. We show how some introductory results in [gadeaD1vsD2] generalize to any commutative semisimple algebra of finite dimension over $\mathbb{R}$. Ultimately the section demonstrates why we prefer the definition of $\mathcal{A}$-differentiability given in Definition 6.1.1 as opposed to the deleted-difference quotient definition. It is helpful to have a precise and abbreviated terminology for the discussion which follows:

Definition 6.4.1. *Let* $f : dom(f) \to \mathcal{A}$ *be a function where* $dom(f)$ *is open and* $p \in dom(f)$.

(1.) *If* f *is* $\mathcal{A}$*-differentiable at* p *then* f *is* D_1 *at* p. .

(2.) *If* $\displaystyle\lim_{\mathbf{U}(\mathcal{A}) \ni \zeta \to p} \frac{f(\zeta) - f(p)}{\zeta - p}$ *exists then* f *is* D_2 *at* p.

If f *is* $D_1(D_2)$ *for each* $p \in U$ *then* f *is* $D_1(D_2)$ *on* U.

If we fix our attention to a point then the class of D_1 and D_2 functions at p are inequivalent.

Example 6.4.2. *In the spirit of Dirichlet we define* $f(z) = \begin{cases} 0 & \text{if } \zeta \in \mathbf{U}(\mathcal{A}) \cup \{0\} \\ 1 & \text{if } \zeta \in \mathbf{zd}(\mathcal{A}) - \{0\} \end{cases}$. *Since* $f(\zeta) = 0$ *for all* $\zeta \in \mathbf{U}(\mathcal{A})$ *we find* $\frac{f(\zeta)-f(0)}{\zeta} = 0$ *for all* $\zeta \in \mathbf{U}(\mathcal{A})$. *Thus* f *is* D_2*-differentiable over* $\mathcal{A}$ *at* $p = 0$. *Notice,* $\mathcal{A}$*-differentiability implies real differentiability and thus continuity. Clearly* f *is not continuous at* $p = 0$ *thus* f *is not* D_1 *differentiable at* $p = 0$.

In fact, D_1 on $\mathcal{A}$ at a point need not imply D_2 at a the given point in $\mathcal{A}$. For an explicit demonstration of this in the case of $\mathcal{A} = \mathcal{H}$ see Example 2.2 part (2) of [gadeaD1vsD2].

[11] suitably modified to avoid zero-divisors

[12] where $\mathcal{A}$-differentiability is imposed by an algebraic condition on the differential

Theorem 6.4.3. *Let f be a function on $\mathcal{A}$ with $\zeta_o \in dom(f)$ where $dom(f)$ is open in $\mathcal{A}$. If f is D_2 differentiable at ζ_o then*

$$\lim_{\substack{\zeta \to \zeta_o \\ \zeta - \zeta_o \in \mathbf{U}(\mathcal{A})}} \frac{||f(\zeta) - f(\zeta_o) - \lambda \star (\zeta - \zeta_o)||}{||\zeta - \zeta_o||} = 0 \tag{6.44}$$

Proof: since f is D_2 differentiable at ζ_o there exists $\lambda \in \mathcal{A}$ to which the deleted-difference quotient of f converges at ζ_o. In particular, for each $\epsilon > 0$ there exists $\delta > 0$ for which $\zeta - \zeta_o \in \mathbf{U}(\mathcal{A})$ with $0 < ||\zeta - \zeta_o|| < \delta$ implies

$$\left|\left|\frac{f(\zeta) - f(\zeta_o)}{\zeta - \zeta_o} - \lambda\right|\right| < \epsilon \quad \Rightarrow \quad \left|\left|\frac{f(\zeta) - f(\zeta_o) - \lambda \star (\zeta - \zeta_o)}{\zeta - \zeta_o}\right|\right| < \epsilon \tag{6.45}$$

Apply Corollary 4.1.3,

$$\frac{||f(\zeta) - f(\zeta_o) - \lambda \star (\zeta - \zeta_o)||}{||\zeta - \zeta_o||} \leq m_{\mathcal{A}} \left|\left|\frac{f(\zeta) - f(\zeta_o) - \lambda \star (\zeta - \zeta_o)}{\zeta - \zeta_o}\right|\right| < m_{\mathcal{A}}\epsilon \tag{6.46}$$

Thus $\dfrac{||f(\zeta) - f(\zeta_o) - \lambda \star (\zeta - \zeta_o)||}{||\zeta - \zeta_o||} \to 0$ as $\zeta \to \zeta_o$ for $\zeta - \zeta_o \in \mathbf{U}(\mathcal{A})$. $\square$

The deleted limit in Equation 6.44 almost provides D_1-differentiability at ζ_o. To overcome the difficulty of Example 6.4.2 it suffices to assume continuity of f near ζ_o.

Theorem 6.4.4. *Let f be a function on $\mathcal{A}$ which is continuous in some open set containing ζ_o. If f is D_2 differentiable at ζ_o then f is D_1 differentiable at ζ_o.*

Proof: let $\epsilon > 0$ and use Theorem 6.4.3 to choose $\delta > 0$ such that $\zeta - \zeta_o \in \mathbf{U}(\mathcal{A})$ and $||\zeta - \zeta_o|| < \delta$ implies $||f(\zeta) - f(\zeta_o) - \lambda \star (\zeta - \zeta_o)|| < \epsilon||\zeta - \zeta_o||$. It remains to show $||f(\zeta) - f(\zeta_o) - \lambda \star (\zeta - \zeta_o)|| < \epsilon||\zeta - \zeta_o||$ for $\zeta - \zeta_o \notin \mathbf{U}(\mathcal{A})$. We begin by making δ smaller (if necessary) such that f is continuous on $U = \{\zeta \in \text{dom}(f) \mid ||\zeta - \zeta_o|| < \delta\}$. Note $U \cap \mathbf{U}(\mathcal{A})$ is dense in U. Consequently, if $\zeta_1 - \zeta_o \in \mathbf{zd}(\mathcal{A}) \cap U$ then $\zeta_1 - \zeta_o$ is a limit point of $U \cap \mathbf{U}(\mathcal{A})$. Hence there exists a sequence of points $\zeta_n - \zeta_o \in U \cap \mathbf{U}(\mathcal{A})$ for which $\zeta_n - \zeta_o \to \zeta_1 - \zeta_o$. Hence, $\zeta_n \to \zeta_1$ and by continuity of f near ζ_o we find $f(\zeta_n) \to f(\zeta_1)$. Observe, as $\zeta_n - \zeta_o \in U \cap \mathbf{U}(\mathcal{A})$ we have the estimate $||f(\zeta_n) - f(\zeta_o) - \lambda \star (\zeta_n - \zeta_o)|| < \epsilon||\zeta_n - \zeta_o||$. Hence, as $n \to \infty$ we find $||f(\zeta_1) - f(\zeta_o) - \lambda \star (\zeta_1 - \zeta_o)|| < \epsilon||\zeta_1 - \zeta_o||$. But, as ζ_1 was an arbitrary zero-divisor near ζ_o we find $\lim_{\zeta \to \zeta_o} \frac{||f(\zeta) - f(\zeta_o) - \lambda \star (\zeta - \zeta_o)||}{||\zeta - \zeta_o||} = 0$. Thus the Frechet derivative of f at ζ_o exists and the differential $d_{\zeta_o} f \in \mathcal{R}_{\mathcal{A}}$ since $d_{\zeta_o} f(h) = \lambda \star h$. We conclude f is D_1 at ζ_o. $\square$

Theorem 6.4.5. *Let $U \subset \mathcal{A}$ be open. If f is D_2 at each point in U then f is continuous on U*

Proof: Suppose f is D_2 at each point of the open set U. Let $\zeta_o \in U$. By Theorem 6.4.3

$$\lim_{\substack{\zeta \to \zeta_o \\ \zeta - \zeta_o \in \mathbf{U}(\mathcal{A})}} \frac{||f(\zeta) - f(\zeta_o) - \lambda \star (\zeta - \zeta_o)||}{||\zeta - \zeta_o||} = 0 \tag{6.47}$$

But, $\lim_{\substack{\zeta \to \zeta_o \\ \zeta - \zeta_o \in \mathbf{U}(\mathcal{A})}} ||\zeta - \zeta_o|| = 0$ and $\lim_{\substack{\zeta \to \zeta_o \\ \zeta - \zeta_o \in \mathbf{U}(\mathcal{A})}} ||\lambda \star (\zeta - \zeta_o)|| = 0$ hence we deduce

$$\lim_{\substack{\zeta \to \zeta_o \\ \zeta - \zeta_o \in \mathbf{U}(\mathcal{A})}} ||f(\zeta) - f(\zeta_o)|| = 0. \tag{6.48}$$

It remains to show ζ for which $\zeta \to \zeta_o$ with $\zeta - \zeta_o \in \mathbf{zd}(\mathcal{A})$ also have $||f(z) - f(\zeta_o)|| \to 0$. Let $\epsilon > 0$ and choose $\delta > 0$ with $\{\zeta \mid ||\zeta - \zeta_o||\} \subset U$ and for which $\zeta - \zeta_o \in \mathbf{U}(\mathcal{A})$ and $||\zeta - \zeta_o|| < \delta$ imply $||f(\zeta) - f(\zeta_o)|| < \epsilon/2$. Selection of such $\delta > 0$ is possible by Equation 6.48. Form a triangle with vertices $\zeta_o, \zeta_2, \zeta_1$ where $||\zeta_2 - \zeta_o|| < ||\zeta_1 - \zeta_o|| < \delta$. By construction $\zeta_1 \in U$ thus f is D_2 at ζ_1. Hence, following the thought behind Equation 6.48 once more, we find there exists $\delta' > 0$ for which $||\zeta_2 - \zeta_1|| < \delta'$ and $\zeta_2 - \zeta_1 \in \mathbf{U}(\mathcal{A})$ imply $||f(\zeta_2) - f(\zeta_1)|| < \epsilon/2$. Since $\mathbf{U}(\mathcal{A})$ is dense in $\mathcal{A}$ we are free to move ζ_2 as close as we wish to ζ_1 while maintaining $\zeta_2 - \zeta_1 \in \mathbf{U}(\mathcal{A})$ and $\zeta_2 - \zeta_o \in \mathbf{U}(\mathcal{A})$. Hence, for ζ_1 such that $\zeta_1 - \zeta_o \in \mathbf{zd}(\mathcal{A})$ with $||\zeta_1 - \zeta_o|| < \delta$ we find

$$||f(\zeta_1) - f(\zeta_o)|| \leq ||f(\zeta_1) - f(\zeta_2)|| + ||f(\zeta_2) - f(\zeta_o)|| < \epsilon/2 + \epsilon/2 = \epsilon. \tag{6.49}$$

Therefore, f is continuous at ζ_o and hence f is continuous on U. $\square$

Given Theorem 6.4.5 and Theorem 6.4.4 we obtain the main result of this section:

Theorem 6.4.6. *Let U be an open set in $\mathcal{A}$. If f is D_2 at each point in U then f is D_1 on U.*

In other words, functions which are D_2-holomorphic are necessarily D_1-holomorphic. We will see the converse need not be true. There are functions which are D_1 on an open set and yet fail to be D_2 at even a single point in the set.

Example 6.4.7. *Let $\mathcal{A} = \mathbb{R} \oplus \epsilon\mathbb{R}$ where $\epsilon^2 = 0$. In this algebra, $(a + b\epsilon)\epsilon = a\epsilon$ hence a typical matrix in $M_{\mathcal{A}}$ has the form $\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$. If $f = u + \epsilon v : \mathcal{A} \to \mathcal{A}$ is $\mathcal{A}$-differentiable in the D_1 sense then $u_x = v_y$ and $u_y = 0$. Conversely, if u, v are continuously differentiable on $\mathcal{A}$ and satisfy $u_x = v_y$ and $u_y = 0$ then $f = u + \epsilon v$ is $\mathcal{A}$-differentiable in the D_1 sense on $\mathcal{A}$. Observe $u = c_1$ and $v = c_2 + y\frac{dc_1}{dx}$ where both c_1 and c_2 are real-valued functions of x alone describe the general form of a $\mathcal{A}$-differentiable function in the D_1 sense.*

For example, setting $c_1 = x$ and $c_2 = 0$ provides the function $f(x+\epsilon y) = x+y\epsilon$. The D_1 derivative is simply the constant function $f' = 1$ on $\mathcal{A}$. Let us study the D_2 differentiability of f at $z_o = x_o+y_o\epsilon$. First, note $(a - b\epsilon/a)(a + b\epsilon) = a^2$ hence for $a \neq 0$

$$\frac{1}{a + b\epsilon} = \frac{a - b\epsilon/a}{a^2}. \tag{6.50}$$

We use this identity to begin the calculation below: for $x \neq x_o$,

$$\begin{aligned} \frac{f(x + y\epsilon) - f(x_o + y_o\epsilon)}{(x - x_o) + \epsilon(y - y_o)} &= \frac{[x - x_o + (y - y_o)\epsilon]\,[x - x_o - (y - y_o)\epsilon/(x - x_o)]}{(x - x_o)^2} \\ &= 1 + \epsilon\left[\frac{y - y_o}{x - x_o} - \frac{y - y_o}{(x - x_o)^2}\right] \end{aligned} \tag{6.51}$$

Notice Equation 6.50 shows $(\mathbb{R} \oplus \epsilon\mathbb{R})^\times = \{x + y\epsilon \mid x \neq 0\}$. Thus, we study how the difference quotient of f behaves as $x + y\epsilon \to x_o + \epsilon y_o$ for $x \neq x_o$. Observe the 1 agrees with the D_1 derivative. However, the remaining terms do not converge in the deleted limit hence f is not D_2 at $x_o + \epsilon y_o$. But, $x_o + y_o\epsilon$ is arbitrary so we have shown f is **nowhere** *D_2.*

It seems for general finite dimensional commutative unital algebras over $\mathbb{R}$ it may be difficult or even impossible to obtain nontrivial functions of D_2 type. Fortunately, we are free to study D_1-differentiability as it includes D_2-functions when they exist.

Much of the literature on hypercomplex variables is largely centered on semisimple algebras. Upto isomorphism in the commutative case we face $\mathcal{A} = \mathbb{R}^n \times \mathbb{C}^m$. In such a context, it can be shown the set of D_1 and D_2 differentiable functions on an open set coincide.

Theorem 6.4.8. *Let U be an open set in a commutative semisimple finite dimensional real algebra $\mathcal{A}$. The set of D_1 functions on U coincides with the set of D_2 functions on U.*

Proof: Suppose $\mathcal{A}$ is a commutative semisimple finite dimensional real algebra and $U \subseteq \mathcal{A}$ is open. In Theorem 6.4.6 we showed that the set of D_2-differentiable functions on U are a subset of the D_1-differentiable functions on U. It remains to show D_1 functions on U are necessarily D_2 functions on U. Our proof involves several steps. First, we show the results hold for the direct product algebra $\mathbb{R}^n$. Second, we show the result holds for the direct product $\mathbb{C}^m$. Third, Wedderburn's Theorem tells us $\mathcal{A} \approx \mathbb{R}^n \times \mathbb{C}^m$ and we show how our result filters naturally through the isomorphism to complete the proof.

Consider $\mathcal{A}_R = \mathbb{R}^n$ with U_R open in $\mathcal{A}_R$. Suppose $f = (f_1, \dots, f_n)$ is D_1-differentiable on $\mathcal{A}_R$. Then f is differentiable on U_R and as the regular representation of $\mathcal{A}_R$ is formed by diagonal matrix we find the Cauchy Riemann equations simply indicate that f_j is a function of x_j alone[13]. In total,

$$f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n)). \tag{6.52}$$

Moreover, differentiability on U_R implies the partial derivatives of f likewise exist on U_R hence f_j is a real-differentiable function of x_j for $j = 1, 2, \dots, n$. Notice, $\mathbb{1} \in \mathbb{R}^n$ has the explicit form $\mathbb{1} = (1, 1, \dots, 1)$ and it follows for $h = (h_1, h_2, \dots, h_n) \neq 0$

$$\frac{1}{h} = \left(\frac{1}{h_1}, \frac{1}{h_2}, \dots, \frac{1}{h_n}\right). \tag{6.53}$$

Consider the difference quotient at $p = (p_1, p_2, \dots, p_n)$. Define $\triangle_j f = f_j(p_j + h_j) - f_j(p_j)$

$$\begin{aligned} \frac{f(p+h) - f(p)}{h} &= \left(\frac{1}{h_1}, \frac{1}{h_2}, \dots, \frac{1}{h_n}\right)(\triangle_1 f, \triangle_2 f, \dots, \triangle_n f) \\ &= \left(\frac{\triangle_1 f}{h_1}, \frac{\triangle_2 f}{h_2}, \dots, \frac{\triangle_n f}{h_n}\right) \end{aligned} \tag{6.54}$$

To prove f is D_2 at p we must show the limit of the difference quotient exists as $h \to 0$ for $h \in \mathbf{U}(\mathcal{A})_R$. The condition $h \in \mathbf{U}(\mathcal{A})_R$ simply requires $h_j \neq 0$ for all $j = 1, 2, \dots, n$. Differentiability of f_j at p_j gives

$$\lim_{h_j \to 0} \frac{\triangle_j f}{h_j} = \lim_{h_j \to 0} \frac{f_j(p_j + h_j) - f_j(p_j)}{h_j} = f_j'(p_j). \tag{6.55}$$

Notice, the condition that $h_i \neq 0$ for $i = 1, 2, \dots, n$ has no bearing on the limit above. Hence,

$$\lim_{\substack{h \to 0 \\ h \in \mathbf{U}(\mathcal{A})_R}} \frac{\triangle_j f}{h_j} = f_j'(p_j). \tag{6.56}$$

Since this holds for each component of $\triangle f / h$ we find

$$\lim_{\substack{h \to 0 \\ h \in \mathbf{U}(\mathcal{A})_R}} \frac{f(p+h) - f(p)}{h} = \left(f_1'(p_1), f_2'(p_2), \dots, f_n'(p_n)\right) \tag{6.57}$$

[13]as is often notated $f_j = f(x_j)$.

Therefore, f is D_2 at $p \in U_R$. But, p was arbitrary hence f is D_2 on U_R.

Next, if $\mathcal{A}_C = \mathbb{C}^m$ and U_C is open in $\mathcal{A}_C$ we consider g which is D_1 on U_C. Extending the result seen in Example 3.1.23 we find the Jacobian matrix of g will be block-diagonal with m-blocks of the form $\left[\begin{array}{cc} a_j & -b_j \\ b_j & a_j \end{array}\right]$ for $j = 1, 2, \dots, m$. The j-th diagonal block serves to give the ordinary Cauchy Riemann equations for g_j. The zero blocks in for g_j serve to indicate g_j is a function of z_j alone. Here we use $(z_1, z_2, \dots, z_m)$ as the variable on $\mathcal{A}_C$. In summary, $g = (g_1, g_2, \dots, g_m)$ where g_j is a complex differentiable function of z_j alone. Moreover, we may follow the arguments for $\mathcal{A}_R$ simply replacing real with complex limits. We find,

$$\lim_{\substack{h \to 0 \\ h \in \mathbf{U}(\mathcal{A})_C}} \frac{g(p+h) - g(p)}{h} = \big(g_1'(p_1), g_2'(p_2), \dots, g_m'(p_m)\big) \tag{6.58}$$

where $g_j' = \frac{dg_j}{dz_j}$ are complex derivatives.

If $\mathcal{B} = \mathbb{R}^n \times \mathbb{C}^m$ then we can fit together our result for $\mathbb{R}^n$ and $\mathbb{C}^m$ if we make the usual identification that $x \in \mathbb{R}^n$ and $z \in \mathbb{C}^m$ gives $(x, z) \in \mathbb{R}^n \times \mathbb{C}^m$. Notice $(x, z) \in \mathcal{B}^\times$ only if $x \in (\mathbb{R}^n)^\times$ and $z \in (\mathbb{C}^m)^\times$. Moreover,

$$\frac{1}{(x, z)} = \left(\frac{1}{x}, \frac{1}{z}\right) = \left(\frac{1}{x_1}, \dots, \frac{1}{x_n}, \frac{1}{z_1}, \dots, \frac{1}{z_m}\right). \tag{6.59}$$

It follows that if (f, g) is D_1 differentiable on U open in $\mathcal{B}$ then (f, g) is D_2 differentiable with $(f, g)' = (f_1', \dots, f_n', g_1', \dots, g_m')$ on U.

Finally, if $\mathcal{A}$ is commutative and semisimple associative algebra over $\mathbb{R}$ then Wedderburn's Theorem[14] provides an isomorphism of $\mathcal{A}$ and $\mathcal{B} = \mathbb{R}^n \times \mathbb{C}^m$ for some $n, m \in \mathbb{N}$. Suppose $\Psi : \mathcal{A} \to \mathcal{B}$ provides the isomorphism. If U is open in $\mathcal{A}$ then $\Psi(U) = U'$ is open in $\mathcal{B}$. Furthermore, suppose F is D_1 with respect to $\mathcal{A}$ on U. Apply Theorem 6.1.11 at each point in U to find that $G = \Psi \circ f \circ \Psi^{-1}$ is D_1 with respect to $\mathcal{B}$ at each point in U'. Therefore, G is D_2 differentiable on U' as we have already shown D_1 implies D_2 for an open subset of $\mathbb{R}^n \times \mathbb{C}^m$. It is simple[15] to verify that Ψ preserves differences and multiplicative inverses and $\Psi \circ f = G \circ \Psi$ thus:

$$\Psi\left(\frac{f(p+h) - f(p)}{h}\right) = \frac{\Psi(f(p+h)) - \Psi(f(p))}{\Psi(h)} = \frac{G(\Psi(p+h)) - G(\Psi(p))}{\Psi(h)}. \tag{6.60}$$

Consequently,

$$\frac{f(p+h) - f(p)}{h} = \Psi^{-1}\left(\frac{G(\Psi(p) + \Psi(h)) - G(\Psi(p))}{\Psi(h)}\right) \tag{6.61}$$

If $p \in U$ then $\Psi(p) \in U'$ where G is D_2 differentiable. Note $h \to 0$ with $h \in \mathbf{U}(\mathcal{A})$ implies $\Psi(h) \to 0$ with $\Psi(h) \in \mathcal{B}^\times$. Using continuity of Ψ^{-1} and that G is D_2 at $\Psi(p)$ we find

$$\lim_{\substack{h \to 0 \\ h \in \mathbf{U}(\mathcal{A})}} \frac{f(p+h) - f(p)}{h} = \Psi^{-1}\left(\lim_{\substack{\Psi(h) \to 0 \\ \Psi(h) \in \mathcal{B}^\times}} \frac{G(\Psi(p) + \Psi(h)) - G(\Psi(p))}{\Psi(h)}\right) = \Psi^{-1}(G'(\Psi(p))). \tag{6.62}$$

[14]see Dummit and Foote page 854-855, Theorem 4 part (5.) in [DF]

[15]if you disagree, then perhaps read this section in my paper [cookAcalculusI] where there is a link to an earlier Theorem which I have omitted in these notes to avoid abstract distraction

Therefore, f is D_2 at p with $f'(p) = \Psi^{-1}(G'(\Psi(p)))$. $\square$

In conclusion, the distinction between D_1 and D_2 differentiability is lost in the commutative semisimple case. This is reflected in Definition 2.11 of [gadeaD1vsD2]. However, if we drop the semisimple condition and just consider general real associative algebras then we argue from Theorem 6.4.6 and Example 6.4.7 that D_1-differentiability provides a more general concept of differentiation over an algebra. For these reasons we take Definition 6.1.1 as primary.

6.5 Higher $\mathcal{A}$-derivatives

The calculus of higher derivatives for functions on $\mathbb{R}^n$ requires the study of symmetric multilinear maps[16]. However, in $\mathcal{A}$-calculus this is avoided due to a fortunate isomorphism between $\mathcal{A}$ and symmetric multi-$\mathcal{A}$-linear mappings of $\mathcal{A}$. Let us begin by generalizing $\mathcal{R}_{\mathcal{A}}$ to its multilinear analog:

Definition 6.5.1. *We say $T : \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{k} \to \mathcal{A}$ is a k-**linear map on** $\mathcal{A}$ if T is right-$\mathcal{A}$-linear in each of its arguments. That is, T is additive in each entry and $T(z_1, \dots, z_j \star w, \dots, z_n) = T(z_1, \dots, z_j, \dots, z_n) \star w$. for all $z_1, \dots, z_n, w \in \mathcal{A}$.*

If $T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = T(v_1, \dots, v_k)$ for all permutations σ then T is **symmetric**. We continue to assume $\mathcal{A}$ is a unital, associative and finite-dimensional algebra over $\mathbb{R}$.

Theorem 6.5.2. *The set of symmetric k-linear maps on $\mathcal{A}$ is isomorphic to $\mathcal{A}$.*

Proof: the sum and scalar multiple of symmetric k-linear map is once more k-linear and symmetric. Since $v_j = \mathbb{1} \star v_j$ we find:

$$T(v_1, \dots, v_k) = T(\mathbb{1}, \dots, \mathbb{1}) \star v_1 \star \cdots \star v_k. \tag{6.63}$$

thus T is uniquely fixed by k-linearity on $\mathcal{A}$ together with its value on $(\mathbb{1}, \dots, \mathbb{1})$. $\square$

This calculation above reminds us a similar calculation which was required to understand the connection between $\mathcal{R}_{\mathcal{A}}$ and $\mathrm{M}_{\mathcal{A}}$.

Definition 6.5.3. *Suppose f is a function on $\mathcal{A}$ for which the derivative function f' is $\mathcal{A}$-differentiable at p then we define $f''(p) = (f')'(p)$. Furthermore, supposing the derivatives exist, we define $f^{(k)}(p) = (f^{(k-1)})'(p)$ for $k = 2, 3, \dots$.*

Naturally we define functions $f'', f''', \dots, f^{(k)}$ in the natural pointwise fashion for as many points as the derivatives exist. Furthermore, with respect to $\beta = \{v_1, \dots, v_n\}$ where $v_1 = \mathbb{1}$, we have $f'(p) = d_p f(\mathbb{1}) = \frac{\partial f}{\partial x_1}(p)$. Thus, $f' = \frac{\partial f}{\partial x_1}$. Suppose $f''(p)$ exists. Note,

$$f''(p) = (f')'(p) = \#(d_p f'(\mathbb{1})) = \frac{\partial f'}{\partial x_1}(p) = \frac{\partial^2 f}{\partial x_1^2}(p). \tag{6.64}$$

Thus, $f'' = \frac{\partial^2 f}{\partial x_1^2}$. By induction, we find the following theorem:

Theorem 6.5.4. *If $f : \mathcal{A} \to \mathcal{A}$, $\beta = \{\mathbb{1}, \dots, v_n\}$ a basis, and $f^{(k)}$ exists then $f^{(k)} = \frac{\partial^k f}{\partial x_1^k}$.*

[16] see, for example, Zorich, Mathematical Analysis II, see Section 10.5 pages 80-87. The results we claim without proof from advanced calculus can all be found in [zorich].

The algebra derivatives naturally dovetail with the iterated-symmetric-Frechet differentials which are used to describe higher derivatives of a map on normed linear spaces[17].

Theorem 6.5.5. *Suppose $f : \mathcal{A} \to \mathcal{A}$ is a function for which $f^{(k)}(p)$ exists. Then the iterated k-th Frechet differential exists and is related to the k-th $\mathcal{A}$ derivative as follows:*

$$d_p^k f(v_1, \dots, v_k) = f^{(k)}(p) \star v_1 \star \cdots \star v_k.$$

for all $v_1, \dots, v_k \in \mathcal{A}$.

Proof: Suppose $f : \mathcal{A} \to \mathcal{A}$ is a function for which $f^{(k)}(p)$ exists. The existence of the iterated $\mathcal{A}$-derivatives implies that f is also k-fold $\mathbb{R}$-differentiable and thus $d_p^k f : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathcal{A}$ exists and is a real symmetric k-linear map. Let $\beta = \{v_1, \dots, v_n\}$ with $v_1 = \mathbb{1}$ be a basis for $\mathcal{A}$ with coordinates $x_1, \dots, x_n$. The iterated k-th Frechet differential and iterated partial derivatives are related by:

$$d_p^k f(v_{i_1}, v_{i_2}, \dots, v_{i_k}) = \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}. \tag{6.65}$$

Differentiating the $\mathcal{A}$-CR equations $\frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial x_1} \star v_j$ with respect to x_i yields:

$$\begin{aligned} \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left[\frac{\partial f}{\partial x_j}\right] &= \frac{\partial}{\partial x_i}\left[\frac{\partial f}{\partial x_1} \star v_j\right] \\ &= \frac{\partial}{\partial x_1}\left[\frac{\partial f}{\partial x_i}\right] \star v_j \\ &= \frac{\partial^2 f}{\partial x_1^2} v_i \star v_j \end{aligned} \tag{6.66}$$

Apply Equation 6.66 repeatedly as to exchange partial derivatives with respect to x_{i_j} for partial derivatives with respect to x_1 and multiplication by v_{i_j} obtain:

$$\frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} = \frac{\partial^k f}{\partial x_1^k} \star v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k} = f^{(k)} \star v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k} \tag{6.67}$$

We used Theorem 6.5.4 in the last step. Compare Equations 6.65 and 6.67 to conclude the proof. □

In fact, the first equality in Equation 6.67 should be emphasized:

Theorem 6.5.6. *If $f : \mathcal{A} \to \mathcal{A}$ is k-times $\mathcal{A}$-differentiable then*

$$\frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} = \frac{\partial^k f}{\partial x_1^k} \star v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k}.$$

Theorem 6.5.5 and 6.5.6 provide the basis for both the formulation of an $\mathcal{A}$-variable Taylor Theorem as well as a lucid derivation of generalized Laplace Equations.

[17] The iterated-differentials are developed in many advanced calculus texts. See [zorich] where the theory of real higher derivatives is developed in Section 10.5 pages 80-87.

6.5.1 The $\mathcal{A}$-harmonic equations

In the case of complex analysis the second order differential consequences of the Cauchy Riemann equations include the Laplace equations. It is interesting to determine what equations form the analog to Laplace's Equation for $\mathcal{A}$. In 1948 Wagner derived **generalized Laplace Equations** in [wagner1948] via calculations performed through the lens of the *paraisotropic matrix*. Then, in 1992, Waterhouse derived the same results by using the trace on a Frobenius algebra [waterhouseII]. In both cases, the argument is essentially a pairing of the commutativity of mixed real partial derivatives and the generalized Cauchy Riemann equations.

Theorem 6.5.7. *Let U be open in $\mathcal{A}$ and suppose $f : U \to \mathcal{A}$ is twice $\mathcal{A}$-differentiable on U. If there exist $B_{ij} \in \mathbb{R}$ for which $\sum_{i,j} B_{ij} v_i \star v_j = 0$ then $\sum_{i,j} B_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} = 0$.*

Proof: suppose f is twice continuously $\mathcal{A}$-differentiable on $U \subset \mathcal{A}$ and suppose there exist $B_{ij} \in \mathbb{R}$ for which $\sum_{i,j} B_{ij} v_i \star v_j = 0$. Multiply the given equation by $\frac{\partial^2 f}{\partial x_1^2}$ to obtain:

$$\sum_{i,j} B_{ij} \frac{\partial^2 f}{\partial x_1^2} \star v_i \star v_j = 0. \tag{6.68}$$

Then, by Equation 6.67 we deduce $\frac{\partial^2 f}{\partial x_1^2} \star v_i \star v_j = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Therefore, $\displaystyle\sum_{i,j} B_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} = 0$. $\square$

Theorem 6.5.7 essentially says that a quadratic equation in $\mathcal{A}$ yields a corresponding Laplace-type equation for $\mathcal{A}$-differentiable functions. Hence we find:

Corollary 6.5.8. *Generalized Laplace equations can be assembled by mimicking patterns in the multiplication table for $\mathcal{A}$ to matching patterns in the Hessian matrix. Moreover, each component of an $\mathcal{A}$-differentiable function is a solution to the generalized Laplace equations.*

This result was given by Wagner in [wagner1948].

Example 6.5.9. *Consider $\mathcal{A} = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R}$ where $j^3 = 1$. Notice, we have multiplication table and Hessian matrix*

$$\begin{array}{c||c|c|c|} & 1 & j & j^2 \\ \hline\hline 1 & 1 & j & j^2 \\ \hline j & j & j^2 & 1 \\ \hline j^2 & j^2 & 1 & j \\ \hline \end{array} \quad \& \quad \begin{array}{c||c|c|c|} & x & y & z \\ \hline\hline x & f_{xx} & f_{xy} & f_{xz} \\ \hline y & f_{yx} & f_{yy} & f_{yz} \\ \hline z & f_{zx} & f_{yz} & f_{zz} \\ \hline \end{array} \tag{6.69}$$

Theorem 6.5.8 allows us to find the following generalized Laplace equations by inspection of the tables above:

$$f_{xx} = f_{yz}, \quad f_{xy} = f_{zz}, \quad f_{xz} = f_{yy} \tag{6.70}$$

You might recognize these from Example 6.3.9.

Example 6.5.10. *Consider $\mathcal{A} = \mathbb{R} \oplus i\mathbb{R}$ where $i^2 = -1$. Notice, we have multiplication table and Hessian matrix*

$$\begin{array}{c||c|c|} & 1 & i \\ \hline\hline 1 & 1 & i \\ \hline i & i & -1 \\ \hline \end{array} \quad \& \quad \begin{array}{c||c|c|} & x & y \\ \hline\hline x & f_{xx} & f_{xy} \\ \hline y & f_{yx} & f_{yy} \\ \hline \end{array} \tag{6.71}$$

Theorem 6.5.8 allows us to find the Laplace equations by inspection of the tables above: if $f = u+iv$ then

$$f_{xx} = -f_{yy}, \quad \Rightarrow \quad u_{xx} + u_{yy} = 0 \;\; \& \;\; v_{xx} + v_{yy} = 0. \tag{6.72}$$

If we consider the component formulation of the Cauchy Riemann equations then differentiation of these equations will produce second order homogeneous PDEs which include the generalized Laplace Equations and other less elegant equations coupling distinct components. The elegance of the generalized Laplace equations is seen in the fact that **every** component of an $\mathcal{A}$-differentiable function is what we may call $\mathcal{A}$-harmonic. In other words, the algebra $\mathcal{A}$ provides a natural function theory to study $\mathcal{A}$-harmonic functions. Notice, the concept of $\mathcal{A}$-harmonicity involves solving a system of PDEs. When is it possible to find an $\mathcal{A}$ for which a given system of real PDEs for the $\mathcal{A}$-harmonic equations for $\mathcal{A}$? Theorem 6.5.8 gives at least a partial answer. If we replicate patterns imposed on the Hessian matrix to produce a multiplication table then we can test if the table is a possible multiplication table for an algebra. It is interesting to note that Ward already solved the corresponding problem for generalized Cauchy Riemann equations in 1952. In particular, Ward showed in [ward1952] how to construct an algebra $\mathcal{A}$ which takes a given set of $n^2 - n$ independent PDEs as its generalized $\mathcal{A}$-CR equations.

The next example was inspired by Example 4.6 in [waterhouseII]. It demonstrates how algebraic insight can be wielded to produce solutions to PDEs.

Example 6.5.11. *Consider the wave equation $c^2 u_{xx} = u_{tt}$ where c is a positive constant which characterizes the speed of the transverse waves modelled by this PDE. Let us find an algebra $\mathcal{W}_c$ which takes the speed-c wave equation as its generalized Laplace Equation. Let $(x, t) = x + kt$ form a typical point in the algebra. What rule should we give to k? Following Corollary 6.5.8 we should consider the correspondence:*

$$c^2 u_{xx} = u_{tt} \quad \leftrightarrow \quad c^2 = k^2 \tag{6.73}$$

thus set $k^2 = c^2$. The algebra $\mathcal{W}_c = \mathbb{R} \oplus k\mathbb{R}$ with $k^2 = c^2$ has $\mathcal{W}_c$-differentiable functions $f = u + kv$ for which $c^2 u_{xx} = u_{tt}$. Observe $\Gamma : \mathcal{W}_c \to \mathcal{H}$ defined by $\Gamma(x+kt) = x+cjt$ serves as an isomorphism of $\mathcal{W}_c$ and the hyperbolic numbers of Example 3.1.14. Combine $\Psi^{-1}(x + jy) = (x + y, x - y)$ of Example 3.1.14 with Γ to construct the isomorphism $\Phi = \Psi^{-1} \circ \Gamma$ from $\mathcal{W}_c$ to $\mathbb{R} \times \mathbb{R}$. In particular,

$$\Phi(x + kt) = \Psi^{-1}(\Gamma(x + kt)) = \Psi^{-1}(x + cjt) = (x + ct, x - ct) \tag{6.74}$$

Following the insight of Theorem 6.1.11 we associate to each $\mathcal{W}_c$-differentiable function $f : \mathcal{W}_c \to \mathcal{W}_c$ a corresponding $\mathbb{R} \times \mathbb{R}$ differentiable function F as follows:

$$f = \Phi^{-1} \circ F \circ \Phi \tag{6.75}$$

where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$. The structure of $\mathbb{R} \times \mathbb{R}$-differentiable functions is rather simple; $F(a, b) = (F_1(a), F_2(b))$ where F_1, F_2 are differentiable functions on $\mathbb{R}$. Thus,

$$\begin{aligned} f(x + kt) &= \Phi^{-1}(F(\Phi(x + kt))) \\ &= \Phi^{-1}(F((x + ct, x - ct))) \\ &= \Phi^{-1}(F_1(x + ct), F_2(x - ct)) \\ &= \frac{1}{2}(F_1(x + ct) + F_2(x - ct)) + \frac{k}{2c}(F_1(x + ct) - F_2(x - ct)) \end{aligned} \tag{6.76}$$

We have shown that $\mathcal{A}$-differentiable functions $f = u + kv$ have (using $x + kt = (x, t)$ to make the formulas more recognizable)

$$u(x, t) = \frac{1}{2}(F_1(x + ct) + F_2(x - ct)) \qquad \& \qquad v(x, t) = \frac{k}{2}(F_1(x + ct) - F_2(x - ct)). \tag{6.77}$$

We've shown how d'Alembert's solution to the wave-equation appears naturally in the function theory of $\mathcal{W}_c$.

Naturally, there are higher order versions of the Laplace Equations.

Theorem 6.5.12. *Let U be open in $\mathcal{A}$ and suppose $f : U \to \mathcal{A}$ is k-times $\mathcal{A}$-differentiable. If there exist $B_{i_1 i_2 \dots i_k} \in \mathbb{R}$ for which $\sum_{i_1 i_2 \dots i_k} B_{i_1 i_2 \dots i_k} v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k} = 0$ then*

$$\sum_{i_1 i_2 \dots i_k} B_{i_1 i_2 \dots i_k} \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} = 0.$$

Proof: multiply the assumed relation by $\frac{\partial^k f}{\partial x_1^k}$ and apply Theorem 6.5.6. $\square$

6.5.2 $\mathcal{A}$-variate Taylor's Theorem

If we are given that $f : \mathcal{A} \to \mathcal{A}$ is smooth in the sense of real analysis then it is simple to show that the existence of the first $\mathcal{A}$-derivative implies the existence of all higher $\mathcal{A}$-derivatives.

Theorem 6.5.13. *Let $\mathcal{A}$ be a commutative unital finite dimensional algebra over $\mathbb{R}$. Suppose $f : \mathcal{A} \to \mathcal{A}$ has arbitrarily many continuous real derivatives at p and suppose f is once $\mathcal{A}$-differentiable at p then $f^{(k)}(p)$ exists for all $k \in \mathbb{N}$.*

Proof: suppose f is smooth and once $\mathcal{A}$-differentiable at p. We assume $\mathcal{A}$ is a commutative unital algebra over $\mathbb{R}$ with basis $\beta = \{\mathbb{1}, \dots, v_n\}$. Assume inductively that $f^{(k)}(p)$ exists hence Theorem 6.5.4 provides $f^{(k)}(p) = \frac{\partial^k f(p)}{\partial x_1^k}$. Consider, omitting p to reduce clutter,

$$\frac{\partial f^{(k)}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\frac{\partial^k f}{\partial x_1^k}\right] = \frac{\partial^k}{\partial x_1^k}\left[\frac{\partial f}{\partial x_j}\right] = \frac{\partial^k}{\partial x_1^k}\left[\frac{\partial f}{\partial x_1}\right] \star v_j = \frac{\partial f^{(k)}}{\partial x_1} \star v_j. \tag{6.78}$$

Thus $f^{(k)}$ is $\mathcal{A}$-differentiable at p which proves $f^{(k+1)}(p)$ exists. $\square$

In [zorich] a multivariate Taylor's Theorem over a finite dimensional normed linear space is given. In particular, if $f : V \to V$ is **real analytic** then f is represented by its multivariate Taylor series on some open set containing p. The multivariate Taylor series of f centered at p is given, for h sufficiently small, by the convergent series[18]:

$$f(p+h) = f(p) + d_p f(h) + \frac{1}{2} d_p^2 f(h,h) + \frac{1}{3!} d_p^3 f(h,h,h) + \cdots \tag{6.79}$$

Notice, for $(h, \dots, h) \in \mathcal{A}^k$ we may expand $h = \sum_{i_j} h_{i_j} v_{i_j}$ for $j = 1, 2, \dots, k$,

$$d_p^k f(h, \dots, h) = \sum_{i_1, \dots, i_k} h_{i_1} h_{i_2} \cdots h_{i_k} d_p^k f(v_{i_1}, v_{i_2}, \dots, v_{i_k}) \tag{6.80}$$

Compare the formulas above to the form of the k-term in the Taylor expansion for an $\mathcal{A}$-differentiable function given below. This simplification has Theorem 6.5.2 at its root.

Theorem 6.5.14. *Let $\mathcal{A}$ be a commutative, unital, associative algebra over $\mathbb{R}$. If f is real analytic at $p \in \mathcal{A}$ then*

$$f(p+h) = f(p) + f'(p) \star h + \frac{1}{2} f''(p) \star h^2 + \cdots + \frac{1}{k!} f^{(k)}(p) \star h^k + \cdots$$

where $h^2 = h \star h$ and $h^{k+1} = h^k \star h$ for $k \in \mathbb{N}$.

[18] equivalently, $f(p+h) = f(p) + \sum_{k=1}^{\infty} \sum_{i_1, \dots, i_k} \frac{1}{k!} \frac{\partial^k f(p)}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} h_{i_1} h_{i_2} \cdots h_{i_k}$.

Proof: Suppose f is real analytic and $\mathcal{A}$-differentiable. Since real analytic implies f is smooth over $\mathbb{R}$ we apply Theorem 6.5.13 to see f is smooth over $\mathcal{A}$. Therefore, we may follow the proof of Theorem 6.5.5 and obtain:

$$d_p^k f(v_{i_1}, v_{i_2}, \dots, v_{i_k}) = f^{(k)}(p) \star v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k}. \tag{6.81}$$

Observe the k-th power of $h \in \mathcal{A}$ is given by

$$h^k = \sum_{i_1,\dots,i_k} h_{i_1} h_{i_2} \cdots h_{i_k} v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k}. \tag{6.82}$$

Therefore, combining Equations 6.80, 6.81, and 6.82 we find

$$\begin{aligned} d_p^k f(h, \dots, h) &= \sum_{i_1,\dots,i_k} h_{i_1} h_{i_2} \cdots h_{i_k} d_p^k f(v_{i_1}, v_{i_2}, \dots, v_{i_k}) \\ &= \sum_{i_1,\dots,i_k} h_{i_1} h_{i_2} \cdots h_{i_k} f^{(k)}(p) \star v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k}. \\ &= f^{(k)}(p) \star \sum_{i_1,\dots,i_k} h_{i_1} h_{i_2} \cdots h_{i_k} v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k} \\ &= f^{(k)}(p) \star h^k. \end{aligned} \tag{6.83}$$

We conclude, $f(p+h) = f(p) + f'(p) \star h + \frac{1}{2} f''(p) \star h^2 + \cdots + \frac{1}{k!} f^{(k)}(p) \star h^k + \cdots$. $\square$

We study the theory of convergence series in $\mathcal{A}$ in the sequel to this paper which is a joint work with Daniel Freese [cookfreese].

6.6 An approach to the inverse problem

The inverse problem of $\mathcal{A}$-calculus is roughly this:

> When can we translate a problem of real calculus to a corresponding problem of $\mathcal{A}$-calculus ?

Naturally, this raises a host of questions. What kind of real calculus problems? How do we choose $\mathcal{A}$? In [pagr2012] and [pagr2015] the authors study how to modify certain ODEs in terms of $\mathcal{A}$-calculus. In contrast, our study on the inverse problem has centered around systems of PDEs. In particular, we seek to answer the following question:

> When can we find solutions to a system of PDEs which are simultaneously solutions to the generalized Laplace equations of some algebra $\mathcal{A}$?

Our hope is that if the answer to the question above is affirmative then it may be possible to rewrite the system of PDEs in an $\mathcal{A}$-based notation where the PDE in many variables simply becomes an $\mathcal{A}$-ODE in a single algebra variable. Of course, this is just an initial conjecture, there are many directions we could explore at the level of algebra-based differential equations[19]. Let us examine a simple example of how an $\mathcal{A}$-ODE can replace a system of PDEs.

[19] In fact at the time this paper is prepared the author has already shown how to solve many $\mathcal{A}$-ODEs. The joint work [cookbedell] with Nathan BeDell is currently under preparation. Nathan BeDell has three other papers [bedellI],[bedellII],[bedellIII] in preparation which discuss zero-divisors and basic algebra, the construction of logarithms, and identities for generalized trigonmetric functions and many other algebraic preliminaries.

Example 6.6.1. *Let $z = x + jy$ denote an independent hyperbolic variable and $w = u + jv$ the solution of $\frac{dw}{dz} = w^2$. Separating variables gives $\frac{dw}{w^2} = dz$ hence $\frac{-1}{w} = z + c$ where $c = c_1 + jc_2$ is a hyperbolic constant. Thus, $w = \frac{-1}{z+c}$ is the solution. What does this mean at the level of real calculus? Note,*

$$\frac{dw}{dz} = w^2 \quad \Rightarrow \quad u_x + jv_x = (u + jv)^2 = u^2 + v^2 + 2juv \tag{6.84}$$

In other words, the $\mathcal{A}$-ODE is the nonlinear system of PDEs

$$u_x = u^2 + v^2, \quad \& \quad v_x = 2uv \tag{6.85}$$

paired with the $\mathcal{A}$-CR equations $u_x = v_y$ and $u_y = v_x$. We have the solution already from direct calculus on $\mathcal{A}$,

$$w = \frac{-1}{z+c} \quad \Rightarrow \quad u + jv = -\frac{x + c_1 - j(y + c_2)}{(x + c_1)^2 - (y + c_2)^2} \tag{6.86}$$

Thus, $u = -\frac{x+c_1}{(x+c_1)^2-(y+c_2)^2}$ and $v = \frac{y+c_2}{(x+c_1)^2-(y+c_2)^2}$ are the real solutions to 6.85 and you can check that $u_x = v_y$ and $u_y = v_x$ as well.

6.6.1 optional section on tableau for $\mathcal{A}$-calculus

Our first goal in understanding the inverse problem was to decide when it is possible to pair a system like 6.85 with the $\mathcal{A}$-CR equations of an appropriate algebra $\mathcal{A}$. Essentially, our first concern is whether there is at least an algebra whose $\mathcal{A}$-CR equations and their *differential consequences* are not inconsistent with a given system of PDEs.

We construct the **generic tableau** $\mathcal{T} = \mathcal{A} \oplus \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \cdots \oplus \mathcal{T}_k \oplus + \cdots$

$$\mathcal{T}_k = \text{span}\{dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k} \otimes v_j \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n, 1 \leq j \leq n\} \tag{6.87}$$

We use these spaces to account for dependencies amongst variables and their derivatives. In particular, we focus our attention on PDEs which are formed from n-dependent and n-independent variables. Notice, we only need increasing indices since these symbols represent partial derivatives which we can commute to be in increasing order[20]. The **Gauss map** of a function $f : \mathcal{A} \to \mathcal{A}$ into the generic tableau is formed as follows: if $f = \sum u^j v_j$

$$\gamma(f) = f \oplus \left(\sum_{i,j} \frac{\partial u^j}{\partial x_i} v_j \otimes dx^i \right) \oplus \left(\sum_{i_1 \leq i_2} \sum_j \frac{\partial^2 u^j}{\partial x_{i_1} \partial x_{i_2}} v_j \otimes dx^{i_1} \otimes dx^{i_2} \right) \oplus \cdots \tag{6.88}$$

For the k-th term,

$$\gamma(f) = \cdots \oplus \left(\sum_{i_1 \leq i_2 \leq \cdots \leq i_k} \sum_j \frac{\partial^k u^j}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} v_j \otimes dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k} \right) \oplus \cdots \tag{6.89}$$

Or, more concisely,

$$\begin{aligned} \gamma(f) = f + \sum_i (\partial_i f)\, dx^i + \sum_{i_1 \leq i_2} (\partial_{i_1} \partial_{i_2} f)\, dx^{i_1} \otimes dx^{i_2} + \\ \cdots + \sum_{i_1 \leq \cdots \leq i_k} (\partial_{i_1} \cdots \partial_{i_k} f) dx^{i_1} \otimes \cdots \otimes dx^{i_k} + \cdots \end{aligned} \tag{6.90}$$

[20]this construction was inspired by a more sophisticated, but similar, construction in Chapter 4 of [cartan4beginners]

We seek to represent systems of PDEs as particular subspaces inside $\mathcal{T}$. The PDEs we study have finite order and hence the calculation ultimately amounts to a question of finite dimensional linear algebra. We should note the subspace of $\mathcal{T}$ which is given by the $\mathcal{A}$-CR equations is particularly simple.

Theorem 6.6.2. *Let $\mathcal{T}_{\mathcal{A}} \leq \mathcal{T}$ denote the subspace of $\mathcal{T}$ generated by infinitely $\mathcal{A}$-differentiable functions then $\gamma \in \mathcal{T}_{\mathcal{A}}$ has the form*

$$\gamma = \alpha_o + \alpha_1 \star \sum_i v_i \otimes dx^i + \alpha_2 \star \sum_{i_1 \leq i_2} v_{i_1} \star v_{i_2} \otimes dx^{i_1} \otimes dx^{i_2} + \cdots + \alpha_k \star \sum_{|I|} v_I \otimes dx^I + \cdots$$

where $|I|$ indicates the sum over increasing k-tuples of indices taken from $\{1, 2, \ldots, n\}$ and $v_I = v_{i_1} \star v_{i_2} \star \cdots \star v_{i_k}$ and $dx^I = dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k}$.

Proof: If f has arbitrarily many $\mathcal{A}$-derivatives then we find

$$\begin{aligned} \partial_i f &= (\partial_1 f) \star v_i, \\ \partial_{i_1} \partial_{i_2} f &= (\partial_1^2 f) \star v_{i_1} \star v_{i_1} \\ \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f &= (\partial_1^k f) \star v_{i_1} \star v_{i_1} \star \cdots \star v_{i_k}. \end{aligned} \tag{6.91}$$

Then by Equation 6.90 we find γ_f has the form given in the Theorem. $\square$

Chapter 7

Complex Functions

Remark: at this point I make a rather drastic turn. For the most part I abandon discussion of abstract hypercomplex analysis and we focus almost entirely on $\mathcal{A} = \mathbb{C}$. I will at times take a few minutes to insert a comment in class about what I've learned for general $\mathcal{A}$-Calculus, but these notes will probably not reflect those comments. That said, I will try to continue developing some ideas in $\mathcal{A}$-calculus in the homework as appropriate. I'm hopeful this makes the remainder of the course a bit easier to follow. My apologies for this hard segway, I have to admit my time is too limited to do a full edit past this point justice.

7.1 Review of Results for Calculus on $\mathbb{C}$

As we discussed, the following Definition is equivalent to that offered for $\mathcal{A}$-differentiability in the specific and very nice case $\mathcal{A} = \mathbb{C}$. I leave this as a Definition in these notes in order to maintain the organization which follows from this point. I am trying hard to not talk much about $\mathcal{A}$-Calculus until much later, so, these notes are largely the same as those I gave in 2015. However, I tried to remove derivations which are not relevant since we already did lot of work in the abstract case. For example, we already know the sum, product and chain rule for $\mathcal{A}$-Calculus hence we need not prove those again here in the special context of complex calculus.

Definition 7.1.1. *If $\displaystyle\lim_{z\to z_o} \frac{f(z)-f(z_o)}{z-z_o}$ exists then we say f is* **complex differentiable at $\mathbf{z}_o$** *and we denote $f'(z_o) = \displaystyle\lim_{z\to z_o} \frac{f(z)-f(z_o)}{z-z_o}$. Furthermore, the mapping $z \mapsto f'(z)$ is the* **complex derivative of f.**

We continue to use many of the same notations as in first semester calculus. In particular, $f'(z) = df/dz$ and $d/dz(f(z)) = f'(z)$. My language differs slightly from Gamelin here in that I insist we refer to the *complex differentiability* of f.

Theorem 7.1.2. *Given functions f, g, w which are complex differentiable (and nonzero for g in the quotient) we have:*

$$\frac{d}{dz}(f+g) = \frac{df}{dz} + \frac{dg}{dz}, \qquad \frac{d}{dz}(cf) = c\frac{df}{dz}, \qquad \frac{d}{dz}\big(f(w)\big) = \frac{df}{dw}\frac{dw}{dz}$$

where the notation $\frac{df}{dw}$ indicates we take the derivative function of f and evaluate it at the value of the inside function w; that is, $\frac{df}{dw}(z) = f'(w(z))$.

We also know the power rule in complex calculus.

Theorem 7.1.3. Power law for integer powers: *let $n \in \mathbb{Z}$ then $\frac{d}{dz}\big(z^n \big) = nz^{n-1}$.*

Non-integer power functions have phase functions which bring the need for branch cuts. It follows that we ought to discuss derivatives of exponential and log functions before we attempt to extend the power law to other than integer powers. That said, nothing terribly surprising happens. It is in fact the case $\frac{d}{dz}z^n = nz^{n-1}$ for $n \in \mathbb{C}$ however we must focus our attention on just one branch of the function.

Let us attempt to find $\frac{d}{dz}e^z$ directly from Definition 7.1.1. Notice $e^{z+h} = e^z e^h$ hence

$$\frac{e^{z+h} - e^z}{h} = e^z \cdot \frac{e^h - 1}{h}$$

it follows that the value of $\frac{d}{dz}e^z$ rests on the limit of $\frac{e^h-1}{h}$ as $h \to 0$. Let $h = a + ib$ and note $e^{a+ib} = e^a \cos b + ie^a \sin b$ thus

$$\frac{e^h - 1}{h} = \frac{e^a \cos b + ie^a \sin b - 1}{a + ib} = \frac{e^a(\cos b - 1 + i \sin b)(a - ib)}{a^2 + b^2}$$

Therefore,

$$\frac{e^h - 1}{h} = e^a \cdot \frac{a(\cos b - 1) + b \sin b}{a^2 + b^2} + ie^a \cdot \frac{b(1 - \cos b) + a \sin b}{a^2 + b^2}$$

The limit $h = a + ib \to 0$ amounts to $(a, b) \to 0$ in the real notation. Notice both the real and imaginary components of $\frac{e^h-1}{h}$ are somewhat formiddable indeterminant forms. This is the trouble we face if we are so bold as to use Definition 7.1.1 directly. In contrast, we will dispatch the problem with ease given the Theorems of the next section.

If a function is complex differentiable over a domain of points it turns out that the complex derivative function **must** be continuous. Not all texts would include this fact in the definition of **analytic**, but, I'll follow Gamelin and make some comments later when we can better appreciate why this is not such a large transgression (if it's one at all). See pages 56-57 of [R91] for a definition without the inclusion of the continuity of f'. Many other texts use the term **holomorphic** in the place of analytic and I will try to use both appropriately. Note carefully the distinction between at a point, on a set and for the whole function. There is a distinction between complex differentiability at a point and holomorphicity at a point.

Definition 7.1.4. *We say f* **is holomorphic on domain** *D if f is complex differentiable at each point in D. We say f* **is holomorphic at** *z_o if there exists an open disk D centered at z_o on which $f|_D$ is holomorphic.*

Given our calculations thus far we can already see that polynomial functions are holomorphic on $\mathbb{C}$. Furthermore, if $p(z), q(z) \in \mathbb{C}[z]$ then p/q is holomorphic on $\mathbb{C} - \{z \in \mathbb{C} \mid q(z) = 0\}$. We discover many more holomorphic functions via the Cauchy Riemann equations of the next section. It is also good to have some examples which show not all functions on $\mathbb{C}$ are holomorphic.

Example 7.1.5. *Let $f(z) = \bar{z}$ then the difference quotient is $\dfrac{\bar{z} - \bar{a}}{z - a}$. If we consider the path $z = a+t$ where $t \in \mathbb{R}$ then*

$$\frac{\bar{z} - \bar{a}}{z - a} = \frac{\bar{a} + t - \bar{a}}{a + t - a} = 1$$

hence as $t \to 0$ we find the difference quotient tends to 1 along this horizontal path through a. On the other hand, if we consider the path $z = a + it$ then

$$\frac{\bar{z} - \bar{a}}{z - a} = \frac{\bar{a} - it - \bar{a}}{a + it - a} = -1$$

hence as $t \to 0$ we find the difference quotient tends to -1 along this vertical path through a. But, this shows the limit $z \to a$ of the difference quotient does not exist. Moreover, as a was an arbitrary point in $\mathbb{C}$ we have shown that $f(z) = \bar{z}$ is **nowhere complex differentiable** *on $\mathbb{C}$.*

The following example is taken from [R91] on page 57. I provide proof of the claims made below in the next section as the Cauchy Riemann equations are far easier to calculate that limits.

Example 7.1.6. *Let $f(z) = x^3y^2 + ix^2y^3$ where $z = x + iy$. We can show that f is complex differentiable where $x = 0$ or $y = 0$. In other words, f is complex differentiable on the coordinate axes. It follows this function is* **nowhere holomorphic** *on $\mathbb{C}$ since we cannot find any point about which f is complex differentiable on an whole open disk.*

7.2 The Cauchy-Riemann Equations

I have edited this Section drastically in comparison to the 2014 notes since we have already introduced and discussed the $\mathcal{A}$-Cauchy Riemann Equations in a lot of detail. Our goal here is to appreciate their specific application to $\mathcal{A} = \mathbb{C}$. In fact, there is much to say.

Let's examine what makes complex differentiability so special from a real viewpoint. Let $h = x+iy$ and $f'(z_o) = a + ib$

$$df_{z_o}(h) = f'(z_o)h = (a + ib)(x + iy) = ax - by + i(bx + ay).$$

If I write this as a matrix multiplication using $1 = (1, 0)$ and $i = (0, 1)$ the calculation above is written as

$$df_{z_o}(h) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

However, the Jacobian matrix is unique and thus

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \quad \Rightarrow \quad \boxed{u_x = v_y, \quad u_y = -v_x}$$

The boxed equations are the **Cauchy Riemann** or (CR) equations for $f = u + iv$.

Definition 7.2.1. *Let $f = u + iv$ then $u_x = v_y$ and $u_y = -v_x$ are the* **Cauchy Riemann or (CR)**-*equations for f.*

We have shown **if a function $f = u+iv$ is complex differentiable then it is real differentiable and the component functions satisfy the CR-equations.**

Example 7.2.2. . *At this point we can return to my claim in Example 7.1.6. Let $f(z) = x^3y^2 + ix^2y^3$ where $z = x + iy$ hence $u = x^3y^2$ and $v = x^2y^3$ and we calculate:*

$$u_x = 3x^2y^2, \quad u_y = 2x^3y, \quad v_x = 2xy^3, \quad v_y = 3x^2y^2.$$

If f is holomorphic on some open set disk D then it is complex differentiable at each point in D. Hence, by our discussion preceding this example it follows $u_x = v_y$ and $v_x = -u_y$. The only points in $\mathbb{C}$ at which the CR-equations hold are where $x = 0$ or $y = 0$. Therefore, it is impossible for f to be complex differentiable on any open disk. Thus our claim made in Example 7.1.6 is true; f is nowhere holomorphic.

Now, let us investigate the converse direction. Let us see that if the CR-equations hold for continuously real differentiable function on a domain then the function is holomorphic on that domain. We assume continuously differentiable on a domain for our expositional convenience. See pages 58-59 of [R91] where he mentions a number of weaker conditions which still are sufficient to guarantee complex differentiability at a given point.

Suppose $f = u + iv$ is continuously real differentiable mapping on a domain D where the CR equations hold throughout D. That is for each $x + iy \in D$ the real-valued component functions u, v satisfy $u_x = v_y$ and $v_x = -u_y$.

$$\begin{aligned} \mathrm{J}_f h &= \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &= \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &= \begin{bmatrix} u_x h_1 - v_x h_2 \\ v_x h_1 + u_x h_2 \end{bmatrix} \\ &= (u_x h_1 - v_x h_2) + i(v_x h_1 + u_x h_2) \\ &= (u_x + i v_x)(h_1 + i h_2). \end{aligned}$$

Let $z_o \in D$ and define $u_x(z_o) = a$ and $v_x(z_o) = b$. The calculation above shows the CR-equations allow the (real) differential of f as multiplication by the complex number $a + ib$. We propose $f'(z_o) = a + ib$. We can derive the needed difference quotient by analyzing the Frechet quotient with care. We are given[1]:

$$\lim_{h \to 0} \frac{f(z_o + h) - f(z_o) - (a + ib)h}{h} = 0.$$

Notice, $\lim_{h \to 0} \frac{(a+ib)h}{h} = a + ib$ thus[2]

$$\lim_{h \to 0} \frac{f(z_o + h) - f(z_o)}{h} - \lim_{h \to 0} \frac{(a + ib)h}{h} = 0.$$

Therefore,

$$a + ib = \lim_{h \to 0} \frac{f(z_o + h) - f(z_o)}{h}$$

which verifies our claim $f'(z_o) = a + ib$. Let us gather the results:

Theorem 7.2.3. *We have shown:*

1. *If $f = u + iv : U \subseteq \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z_o \in U$ then f is real differentiable at z_o and $u_x = v_y$ and $v_x = -u_y$ at z_o.*

[1] I'm cheating, see your homework (Problem 20) where you show $\lim_{h \to 0} g(h)/|h| = 0$ implies $\lim_{h \to 0} g(h)/h = 0$.

[2] I encourage the reader to verify the little theorem: if $\lim(f - g) = 0$ and $\lim g$ exists then $\lim f = \lim g$.

2. *If $f = u + iv : U \subseteq \mathbb{C} \to \mathbb{C}$ is continuously real differentiable at $z_o \in U$ and $u_x = v_y$ and $v_x = -u_y$ at z_o then f is complex differentiable at z_o.*

3. *If $f = u+iv$ is continuously differentiable on a domain D and the CR-equations hold throughout D then f is holomorphic on D.*

Note that (3.) aligns with the theorem given on page 47 of Gamelin. I reader might note the proof I offered here differs significantly in style from that of page 48 in Gamelin. We should note when f is complex differentiable we have the following identities:

$$f'(z) = u_x + iv_x = v_y - iu_y \quad \Rightarrow \quad \frac{df}{dz} = \frac{\partial f}{\partial x} \;\; \& \;\; \frac{df}{dz} = -i\frac{\partial f}{\partial y}$$

where the differential identities hold **only** for holomorphic functions. The corresponding identities for arbitrary functions on $\mathbb{C}$ are discussed on pages 124-126 of Gamelin.

As promised, we can show the other elementary functions are holomorphic in the appropriate domain. Let us begin with the complex exponential.

Example 7.2.4. *Let $f(z) = e^z$ then $f(x+iy) = e^x(\cos y + i \sin y)$ hence $u = e^x \cos y$ and $v = e^x \sin y$. Observe u, v clearly have continuous partial derivatives on $\mathbb{C}$ and*

$$u_x = e^x \cos y, \quad v_x = e^x \sin y, \quad u_y = -e^x \sin y, \quad v_y = e^x \cos y.$$

Thus $u_x = v_y$ and $v_x = -u_y$ for each point in $\mathbb{C}$ and we find $f(z) = e^z$ is holomorphic on $\mathbb{C}$. Moreover, as $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y$ we find the comforting result $\frac{d}{dz}e^z = e^z$.

Definition 7.2.5. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$ then f is an* **entire function**. *The set of entire functions on $\mathbb{C}$ is denoted $\mathcal{O}(C)$*

The complex exponential function is entire. Functions constructed from the complex exponential are also entire. In particular, it is a simple exercise to verify $\sin z, \cos z, \sinh z, \cosh z$ are all entire functions. We can either use part (2.) of Theorem 7.2.3 and explicitly calculate real and imaginary parts of these functions, or, we could just use Example 7.2.4 paired with the chain rule. For example:

Example 7.2.6.

$$\begin{aligned}
\frac{d}{dz}\sin z &= \frac{d}{dz}\left[\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right] \\
&= \frac{1}{2i}\frac{d}{dz}\left[e^{iz}\right] - \frac{1}{2i}\frac{d}{dz}\left[e^{-iz}\right] \\
&= \frac{1}{2i}e^{iz}\frac{d}{dz}\left[iz\right] - \frac{1}{2i}e^{-iz}\frac{d}{dz}\left[-iz\right] \\
&= \frac{1}{2i}e^{iz}i - \frac{1}{2i}e^{-iz}(-i) \\
&= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
&= \cos(z).
\end{aligned}$$

Very similar arguments show the hopefully unsurprising results below:

$$\frac{d}{dz}\sin z = \cos z, \quad \frac{d}{dz}\cos z = -\sin z, \quad \frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z.$$

You might notice that Theorem 4.2.10 applies to **real-valued** functions on the plane. The theorem below deals with a complex-valued function.

Theorem 7.2.7. *If f is analytic on a domain D and $f'(z) = 0$ for all $z \in D$ then f is constant.*

Proof: observe $f'(z) = u_x + iv_x = 0$ thus $u_x = 0$ and $v_x = 0$ thus $v_y = 0$ and $u_y = 0$ by the CR-equations. Thus $\nabla u = 0$ and $\nabla v = 0$ on a connected open set so we may apply Theorem 4.2.10 to see $u(z) = a$ and $v(z) = b$ for all $z \in D$ hence $f(z) = a + ib$ for all $z \in D$. $\square$

There are some striking, but trivial, statements which follow from the Theorem above. For instance:

Theorem 7.2.8. *If f is holomorphic and real-valued on a domain D then f is constant.*

Proof: Suppose $f = u + iv$ is holomorphic on a domain D then $u_x = v_y$ and $v_x = -u_y$ hence $f'(z) = u_x + iv_x = v_y + iv_x$. Yet, $v = 0$ since f is real-valued hence $f'(z) = 0$ and we find f is constant by Theorem 7.2.8. $\square$

You can see the same is true of f which is imaginary and analytic. We could continue this section to see how to differentiate the reciprocal trigonometric or hyperbolic functions such as $\sec z, \csc z, \mathrm{csch} z, \mathrm{sech} z, \tan z, \tanh z$ however, I will refrain as the arguments are the same as you saw in first semester calculus. It seems likely I ask some homework about these. You may also recall, we needed **implicit differentiation** to find the derivatives of the inverse functions in calculus I. The same is true here and that is the topic of the next section.

The set of holomorphic functions over a domain is an object worthy of study. Notice, if D is a domain in $\mathbb{C}$ then polynomials, rational functions with nonzero denominators in D are all holomorphic. Of course, the functions built from the complex exponential are also holomorphic. A bit later, we'll see any power series is holomorphic in some domain about its center. Each holomorphic function on D is continuous, but, not all continous functions on D are holomorphic. The **antiholomorphic** functions are also continuous. The quintessential antiholomorphic example is $f(z) = \bar{z}$.

Definition 7.2.9. *The set of all holomorphic functions on a domain $D \subseteq \mathbb{C}$ is denoted $\mathcal{O}(D)$.*

On pages 59-60 of [R91] there is a good discussion of the algebraic properties of $\mathcal{O}(D)$. Also, on 61-62 Remmert discusses the notation $\mathcal{O}(D)$ and the origin of the term **holomorphic** which was given in 1875 by Briot and Bouquet. We will eventually uncover the equivalence of the terms holomorphic, analytic , conformal. These terms are in part tied to the approaches of Cauchy, Weierstrauss and Riemann. I'll try to explain this trichotomy in better detail once we know more. It is the theme of Remmert's text [R91].

7.2.1 CR equations in polar coordinates

If we use polar coordinates to rewrite f as follows:

$$f(x(r,\theta), y(r,\theta)) = u(x(r,\theta), y(r,\theta)) + iv(x(r,\theta), y(r,\theta))$$

we use shorthands $F(r,\theta) = f(x(r,\theta), y(r,\theta))$ and $U(r,\theta) = u(x(r,\theta), y(r,\theta))$ and $V(r,\theta) = v(x(r,\theta), y(r,\theta))$. We derive the CR-equations in polar coordinates via the chain rule from multivariate calculus,

$$U_r = x_r u_x + y_r u_y = \cos(\theta)u_x + \sin(\theta)u_y \quad \text{and} \quad U_\theta = x_\theta u_x + y_\theta u_y = -r\sin(\theta)u_x + r\cos(\theta)u_y$$

Likewise,

$$V_r = x_r v_x + y_r v_y = \cos(\theta)v_x + \sin(\theta)v_y \quad \text{and} \quad V_\theta = x_\theta v_x + y_\theta v_y = -r\sin(\theta)v_x + r\cos(\theta)v_y$$

We can write these in matrix notation as follows:

$$\begin{bmatrix} U_r \\ U_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Multiply these by the inverse matrix: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}^{-1} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & -\sin(\theta) \\ r\sin(\theta) & \cos(\theta) \end{bmatrix}$ to find

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & -\sin(\theta) \\ r\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} U_r \\ U_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)U_r - \frac{1}{r}\sin(\theta)U_\theta \\ \sin(\theta)U_r + \frac{1}{r}\cos(\theta)U_\theta \end{bmatrix}$$

A similar calculation holds for V. To summarize:

$$\boxed{u_x = \cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta} \qquad \boxed{v_x = \cos(\theta)V_r - \tfrac{1}{r}\sin(\theta)V_\theta}$$

$$\boxed{u_y = \sin(\theta)U_r + \tfrac{1}{r}\cos(\theta)U_\theta} \qquad \boxed{v_y = \sin(\theta)V_r + \tfrac{1}{r}\cos(\theta)V_\theta}$$

Another way to derive these would be to just apply the chain-rule directly to u_x,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial \theta}$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$. I leave it to the reader to show you get the same formulas from that approach. The CR-equation $u_x = v_y$ yields:

$$(A.) \quad \cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta = \sin(\theta)V_r + \tfrac{1}{r}\cos(\theta)V_\theta$$

Likewise the CR-equation $u_y = -v_x$ yields:

$$(B.) \quad \sin(\theta)U_r + \tfrac{1}{r}\cos(\theta)U_\theta = -\cos(\theta)V_r + \tfrac{1}{r}\sin(\theta)V_\theta$$

Multiply (A.) by $r\sin(\theta)$ and $(B.)$ by $r\cos(\theta)$ and subtract (A.) from (B.):

$$U_\theta = -rV_r$$

Likewise multiply (A.) by $r\cos(\theta)$ and $(B.)$ by $r\sin(\theta)$ and add (A.) and (B.):

$$rU_r = V_\theta$$

Finally, recall that $z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$ hence

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= (\cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta) + i(\cos(\theta)V_r - \tfrac{1}{r}\sin(\theta)V_\theta) \\ &= (\cos(\theta)U_r + \sin(\theta)V_r) + i(\cos(\theta)V_r - \sin(\theta)U_r) \\ &= (\cos(\theta) - i\sin(\theta))U_r + i(\cos(\theta) - i\sin(\theta))V_r \\ &= e^{-i\theta}(U_r + iV_r) \end{aligned}$$

Theorem 7.2.10. Cauchy Riemann Equations in Polar Form: *If $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$ is a complex function written in polar coordinates r, θ then the Cauchy Riemann equations are written $U_\theta = -rV_r$ and $rU_r = V_\theta$. If $f'(z_o)$ exists then the CR-equations in polar coordinates hold. Likewise, if the CR-equations hold in polar coordinates and all the polar component functions and their partial derivatives with respect to r, θ are continuous on an open disk about z_o then $f'(z_o)$ exists and $f'(z) = e^{-i\theta}(U_r + iV_r)$ which can be written simply as $\dfrac{df}{dz} = e^{-i\theta}\dfrac{\partial f}{\partial r}$.*

Example 7.2.11. *Let $f(z) = z^2$ hence $f'(z) = 2z$ as we have previously derived. That said, lets see how the theorem above works: $f(re^{i\theta}) = r^2 e^{2i\theta}$ hence*

$$f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = e^{-i\theta} 2re^{2i\theta} = 2re^{i\theta} = 2z.$$

Example 7.2.12. *Let $f(z) = Log(z)$ then for $z \in \mathbb{C}^-$ we find $f(re^{i\theta}) = \ln(r) + i\theta$ for $\theta = Arg(z)$ hence*

$$f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = e^{-i\theta}\frac{1}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

I mentioned the polar form of Cauchy Riemann equations in these notes since they can be very useful when we work problems on disks. We may not have much occasion to use these, but it's nice to know they exist.

7.3 Inverse Mappings and the Jacobian

In advanced calculus there are two central theorems of the classical study: the inverse function theorem and the implicit function theorem. In short, the inverse function theorem simply says that if $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable at p and has $\det(F'(p)) \neq 0$ then there exists some neighborhood V of p on which $F|_V$ has a continuously differentiable inverse function. The simplest case of this is calculus I where $f : U \subseteq \mathbb{R} \to \mathbb{R}$ is locally invertible at $p \in U$ if $f'(p) \neq 0$. Note, geometrically this is clear, if the slope were zero then the function will not be $1-1$ near the point so the inverse need not exist. On the other hand, if the derivative is nonzero at a point **and continuous** then the derivative must stay nonzero near the point (by continuity of the derivative function) hence the function is either increasing or decreasing near the point and we can find a local inverse. I remind the reader of these things as they may not have thought through them carefully in their previous course work. That said, I will not attempt a geometric visualization of the complex case. We simply need to calculate the determinant of the derivative matrix and that will allow us to apply the advanced calculus theorem here:

Theorem 7.3.1. *If f is complex differentiable at p then $\det J_f(p) = |f'(p)|^2$.*

Proof: suppose $f = u + iv$ is complex differentiable then the CR equations hold thus:

$$\det \mathrm{J}_f(p) = \det \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} = (u_x)^2 + (v_x)^2 = |u_x + iv_x|^2 = |f'(z)|^2. \qquad \square$$

If $f = u + iv$ is holomorphic on a domain D with $(u_x)^2 + (v_x)^2 \neq 0$ on D then f is locally invertible throughout D. The interesting thing about the theorem which follows is we also learn that the inverse function is holomorphic about some small open disk about the point where $f'(p) \neq 0$.

Theorem 7.3.2. *If $f(z)$ is analytic on a domain D, $z_o \in D$, and $f'(z_o) \neq 0$. Then there is a (small) disk $U \subseteq D$ containing z_o such that $f|_U$ is $1-1$, the image $V = f(U)$ of U is open, and the inverse function $f^{-1} : V \to U$ is analytic and satisfies*

$$(f^{-1})'(f(z)) = 1/f'(z) \qquad \text{for } z \in U.$$

Proof: I will give a proof which springs naturally from advanced calculus. First note that $f'(z_o) \neq 0$ implies $|f'(z_o)|^2 \neq 0$ hence by Theorem 7.3.1 and the inverse function theorem of advanced calculus the exists an open disk U centered about z_o and a function $g : f(U) \to U$ which is the inverse of f restricted to U. Furthermore, we know g is continuously real differentiable. In particular, $g \circ f = Id_U$ and the chain rule in advanced calculus provides $\mathrm{J}_g(f(p))\mathrm{J}_f(p) = I$ for each $p \in U$. Here $I = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$. We already learned that the holomorphicity of f implies we can write $\mathrm{J}_f(p) = \left[\begin{array}{cc} a & -b \\ b & a \end{array}\right]$ where $u_x(p) = a$ and $v_x(p) = b$. The inverse of such a matrix is given by:

$$\left[\begin{array}{cc} a & -b \\ b & a \end{array}\right]^{-1} = \frac{1}{a^2+b^2}\left[\begin{array}{cc} a & b \\ -b & a \end{array}\right].$$

But, the equation $\mathrm{J}_g(f(p))\mathrm{J}_f(p) = I$ already tells us $(\mathrm{J}_f(p))^{-1} = \mathrm{J}_g(f(p))$ hence we find the Jacobian matrix of $g(f(p))$ is given by:

$$\mathrm{J}_g(f(p)) = \left[\begin{array}{cc} a/(a^2+b^2) & b/(a^2+b^2) \\ -b/(a^2+b^2) & a/(a^2+b^2) \end{array}\right]$$

This matrix shows that if $g = m + in$ then $m_x(f(p)) = a/(a^2+b^2)$ and $n_x = -b/(a^2+b^2)$. Thus we have $g' = m_x + in_x$ where

$$g'(f(p)) = \frac{1}{a^2+b^2}(a - ib) = \frac{a-ib}{(a+ib)(a-ib)} = \frac{1}{a+ib} = \frac{1}{f'(p)}. \qquad \square$$

Discussion: I realize some of you have not had advanced calculus so the proof above it not optimal. Thankfully, Gamelin gives an argument on page 52 which is free of matrix arguments. That said, if we understand the form of the Jacobian matrix as it relates the real Jordan form of a matrix then the main result of the conformal mapping section is immediately obvious. In particular, provided $a^2 + b^2 \neq 0$ we can factor as follows

$$\mathrm{J}_f = \left[\begin{array}{cc} a & -b \\ b & a \end{array}\right] = \sqrt{a^2+b^2}\left[\begin{array}{cc} a/\sqrt{a^2+b^2} & -b/\sqrt{a^2+b^2} \\ b/\sqrt{a^2+b^2} & a/\sqrt{a^2+b^2} \end{array}\right].$$

It follows there exists θ for which

$$\mathrm{J}_f = \pm\sqrt{a^2+b^2}\left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right].$$

This shows the Jacobian matrix of a complex differentiable mapping has a very special form. Geometrically, we have a scale factor of $\sqrt{a^2+b^2}$ which either elongates or shrinks vectors. Then the matrix with θ is precisely a rotation by θ. If the $\pm = +$ then in total the Jacobian is just a **dilation** and **rotation**. If the $\pm = -$ then the Jacobian is a **reflection about the origin** followed by a **dilation** and **rotation**. In general, the possible geometric behaviour of 2×2 matrices is much more varied. This decomposition is special to our structure. We discuss the further implications of these observations in Section 7.5.

The application of the inverse function theorem requries less verbosity.

Example 7.3.3. *Note $f(z) = e^z$ has $f'(z) = e^z \neq 0$ for all $z \in \mathbb{C}$. It follows that there exist local inverses for f about any point in the complex plane. Let $w = Log(z)$ for $z \in \mathbb{C}^-$. Since the inverse function theorem shows us $\frac{dw}{dz}$ exists we may calculate as we did in calculus I. To begin, $w = Log(z)$ hence $e^w = z$ then differentiate to obtain $e^w \frac{dw}{dz} = 1$. But $e^w = z$ thus $\frac{d}{dz}Log(z) = \frac{1}{z}$ for all $z \in \mathbb{C}^-$.*

We should remember, it is not possible to to find a global inverse as we know $e^z = e^{z+2\pi im}$ for $m \in \mathbb{Z}$. However, given any choice of logarithm $Log_\alpha(z)$ we have $\frac{d}{dz}Log_\alpha(z) = \frac{1}{z}$ for all z in the slit plane which omits the discontinutiy of $Log_\alpha(z)$. In particular, $Log_\alpha(z) \in \mathcal{O}(D)$ for

$$D = \mathbb{C} - \{re^{i\alpha} \mid r \geq 0\}.$$

Example 7.3.4. *Suppose $f(z) = \sqrt{z}$ denotes the principal branch of the square-root function. In particular, we defined $f(z) = e^{\frac{1}{2}Log(z)}$ thus for[3] $z \in \mathbb{C}^-$*

$$\frac{d}{dz}\sqrt{z} = \frac{d}{dz}e^{\frac{1}{2}Log(z)} = e^{\frac{1}{2}Log(z)}\frac{d}{dz}\frac{1}{2}Log(z) = \sqrt{z}\cdot\frac{1}{2z} = \frac{1}{2\sqrt{z}}.$$

Let $\mathcal{L}(z)$ be some branch of the logarithm and define $z^c = e^{c\mathcal{L}(z)}$ we calculate:

$$\frac{d}{dz}z^c = \frac{d}{dz}e^{c\mathcal{L}(z)} = e^{c\mathcal{L}(z)}\frac{d}{dz}c\mathcal{L}(z) = e^{c\mathcal{L}(z)}\frac{c}{z} = cz^{c-1}.$$

To verify the last step, we note:

$$\frac{1}{z} = z^{-1} = e^{-\mathcal{L}(z)} \quad \Rightarrow \quad \frac{1}{z}e^{c\mathcal{L}(z)} = e^{-\mathcal{L}(z)+c\mathcal{L}(z)} = e^{(c-1)\mathcal{L}(z)} = z^{c-1}.$$

Here I used the adding angles property of the complex exponential which we know[4] arises from the corresponding laws for the real exponential and the sine and cosine functions.

7.4 Harmonic Functions

If a function F has second partial derivatives is continuously differentiable then the order of partial derivatives in x and y may be exchanged. In particular,

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\big(\ F(x,y)\ \big) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\big(\ F(x,y)\ \big)$$

We will learn as we study the finer points of complex function theory that if a function is complex differentiable at each point in some domain[5] then the complex derivative is **continuous**. In other words, there are no merely complex differentiable functions on a domain, there are only continuously complex differentiable functions on a domain. The word "domain" is crucial to that claim as Example 7.2.2 shows that the complex derivaitve may only exist along stranger sets and yet not exist elsewhere (such a complex derivative function is hardly continuous on $\mathbb{C}$).

In addition to the automatic continuity of the complex derivative on domains[6] we will also learn that the complex derivative function on a domain is itself complex differentiable. In other words,

[3]we defined $\sqrt{z}$ for all $z \in \mathbb{C}^\times$, however, we cannot find a derivative on all of the punctured plane since if we did that would imply the $\sqrt{z}$ function is continuous on the punctured plane (which is false). In short, the calculation breaks down at the discontinuity of the square root function

[4]perhaps we can give a more fundamental reason based on self-contained arithmetic later in this course!

[5]as we have discussed, a domain is an open and connected set

[6]Gamelin assumes this point as he defines analytic to include this result on page 45

on a domain, if $z \mapsto f'(z)$ exists then $z \mapsto f''(z)$ exists. But, then by the same argument $f^{(3)}(z)$ exists etc. We don't have the theory to develop this claim yet, but, I hope you don't mind me sharing it here. It explains why if $f = u + iv$ is holomorphic on a domain then the second partial derivatives of u, v must exist and be continuous. I suppose it might be better pedagogy to just say we know the second partial derivatives of the component functions of an analytic function are continuous. But, the results I discuss here are a bit subtle and its not bad for us to discuss them multiple times as the course unfolds. We now continue to the proper content of this section.

Laplace's equation is one of the fundamental equations of mathematical physics. The study of the solutions to Laplace's equation is known as **harmonic analysis**. For $\mathbb{R}^n$ the Laplacian is defined:

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

which gives **Laplace's equation** the form $\triangle u = 0$. Again, this is studied on curved spaces and in generality far beyond our scope.

Definition 7.4.1. *Let x, y be Cartesian coordinates on $\mathbb{C}$ then $u_{xx} + u_{yy} = 0$ is* **Laplace's Equation**. *The solutions of Laplace's Equation are called* **harmonic functions**.

The theorem below gives a very simple way to create new examples of harmonic functions. It also indicates holomorphic functions have very special the component functions.

Theorem 7.4.2. *If $f = u + iv$ is holomorphic on a domain D then u, v are harmonic on D.*

Proof: as discussed at the beginning of this section, we may assume on the basis of later work that u, v have continuous second partial derivatives. Moreover, as f is holomorphic we know u, v solve the CR-equations $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$. Observe

$$\partial_x u = \partial_y v \quad \Rightarrow \quad \partial_x \partial_x u = \partial_x \partial_y v \quad \Rightarrow \quad \partial_x \partial_x u = \partial_y \partial_x v = \partial_y[-\partial_y u]$$

Therefore, $\partial_x \partial_x u + \partial_y \partial_y u = 0$ which shows u is harmonic. The proof for v is similar. $\square$

A fun way to prove the harmonicity of v is to notice that $f = u + iv$ harmonic implies $-if = v - iu$ is harmonic thus $\textbf{Re}(-if) = v$ and we already showed the real component of f is harmonic thus we may as well apply the result to $-if$.

Example 7.4.3. *Let $f(z) = e^z$ then $e^{x+iy} = e^x \cos y + ie^x \sin y$ hence $u = e^x \cos y$ and $v = e^x \sin y$ are solutions of $\phi_x x + \phi_y y = 0$.*

The functions $u = e^x \cos y$ and $v = e^x \sin y$ have a special relationship. In general:

Definition 7.4.4. *If u is a harmonic function on a domain D and $u + iv$ is holomorphic on D then we say v is a* **harmonic conjugate** *of u on D.*

I chose the word "a" in the definition above rather than the word "the" as the harmonic conjugate is not unique. Observe:

$$\frac{d}{dz}\big(u + i(v + v_o)\big) = \frac{d}{dz}\big(u + iv\big).$$

If v is a harmonic conjugate of u then $v + v_o$ is also a harmonic conjugate of u for any $v_o \in \mathbb{R}$.

A popular introductory exercise is the following:

Given a harmonic function u find a harmonic conjugate v on a given domain.

Gamelin gives a general method to calculate the harmonic conjugate on page 56. This is essentially the same problem we faced in calculus III when we derived potential functions for a given conservative vector field.

Example 7.4.5. *Let $u(x,y) = x^2 - y^2$ then clearly $u_{xx} + u_{yy} = 2 - 2 = 0$. Hence u is harmonic on $\mathbb{C}$. We wish to find v for which $u + iv$ is holomorphic on $\mathbb{C}$. This means we need to solve $u_x = v_y$ and $v_x = -u_y$ which yield $v_y = 2x$ and $v_x = 2y$. Integrating yields:*

$$\frac{\partial v}{\partial y} = 2x \quad \Rightarrow \quad v = 2xy + h_1(x)$$

and

$$\frac{\partial v}{\partial x} = 2y \quad \Rightarrow \quad v = 2xy + h_2(y)$$

from $h_1(x), h_2(y)$ are constant functions and a harmonic conjugate has the form $v(x,y) = 2xy + v_o$. In particular, if we select $v_o = 0$ then

$$u + iv = (x^2 - y^2) + 2ixy = (x + iy)^2$$

The holomorphic function here is just our old friend $f(z) = z^2$.

The shape of the domain was not an issue in the example above, but, in general we need to be careful as certain results have a topological dependence. In Gamelin he proves the theorem below for a rectangle. As he cautions, it is not true for regions with holes like the punctured plane $\mathbb{C}^\times$ or annuli. Perhaps I have assigned problem 7 from page 58 which gives explicit evidence of the failure of the theorem for domains with holes.

Theorem 7.4.6. *Let D be an open disk, or an open rectangle with sides parallel to the axes, and let $u(x,y)$ be a harmonic function on D. Then there is a harmonic function $v(x,y)$ on D such that $u + iv$ is holomorphic on D. The harmonic conjugate v is unique, up to adding a constant.*

7.5 Conformal Mappings

A few nice historical remarks on the importance of the concept discussed in this section is given on page 78 of [R91]. Gauss realized the importance in 1825 and it served as a cornerstone of Riemann's later work. Apparently, Cauchy and Weierstrauss did not make much use of conformality.

Following the proof of the inverse function theorem I argued the 2×2 Jacobian matrix of a holomorphic function was quite special. In particular, we observed it was the product of a reflection, dilation and rotation. That said, at the level of complex notation the same observation is cleanly given in terms of the chain rule and the polar form of complex numbers.

Suppose $f : D \to \mathbb{C}$ is holomorphic on the domain D. Let z_o be a point in D and, for some $\varepsilon > 0$, $\gamma : (-\varepsilon, \varepsilon) \to D$ a path with $\gamma(0) = z_o$. The tangent vector at z_o for γ is simply $\gamma'(0)$. Consider f as the mapping $z \mapsto w = f(z)$; we transport points in the $z = x + iy$-plane to points in the $w = u + iv$-plane. Thus, the curve $f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{C}$ is naturally a path in the w-plane and we are free to study how the tangent vector of the transformed curve relates to the initial curve in the z-plane. In particular, differentiate and make use of the chain rule for complex differentiable functions[7]:

$$\frac{d}{dt}\big(f(\gamma(t))\big) = \frac{df}{dz}(\gamma(t))\frac{d\gamma}{dt}.$$

Let $\frac{df}{dz}(\gamma(0)) = re^{i\theta}$ and $\gamma'(0) = v$ we find the vector $\gamma'(0) = v$ transforms to $(f\circ\gamma)'(0) = re^{i\theta}v$. Therefore, the tangent vector to the transported curve is stretched by a factor of $r = |(f\circ\gamma)'(0)|$ and rotated by angle $\theta = Arg((f\circ\gamma)'(0))$.

Now, suppose we have c such that $\gamma_1(0) = \gamma_2(0) = z_o$ then $f\circ\gamma_1$ and $f\circ\gamma_2$ are curves through $f(z_o) = w_o$ and we can compare the angle between the curves $f\circ\gamma_1$ at z_o and the angle between the image curves $f\circ\gamma_1$ and $f\circ\gamma_2$ at w_o. Recall the angle between to curves is measured by the angle between their tangent vectors at the point of intersection. In particular, if $\gamma_1'(0) = v_1$ and $\gamma_2'(0) = v_2$ then note $\frac{df}{dz}(\gamma_1(0)) = \frac{df}{dz}(\gamma_1(0)) = re^{i\theta}$ hence both v_1 and v_2 are rotated and stretched in the same fashion. Let us denote $w_1 = re^{i\theta}v_1$ and $w_1 = re^{i\theta}v_1$. Recall the dot-product defines the angle betwen nonzero vectors by $\theta = \dfrac{\vec{A}\bullet\vec{B}}{||\vec{A}||||\vec{B}||}$. Furthermore, we saw shortly after Definition 1.1.3 that the Euclidean dot-product is simply captured by the formula $\langle v, w\rangle = \mathbf{Re}(z\overline{w})$. Hence, consider:

$$\begin{aligned}
\langle w_1, w_2\rangle &= \langle re^{i\theta}v_1,\ re^{i\theta}v_2\rangle \\
&= \mathbf{Re}\left(re^{i\theta}v_1\overline{re^{i\theta}v_2}\right) \\
&= r^2\mathbf{Re}\left(e^{i\theta}v_1\overline{v_2}e^{-i\theta}\right) \\
&= r^2\mathbf{Re}\left(v_1\overline{v_2}\right) \\
&= r^2\langle v_1, v_2\rangle.
\end{aligned}$$

Note we have already shown $|w_1| = r|v_1|$ and $|w_2| = r|v_2|$ hence:

$$\frac{\langle v_1, v_2\rangle}{|v_1||v_2|} = \frac{r^2\langle v_1, v_2\rangle}{r|v_1|r|v_2|} = \frac{\langle w_1, w_2\rangle}{|w_1||w_2|}.$$

Therefore, the angle between curves is preserved under holomorphic maps.

Definition 7.5.1. *A smooth complex-valued function $g(z)$ is* **conformal at $\mathbf{z_o}$** *if whenever γ_o, γ_1 are curves terminating at z_o with nonzero tangents, then the curves $g\circ\gamma_o$ and $g\circ\gamma_1$ have nonzero tangents at $g(z_o)$ and the angle between $g\circ\gamma_o$ and $g\circ\gamma_1$ at $g(z_o)$ is the same as the angle between γ_o and γ_1 at z_o.*

Therefore, we have the following result from the calculation of the previous page:

Theorem 7.5.2. *If $f(z)$ is holomorphic at z_o and $f'(z_o) \neq 0$ then $f(z)$ is conformal at z_o.*

This theorem gives beautiful geometric significance to holomorphic functions. The converse of the theorem requires we impose an additional condition. The function $f(z) = \bar{z} = x - iy$ has $\mathrm{J}_f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\det(\mathrm{J}_f) = -1 < 0$. This means that the function does not maintain the **orientation** of vectors. On page 74 of [R91] the equivalence of real differentiable, angle-preserving, orientation-preserving maps and nonzero f' holomorphic maps is asserted. The proof is already

[7] I will write a homework (Problem 27) where you derive this

contained in the calculations we have considered.

We all should recognize $x = x_o$ and $y = y_o$ as the equations of vertical and horizontal lines respective. At the point (x_o, y_o) these lines intersect at right angles. It follows that the image of the coordinate grid in the $z = x + iy$ plane gives a family of orthogonal curves in the w-plane. In particular, the lines which intersect at (x_o, y_o) give orthogonal curves which intersect at $f(x_o + iy_o)$. In particular $x \mapsto w = f(x + iy_o)$ and $y \mapsto w = f(x_o + iy)$ are paths in the w-plane which intersect orthogonally at $w_o = f(x_o + iy_o)$.

Example 7.5.3. *Consider $f(z) = z^2$. We have $f(x + iy) = (x + iy)(x + iy) = x^2 - y^2 + 2ixy$. Thus,*

$$t \mapsto t^2 - y_o^2 + 2iy_o t \qquad \& \qquad t \mapsto x_o^2 - t^2 + 2ix_o t$$

Let u, v be coordinates on the w-plane. The image of $y = y_o$ has

$$u = t^2 - y_o^2 \qquad \& \qquad v = 2y_o t$$

If $y_o \neq 0$ then $t = v/2y_o$ which gives $u = \frac{1}{4y_o^2}v^2 - y_o^2$. This is a parabola which opens horizontally to the right in the w-plane. The image of $x = x_o$ has

$$u = x_o^2 - t^2 \qquad \& \qquad v = 2x_o t$$

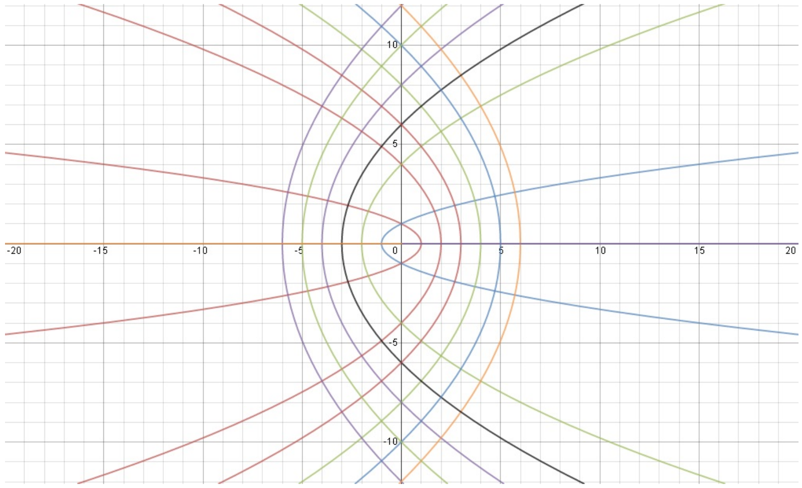
If $x_o \neq 0$ then $t = v/2x_o$ which gives $u = x_o^2 - \frac{1}{4x_o^2}v^2$. This is a parabola which opens horizontally to the left in the w-plane. As level-curves in the w-plane the right-opening parabola is $F(u, v) = u - \frac{1}{4y_o^2}v^2 + y_o^2 = 0$ whereas the left-opening parabola is given by $G(u, v) = u - x_o^2 + \frac{1}{4x_o^2}v^2$. We know the gradients of F and G are normals to the curves. Calculate,

$$\nabla F = \langle 1, -\frac{v}{2y_o^2}\rangle \qquad \& \qquad \nabla G = \langle 1, \frac{v}{2x_o^2}\rangle \qquad \Rightarrow \qquad \nabla F \bullet \nabla G = 1 - \frac{v^2}{4x_o^2 y_o^2}$$

At a point of intersection we have $x_o^2 - \frac{1}{4x_o^2}v^2 = \frac{1}{4y_o^2}v^2 - y_o^2$ from which we find $x_o^2 + y_o^2 = v^2(\frac{1}{4x_o^2} + \frac{1}{4y_o^2})$. Multiply by $x_o^2 y_o^2$ to obtain $x_o^2 y_o^2 (x_o^2 + y_o^2) = \frac{v^2}{4}(y_o^2 + x_o^2)$. But, this gives $1 = \frac{v^2}{4x_o^2 y_o^2}$. Therefore, at the point of intersection we find $\nabla F \bullet \nabla G = 0$. It follows the sideways parabolas intersect orthogonally.

If $x_o = 0$ then $t \mapsto -t^2$ is a parametrization of the image of the y-axis which is the negative real axis in the w-plane. If $y_o = 0$ then $t \mapsto t^2$ is a parametrization of the image of the x-axis which is the positive real axis in the w-plane. The point at which these exceptional curves intersect is $w = 0$ which is the image of $z = 0$. That point, is the only point at which $f'(0) \neq 0$.

I plot several of the curves in the w-plane. You can see how the intersections make right angles at each point except the origin.



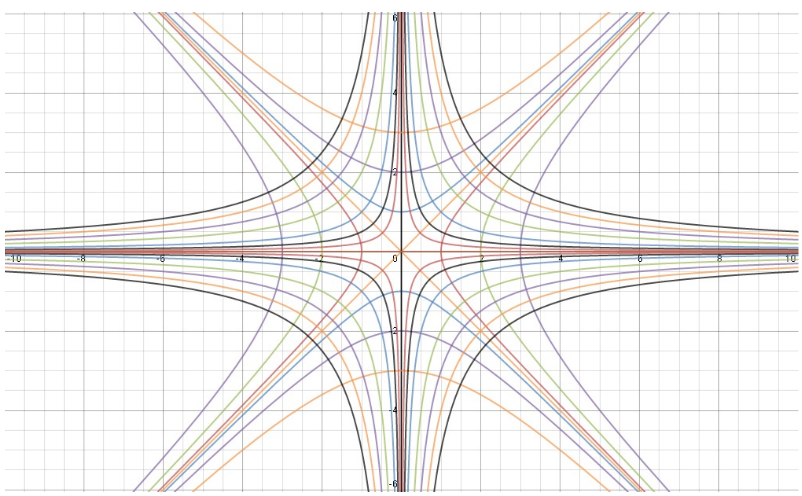
The plot above was produced using www.desmos.com which I whole-heartedly endorse for simple graphing tasks.

We can also study the inverse image of the cartesian coordinate lines $u = u_o$ and $v = v_o$ in the z-plane. In particular,

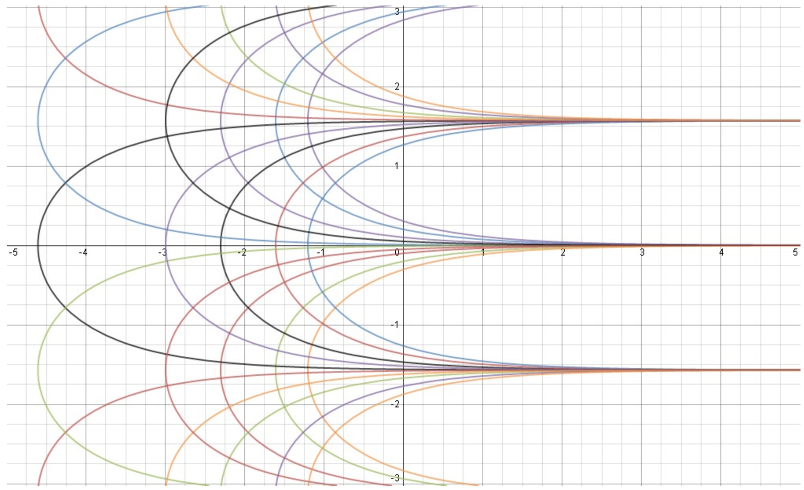
$$u(x, y) = u_o \qquad \& \qquad v(x, y) = v_o$$

give curves in $z = x + iy$-plane which intersect at z_o orthogonally provided $f'(z_o) \neq 0$.

Example 7.5.4. *We return to Example 7.5.3 and as the reverse question: what is the inverse image of $u = u_o$ or $v = v_o$ for $f(z) = z^2$ where $z = x + iy$ and $u = x^2 - y^2$ and $v = 2xy$. The curve $x^2 - y^2 = u_o$ is a hyperbola with asymptotes $y = \pm x$ whereas $2xy = v_o$ is also a hyperbola, but, it's asymptotes are the x, y axes. Note that $u_o = 0$ gives $y = \pm x$ whereas $v_o = 0$ gives the x, y-axes. These meet at the origin which is the one point where $f'(z) \neq 0$.*



Example 7.5.5. *Consider $f(z) = e^z$ then $f(x + iy) = e^x \cos y + ie^x \sin y$. We observe $u = e^x \cos y$ and $v = e^x \sin y$. The curves $u_o = e^x \cos y$ and $v_o = e^x \sin y$ map to the vertical and horizontal lines in the w-plane. I doubt these are familar curves in the xy-plane. Here is a plot of the z-plane with the inverse images of a few select u, v-coordinate lines:*



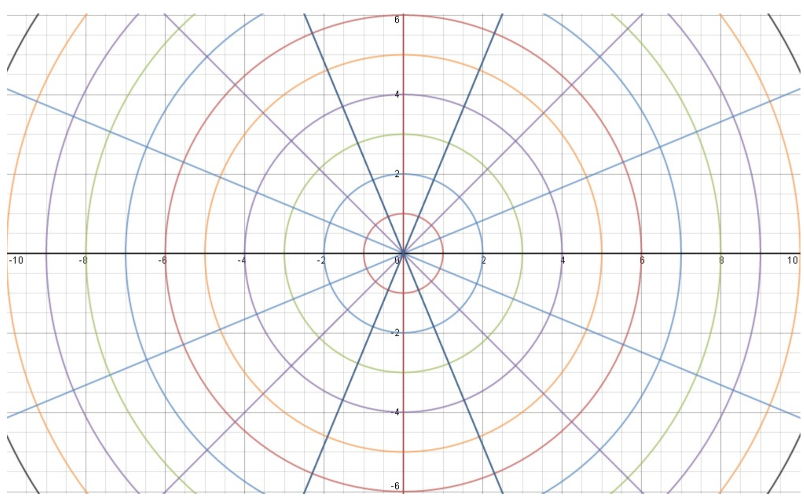
On the other hand, we can study how $z \mapsto w = e^z$ distorts the x, y-coordinate grid. The horizontal line through $x_o + iy_o$ is parametrized by $x = x_o + t$ and $y = y_o$ has image

$$t \mapsto f(x_o + t + iy_o) = e^{x_o+t}e^{iy_o}$$

as t varies we trace out the ray from the origin to ∞ in the w-plane at angle y_o. The vertical line through $x_o + iy_o$ is parametrized by $x = x_o$ and $y = y_o + t$ has image

$$t \mapsto f(x_o + t + iy_o) = e^{x_o}e^{i(y_o+t)}$$

as t varies we trace out a circle of radius e^{x_o} centered at the origin of the w-plane. Therefore, the image of the x, y-coordinate lines in the w-plane is a family of circles and rays eminating from the origin. Notice, the origin itself is not covered as $e^z \neq 0$.



There is another simple calculation to see the orthogonality of constant u or v curves. Calculate $\nabla u = \langle u_x, u_y \rangle$ and $\nabla v = \langle v_x, v_y \rangle$. But, if $f = u + iv$ is holomorphic then $u_x = v_y$ and $v_x = -u_y$. By CR-equations,

$$\nabla u = \langle u_x, u_y \rangle = \langle v_y, -v_x \rangle$$

but, $\nabla v = \langle v_x, v_y \rangle$ hence $\nabla u \bullet \nabla v = 0$. Of course, this is just a special case of our general result on conformality of holomorphic maps.

7.6 Fractional Linear Transformations

Definition 7.6.1. *Let $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. A* **fractional linear transformation** *or* **Mobius transformation** *is a function of the form $f(z) = \dfrac{az+b}{cz+d}$. If $f(z) = az$ then f is a* **dilation**. *If $f(z) = z + b$ then f is a* **translation**. *If $f(z) = az + b$ then f is an* **affine transformation**. *If $f(z) = 1/z$ then f is an* **inversion**.

The quotient rule yields $f'(z) = \dfrac{a(cz+d) - (az+b)c}{(cz+d)^2} = \dfrac{ad-bc}{(cz+d)^2}$ thus the condition $ad - bc \neq 0$ is the requirement that $f(z)$ not be constant. Gamelin shows through direct calculation that if f and g are Mobius transformations then the composite $f \circ g$ is also a mobius transformation. It turns out the set of all Mobius transformations forms a **group** under composition. This group of fractional linear transformations is built from affine transformations and inversions. In particular, consider $f(z) = \dfrac{az+b}{cz+d}$. If $c = 0$ then $f(z) = \frac{a}{d}z + \frac{b}{d}$ which is just an affine transformation. On the other hand, if $c \neq 0$ then

$$f(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}$$

This expression can be seen as the composition of the maps below:

$$f_1(z) = \frac{a}{c} + \frac{bc-ad}{c}z \qquad \& \qquad f_2(z) = \frac{1}{z} \qquad \& \qquad f_3(z) = cz + d$$

In particular, $f = f_1 \circ f_2 \circ f_3$. This provides proof similar to that given in Gamelin page 65:

Theorem 7.6.2. *Every fractional linear transformation is the composition of dilations, translations and inversions.*

Furthermore, we learn that any three points and values in the extended complex plane $\mathbb{C} \cup \{\infty\} = \mathbb{C}^*$ fix a unique Mobius transformation.

Theorem 7.6.3. *Given any distinct triple of points $z_o, z_1, z_2 \in \mathbb{C}^*$ and a distinct triple of values $w_o, w_1, w_2 \in \mathbb{C}^*$ there is a unique fractional linear transformation $f(z)$ for which $f(z_o) = w_o$, $f(z_1) = w_1$ and $f(z_2) = w_2$.*

The arithmetic for the extended complex plane is simply:

$$1/\infty = 0, \qquad \& \qquad c \cdot \infty = \infty$$

expressions of the form ∞/∞ must be carefullly analyzed by a limiting procedure just as we introduced in calculus I. I will forego a careful proof of these claims, but, it is possible.

Example 7.6.4. *Find a mobius transformation which takes $1, 2, 3$ to $0, i, \infty$ respective. Observe $\frac{1}{z-3}$ has $3 \mapsto \infty$. Also, $z - 1$ maps 1 to 0. Hence, $f(z) = A\frac{z-1}{z-3}$ maps $1, 3$ to $0, \infty$. We need only set $f(2) = i$ but this just requires we choose A wisely. Consider:*

$$f(2) = A\frac{2-1}{2-3} = i \quad \Rightarrow \quad A = -i \quad \Rightarrow \quad f(z) = -i\frac{z-1}{z-3}.$$

Example 7.6.5. *Find a mobius transformation which takes $z_o = \infty$, $z_1 = 0$ and $z_2 = 3i$ to $w_o = 1$ and $w_1 = i$ and $w_2 = \infty$ respective. Let us follow idea of page 65 in Gamelin. We place $z - 3i$ in the denominator to map $3i$ to ∞. Hence $f(z) = \dfrac{az+b}{z-3i}$. Now, algebra finishes the job:*

$$f(0) = i \quad \Rightarrow \quad \frac{b}{-3i} = i \quad \Rightarrow \quad b = 3.$$

and

$$f(\infty) = 1 \quad \Rightarrow \quad \frac{a\infty + 3}{\infty - 3i} = 1 \quad \Rightarrow \quad \frac{a + 3/\infty}{1 - 3i/\infty} = 1 \quad \Rightarrow \quad a = 1.$$

Hence $f(z) = \dfrac{z+3}{z-3i}$. Now, perhaps the glib arithmetic I just used with ∞ has sown disquiet in your mathematical soul. Let us double check given the sketchy nature of my unproven assertions: to be careful, what we mean by $f(\infty)$ in the context of $\mathbb{C}$ is just:

$$f(\infty) = \lim_{z\to\infty} \frac{z+3}{z-3i} = \lim_{z\to\infty} \frac{z+3}{z-3i} = \lim_{z\to\infty} \frac{1+3/z}{1-3i/z} = 1.$$

In the calculation above I have used the claim $1/z \to 0$ as $z \to \infty$. This can be rigorously shown once we give a careful definition of $z \to \infty$. Next, consider

$$f(0) = \frac{0+3}{0-3i} = \frac{1}{-i} = i \qquad \& \qquad f(3i) = \frac{6i}{0} = \infty \in \mathbb{C}^*$$

thus $f(z)$ is indeed the Mobius transformation we sought. Bottom line, the arithmetic I used with ∞ is justified by the corresponding arithmetic for limits of the form $z \to \infty$.

There is a nice trick to find the formula which takes $\{z_1,\ z_2,\ z_3\} \subset \mathbb{C}^*$ to $\{w_1,\ w_2,\ w_3\} \subset \mathbb{C}^*$ respectively. We simply write the **cross-ratio** below and solve for w:

$$\frac{(w_1 - w)(w_3 - w_2)}{(w_1 - w_2)(w_3 - w)} = \frac{(z_1 - z)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z)}.$$

This is found in many complex variables texts. I found it in *Complex Variables: Introduction and Applications* second ed. by Mark J. Ablowitz and Athanassios S. Fokas; see their §5.7 on **bilinear transformations**[8]. I tend to consult Ablowitz and Fokas for additional computational ideas. It's a bit beyond what I intend for this course computationally.

Example 7.6.6. *Let us try out this mysterious cross-ratio. We seek the map of $\{i, \infty, 3\}$ to $\{\infty, 0, 1\}$. Consider,*

$$\frac{(\infty - w)(1-0)}{(\infty - 0)(1-w)} = \frac{(i-z)(3-\infty)}{(i-\infty)(3-z)}.$$

This simplifies to:

$$\frac{1}{1-w} = \frac{i-z}{3-z} \quad \Rightarrow \quad 1 - w = \frac{3-z}{i-z} \quad \Rightarrow \quad w = 1 - \frac{3-z}{i-z} = \frac{z-3+i-z}{i-z} = \frac{i-3}{i-z}.$$

Define $f(z) = \frac{i-3}{i-z}$ and observe $f(i) = \infty$, $f(\infty) = 0$ and $f(3) = \frac{i-3}{i-3} = 1$.

[8] I would not use this term, but, some folks use this as yet another label for Mobius transformation or fractional linear transformation. You might wonder why this cross-ratio technique provides the desired fractional linear transformation. I welcome you to explain it to me in office hours.

There are many variations on the examples given above, but, I hope that is enough to get you started. Beyond finding a particular Mobius transformation, it is also interesting to study what happens to various curves for a given Mobius transformation. In particular, the theorem below is beautifully simple and reason enough for us to discuss $\mathbb{C}^*$ in this course:

Theorem 7.6.7. *A fractional linear transformation maps circles to circles in the extended complex plane to circles in the extended complex plane*

Recall that a line is a circle through ∞ in the context of $\mathbb{C}^*$. You might think I'm just doing math for math's sake here[9], but, there is actually application of the observations of this section to the problem of conformal mapping. We later learn that conformal mapping allows us to solve Laplace's equation by transferrring solutions through conformal maps. Therefore, the problem we solve here is one step towards find the voltage function for somewhat complicated boundary conditions in the plane. Or, solving certain problems with fluid flow. Gamelin soon returns to these applications in future chapters, I merely make this comment here to give hope to those who miss applications.

Recall, if D is a domain then $\mathcal{O}(D)$ the set of holomorphic functions from D to $\mathbb{C}$.

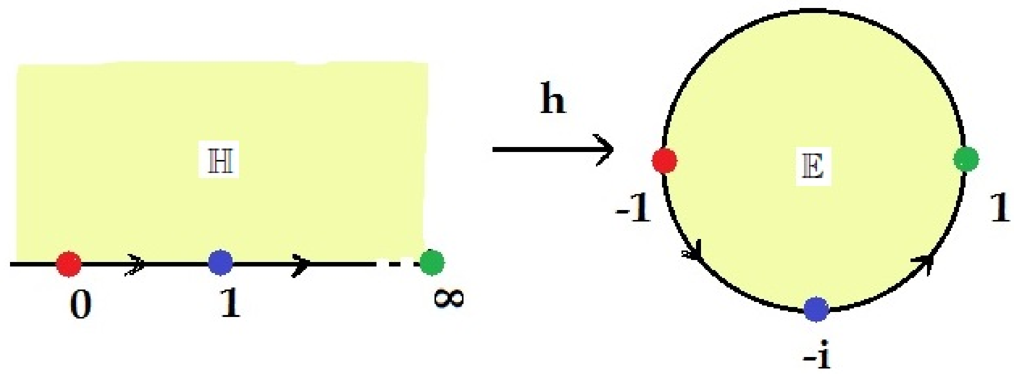
Definition 7.6.8. *Let D be a domain, we say $f \in \mathcal{O}(D)$ is* **biholomorphic** *mapping of D onto D' if $f(D) = D'$ and $f^{-1} : D' \to D$ is holomorphic.*

In other words, a biholomorphic mapping is a bijection which is a holomorphic map with holomorphic inverse map. These sort of maps are important because they essentially describe coordinate change maps for $\mathbb{C}$, or from another perspective, they give us a way to produce new domains from old. The fractional linear transformations are important examples of biholomorphic maps on $\mathbb{C}$. However, the restriction of a fractional linear transformation is also worthy of study. In particular, we find below the restriction of a mobius transformation to a half-plane may give us an image which is a disk.

As is often the case, this construction is due to Cayley. The **Cayley Map** is simply a particular example of a linear fractional transformation. In what follows here I share some insights I found on pages 80-84 of [R91].

Example 7.6.9. *Let $h(z) = \frac{z-i}{z+i}$ be defined for $z \in \mathbb{H}$ where $\mathbb{H} = \{z \in \mathbb{C} \mid \mathbf{Im}(z) > 0\}$ is the open upper half-plane. It can be shown that $h(z) \in \mathbb{E}$ where $\mathbb{E} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the closed unit disk. To see how you might derive this function simply imagine mapping $\{0, 1, \infty\}$ of the boundary of $\mathbb{H}$ to the points $\{-1, -i, 1\}$. Fun fact, if you walk along the boundary of a subset of the plane then the interior of the set is on your left if you are wolking in the positively oriented sense. It is also known that a holomorphic map preserves orientations which implies that boundaries map to boundaries* **and** *points which were locally left of the boundary get mapped to points which are locally left of the image curve. In particular, note $\{0, 1, \infty\}$ on the boundary of $\mathbb{H}$ in the domain map to the points $\{-1, -i, 1\}$ on the unit-circle under the mobius transformations. Furthermore, we see the unit-circle given a CCW orientation. The direction of the curve is implicit within the fact that the triple $\{0, 1, \infty\}$ is in the order in which they are found on $\partial\mathbb{H}$ so likewise $\{-1, -i, 1\}$ are in order (this forces CCW orientiation of the image circle). Perhaps a picture is helpful:*

[9] which is, incidentally, totally fine



I skipped the derivation of $h(z)$ in the example above. We could use the cross-ratio or the techniques discussed earlier. The larger point made by Remmert here is that this transformation is natural to seek out and in some sense explains why we've been studying fractional linear transformations for 200^+ years. Actually, I'm not sure the precise history of these. I think it is fair to conjecture Mobius, Cauchy and Gauss were involved. But, as with any historical conjecture of this period, it seems wise to offer Euler as a possibility, surely he at least looked at these from a real variable viewpoint.

The next thing you might try is to square h^{-1} of the mapping above. If we feed $z \mapsto z^2$ the open half-plane then the image will be a slit-complex plane. In total $z \mapsto \left(\frac{z+1}{z-1}\right)^2 : \mathbb{E} \to \mathbb{C}^-$ is a **surjection** indeed we can even verify this is a biholomorphic mapping. It turns out the slit is a necessary feature, no amount of tinkering can remove it and obtain all of $\mathbb{C}$ while maintaining the biholomorphicity of the map. In fact, Liouville's Theorem forbids a biholomorphic mapping of the disk onto $\mathbb{C}$. But, Riemann in 1851[10] showed that every simply connected proper subset of $\mathbb{C}$ can be biholomorphically mapped onto the unit-disk. In this section, we have simply exposed the machinery to make that happen for simple sets like half-planes. A vast literature exists for more complicated domains.

[10]Riemann's study of complex analysis was centered around the study of conformal mappings, this result is known as "Riemann Mapping Theorem" see apge 295 of Gamelin for further discussion.

Chapter 8

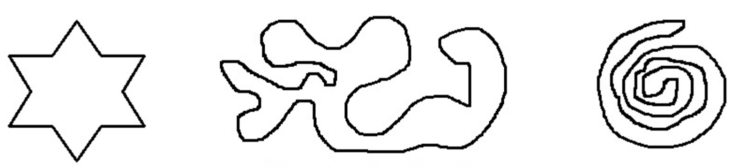
Line Integrals and Harmonic Functions

In this chapter we review and generalize some basic constructions in multivariate calculus. Generalize in the sense that we analyze complex-valued vector fields over $\mathbb{C}$. We remind the reader how Green's Theorem connects to both Stokes' and Gauss' Theorems as the line integral allows us to calculate both circulation and flux in two-dimensions. We analyze the interplay between path-independence, vanishing loop integrals, exact and closed differential forms. Complex analysis enters mainly in our discussion of harmonic conjugates. When $u + iv$ is holomorphic this indicates u, v are both solutions of Laplace's equation **and** they have orthogonal level curves. This simple observation motivates us to use complex analysis to solve Laplace's equation in the plane. In particular, we examine how fluid flow problems may be solved by selecting an appropriate holomorphic function on a given domain. Heat and electrostatics are also briefly discussed.

8.1 Line Integrals and Green's Theorem

The terminology which follows here is not universally used. As you read different books the terms curve and path are sometimes overloaded with particular technical meanings. In my view, this is the case with the term "curve" as defined below.

Definition 8.1.1. *A path $\gamma : I \subseteq \mathbb{R} \to \mathbb{C}$ is a continuous function. We usually either have $I = [a, b]$ or $I = \mathbb{R}$. If γ is a path such that $\gamma(s) \neq \gamma(t)$ for $s \neq t$ then γ is* **simple**. *If the path begins and ends at the same point then γ is said to be* **closed**. *A simple closed path is of the form $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(s) \neq \gamma(t)$ for all $s \neq t$ with $a \leq s, t < b$ and $\gamma(a) = \gamma(b)$. The component functions of $\gamma = x + iy$ are x and y respective. We say γ is smooth if it has smooth component functions. A continuous path which is built by casewise-joining of finitely many smooth paths is a* **piecewise smooth** *path or simply a* **curve**.



some silly simple closed curves

In other texts, the term *curve* is replaced with *arc* or *contour*. In this course I follow Gamelin's terminology[1].

Definition 8.1.2. *A path $\gamma : I \subseteq \mathbb{R} \to \mathbb{C}$ has* **trace** *$\gamma(I)$.*

The trace of a path is the pointset in $\mathbb{C}$ which the path covers.

Definition 8.1.3. *A path $\gamma : [a,b] \to \mathbb{C}$ has reparametrization $\tilde{\gamma}$ if there is a smooth injective function $h : [\tilde{a}, \tilde{b}] \to [a,b]$ such that $\tilde{\gamma}(t) = \gamma(h(t))$. If h is strictly increasing then $\tilde{\gamma}$ shares the same direction as γ. If h is strictly decreasing then $\tilde{\gamma}$ has direction opposite to that of γ.*

The direction of a curve is important since the line-integral is sensitive to the orientation of curves. Furthermore, to be careful, whenever a geometric definition is given we ought to show the definition is independent of the choice of parametrization for the curve involved. The technical details of such soul-searching amounts to taking an arbtrary reparametrization (or perhaps all orientation preserving reparametrizations) and demonstrating the definition naturally transforms.

Definition 8.1.4. *Let P be complex values and continuous near the trace of $\gamma : [a,b] \to \mathbb{C}$. Define:*

$$\int_\gamma P\,dx = \int_a^b P(\gamma(t))\frac{d\mathbf{Re}(\gamma)}{dt}dt \qquad \& \qquad \int_\gamma P\,dy = \int_a^b P(\gamma(t))\frac{d\mathbf{Im}(\gamma)}{dt}dt.$$

If we use the usual notation $\gamma = x + iy$ then the definitions above look like a u-substitution:

$$\int_\gamma P\,dx = \int_a^b P(x(t),y(t))\frac{dx}{dt}dt \qquad \& \qquad \int_\gamma P\,dy = \int_a^b P(x(t),y(t))\frac{dy}{dt}dt$$

These integrals have nice linearity properties.

Theorem 8.1.5. *For f, g continuous near the trace of γ and $c \in \mathbb{C}$:*

$$\int_\gamma (f+cg)dx = \int_\gamma f\,dx + c\int_\gamma g\,dx \qquad \& \qquad \int_\gamma (f+cg)dy = \int_\gamma f\,dy + c\int_\gamma g\,dy.$$

We define sums of the integrals over dx and dy in the natural manner:

$$\int_\gamma P\,dx + Q\,dy = \int_\gamma P\,dx + \int_\gamma Q\,dy.$$

At first glance this seems like it is merely calculus III restated. However, you should notice that P and Q are complex-valued functions. That said, if both P, Q are **real** then $\vec{F} = \langle P, Q\rangle$ is a real-vector field in the plane and the standard line-integral from multivariate calculus is precisely:

$$\int_\gamma \vec{F} \bullet d\vec{r} = \int_\gamma P\,dx + Q\,dy.$$

[1] in other courses, my default is to call the parametrization of a curve a path. For me, a curve is the point-set whereas a path is a mapping from $\mathbb{R}$ into whatever space is considered. Gamelin uses the term "trace" in the place of my usual term "curve"

You should recall the integral above calculates the work done by $\vec{F}$ along the path γ. Continuing, suppose we drop the condition that P, Q be real. Instead, consider $P = P_1 + iP_2$ and $Q = Q_1 + iQ_2$. Consider:

$$\begin{aligned}
\int_\gamma P\,dx + Q\,dy &= \int_\gamma (P_1 + iP_2)\,dx + (Q_1 + iQ_2)\,dy \\
&= \int_\gamma P_1\,dx + i\int_\gamma P_2\,dx + \int_\gamma Q_1\,dy + i\int_\gamma Q_2\,dy \\
&= \left(\int_\gamma P_1\,dx + \int_\gamma Q_1\,dy\right) + i\left(\int_\gamma P_2\,dx + \int_\gamma Q_2\,dy\right) \\
&= \int_\gamma \langle P_1, Q_1\rangle \bullet d\vec{r} + i\int_\gamma \langle P_2, Q_2\rangle \bullet d\vec{r}.
\end{aligned}$$

Therefore, we can interpret the $\int_\gamma P\,dx + Q\,dy$ as the complex sum of the work done by the force $\langle \mathbf{Re}(P), \mathbf{Re}(Q)\rangle$ and i times the work done by $\langle \mathbf{Im}(P), \mathbf{Im}(Q)\rangle$ along γ. Furthermore, as the underlying real integrals are invariant under an orientation preserving reparametrization $\tilde{\gamma}$ of γ it follows $\int_\gamma P dx + Q dy = \int_{\tilde{\gamma}} P dx + Q dy$. In truth, these integrals are not the objects of primary interest in complex analysis. We merely discuss them here to gain the computational basis for the **complex integral** which is defined by $\int_\gamma f dz = \int_\gamma f dx + i\int_\gamma f dy$. We study the complex integral in Chapter 4.

Example 8.1.6. *Let $\gamma_1(t) = \cos t + i\sin t$ for $0 \leq t \leq \pi$. Then $x = \cos t$ and $dx = -\sin t dt$ whereas $y = \sin t$ and $dy = \cos t dt$. Let $P(x+iy) = y + ix^2$ and calculate $\int_{\gamma_1} P\,dx$:*

$$\begin{aligned}
\int_{\gamma_1} (y + ix^2)\,dx &= \int_0^\pi \big(\sin t + i\cos^2 t\big)(-\sin t dt) \\
&= -\int_0^\pi \sin^2 t - i\int_0^\pi \cos^2 t \sin t dt \\
&= -\frac{\pi}{2} + i\frac{u^3}{3}\bigg|_1^{-1} \\
&= -\frac{\pi}{2} - i\frac{2}{3}.
\end{aligned}$$

To integrate along a curve we simply sum the integrals along the smooth paths which join to form the curve. In particular:

Definition 8.1.7. *Let γ be a curve formed by joining the smooth paths $\gamma_1, \gamma_2, \dots, \gamma_n$. In terms of the trace denoted trace$(\gamma) = [\gamma]$ we have $[\gamma] = [\gamma_1] \cup [\gamma_2] \cup \cdots \cup [\gamma_n]$ Let P, Q be complex valued and continuous near the trace of γ. Define:*

$$\int_\gamma P\,dx + Q\,dy = \sum_{j=1}^{n} \int_{\gamma_j} P\,dx + Q\,dy.$$

I assume the reader can define $\int_\gamma P\,dx$ and $\int_\gamma Q\,dy$ for a curve in the same fashion. Let us continue Example 8.1.6.

Example 8.1.8. *Let $\gamma_2(t) = -1 + 2t$ for $0 \leq t \leq 1$. This is the natural parametrization of the line-segment $[-1, 1]$. Let γ_1 be as in Example 8.1.6 and define $\gamma = \gamma_1 \cup \gamma_2$. We seek to calculate*

$\int_\gamma P\,dx$ where $P(x+iy) = y + ix^2$. Let us consider γ_2, in this path we have $x = -1+2t$ hence $dx = 2dt$ whereas $y = 0$ so $dy = (0)dt$. Thus,

$$\int_{\gamma_2} (y+ix^2)dx = \int_0^1 \big(0 + i(-1+2t)^2\big)(2dt) = 2i\int_0^1 \big(1-4t+4t^2\big) = 2i\left(1 - \frac{4}{2} + \frac{4}{3}\right) = \frac{2i}{3}.$$

Thus,

$$\int_\gamma P\,dx = \int_{\gamma_1} P\,dx + \int_{\gamma_2} P\,dx = -\frac{\pi}{2} - i\frac{2}{3} + \frac{2i}{3} = -\frac{\pi}{2}.$$

In order to state the complex-valued version of Green's Theorem we define complex-valued area integrals and partial derivatives of complex-valued functions[2]:

Definition 8.1.9. *Let F and G be real-valued functions on $\mathbb{C} = \mathbb{R}^2$ then we define:*

$$\iint_S (F+iG)dA = \iint_S F\,dA + i\iint_S G\,dA.$$

$$\frac{\partial}{\partial x}\big(F+iG\big) = \frac{\partial F}{\partial x} + i\frac{\partial G}{\partial x}$$

The double integral $\iint_S f\,dA = \iint_S f(x,y)dx\,dy$ and partial derivatives above are discussed in detail in multivariable calculus. We calculate these integrals by iterated integrals over type I or II regions or polar coordinate substitution.

Theorem 8.1.10. *Complex-valued Green's Theorem: Let γ be a simple closed curve which forms the boundary of S in the positively oriented sense; that is, $S \subseteq \mathbb{C}$ and $\partial S = \gamma$:*

$$\int_\gamma P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Proof: let $P = P_1 + iP_2$ and $Q = Q_1 + iQ_2$ where P_1, P_2, Q_1, Q_2 are all real-valued functions. Observe, from our discussion ealier in this section,

$$\int_\gamma P\,dx + Q\,dy = \left(\int_\gamma P_1\,dx + \int_\gamma Q_1\,dy\right) + i\left(\int_\gamma P_2\,dx + \int_\gamma Q_2\,dy\right)$$

however, we also have:

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_S \left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right) dA + i\iint_S \left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right) dA.$$

Finally, by Green's Theorem for real-valued double integrals we obtain:

$$\int_\gamma P_j\,dx + \int_\gamma Q_j\,dy = \iint_S \left(\frac{\partial Q_j}{\partial x} - \frac{\partial P_j}{\partial y}\right) dA.$$

for $j = 1$ and $j = 2$. Therefore, $\displaystyle\int_\gamma P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$ $\square$

Of course, I have taken a rather different path from that in Gamelin. He gives a proof of Green's Theorem based on deriving Green's Theorem for a triangle, stretching the theorem to a curved triangle then summing over a triangulazation of the space. It is a nice, standard, argument. I might go over it in lecture. See pages 357-367 of my 2014 Multivariable Calculus notes for a proof of Green's Theorem based on a rectangularization of a space. I will not replicate it here.

[2]we have used this idea before. For example, when I wrote $\frac{df}{dz} = \frac{\partial f}{\partial x} = u_x + iv_x$.

Example 8.1.11. *We now attack Example 8.1.8 with the power of Green's Theorem. Consider $P = y + ix^2$ and $Q = 0$. Apply Green's Theorem with $Q = 0$ we have (remember S is the upper-half of the unit-disk)*

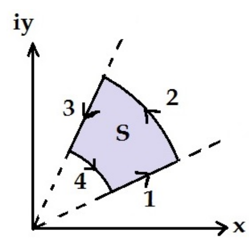
$$\int_\gamma P\,dx = -\iint_S \frac{\partial P}{\partial y}\,dA = -\iint_S dA = -\frac{\pi}{2}.$$

Very well, we have almost good agreement between Example 8.1.11 and Example 8.1.8. I now am sure there is an error in what is currently written. First person to find the error and email me the correction in detail earns 5pts bonus.

A comment from the future: In view of the results of the next section there is a natural reason why $P = y + ix^2$ has different behaviour for $P_1 = y$ and $P_2 = x^2$. In particular, the differential ydx is not an exact form on the half-disk whereas $x^2dx = d(x^3/3)$ hence x^2dx is exact.

In order to calculate integrals in the complex plane we need to be able to parametrize the paths of interest. In particular, you should be ready, willing, and able to parametrize lines, circles and all manner of combinations thereof. The next example illustrates how two dimensional vector problems are nicely simplified by the use of complex notation.

Example 8.1.12. *Let γ be the curve formed by the rays γ_1, γ_3 at $\theta = \pi/6$ and $\theta = \pi/3$ and the arcs γ_4, γ_1 connecting the rays along $|z| = 1$ and $|z| = 2$. Assume γ is positively oriented and the picture below helps explain my choice of numbering:*



In detail: the rays are parametrized by $\gamma_1(t) = te^{i\pi/6} = t(\sqrt{3}+i)/2$ for $1 \leq t \leq 2$ and $\gamma_3(t) = -te^{i\pi/3} = -t(1+i\sqrt{3})/2$ for $-2 \leq t \leq -1$. The arcs are given by $\gamma_2(t) = 2e^{it}$ for $\pi/6 \leq t \leq \pi/3$ and $\gamma_4(t) = e^{-it}$ for $-\pi/3 \leq t \leq -\pi/6$. If we let γ_{-4} denote the reverse of γ_4 then we have the natural parametrization $\gamma_{-4}(t) = e^{it}$ for $\pi/6 \leq t \leq \pi/3$. In practice, it's probably better to use the reversed curve and simply place a minus in to account for the reversed path. I use this idea in what follows. Consider then, for γ_1 we have $x = t\sqrt{3}/2$ and $y = t/2$ hence $dy = dt/2$ and:

$$\int_{\gamma_1} x\,dy = \int_1^2 (t\sqrt{3}/2)(dt/2) = \frac{\sqrt{3}}{4}\frac{t^2}{2}\bigg|_1^2 = \frac{\sqrt{3}}{4}\left[\frac{4}{2} - \frac{1}{2}\right] = \frac{3\sqrt{3}}{8}.$$

For γ_2 observe $x = 2\cos t$ whereas $y = 2\sin t$ hence $dy = 2\cos t\,dt$ and:

$$\begin{aligned}
\int_{\gamma_2} x\,dy &= \int_{\pi/6}^{\pi/3} (2\cos t)(2\cos t dt) \\
&= \int_{\pi/6}^{\pi/3} 4\cos^2 t dt \\
&= 2\int_{\pi/6}^{\pi/3} [1 + \cos(2t)]dt = 2\left(\frac{\pi}{3} - \frac{\pi}{6} + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) - \frac{1}{2}\sin\left(\frac{2\pi}{6}\right)\right) = \frac{\pi}{3}.
\end{aligned}$$

Next, consider $\gamma_{-3}(t) = t(1+i\sqrt{3})/2$ *for* $1 \leq t \leq 2$. *This is the reversal of* γ_3. *We have* $x = t/2$ *and* $y = t\sqrt{3}/2$ *hence* $dy = \sqrt{3}dt/2$ *hence:*

$$\int_{\gamma_{-3}} x\,dy = \int_1^2 (t/2)(\sqrt{3}dt/2) = \frac{\sqrt{3}}{4}\frac{t^2}{2}\bigg|_1^2 = \frac{\sqrt{3}}{4}\left[\frac{4}{2} - \frac{1}{2}\right] = \frac{3\sqrt{3}}{8}.$$

Last, γ_4 *has reversal* $\gamma_{-4}(t) = e^{it} = \cos t + i \sin t$ *thus* $x = \cos t$ *and* $dy = \cos t dt$

$$\begin{aligned}\int_{\gamma_{-4}} x\,dy &= \int_{\pi/6}^{\pi/3} (\cos t)(\cos t dt) \\ &= \int_{\pi/6}^{\pi/3} \cos^2 t dt \\ &= \frac{1}{2}\int_{\pi/6}^{\pi/3} [1+\cos(2t)]dt = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6} + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) - \frac{1}{2}\sin\left(\frac{2\pi}{6}\right)\right) = \frac{\pi}{12}.\end{aligned}$$

In total, we have:

$$\int_\gamma x\,dy = \int_{\gamma_1} x\,dy + \int_{\gamma_2} x\,dy - \int_{\gamma_{-3}} x\,dy - \int_{\gamma_{-4}} x\,dy = \frac{3\sqrt{3}}{8} + \frac{\pi}{3} - \frac{3\sqrt{3}}{8} - \frac{\pi}{12} = \boxed{\frac{\pi}{4}.}$$

Let us check our work by using Green's Theorem on $Q = x$ *and* $P = 0$. *Let* S *be as in the diagram hence* $\gamma = \partial S$ *is the positively oriented boundary for which Green's Theorem applies:*

$$\int_\gamma x\,dy = \iint_S dA = \int_{\pi/6}^{\pi/3}\int_1^2 r\,dr\,d\theta = \left(\int_{\pi/6}^{\pi/3} d\theta\right)\left(\int_1^2 r\,dr\right) = \left[\frac{\pi}{3} - \frac{\pi}{6}\right]\left[\frac{2^2}{2} - \frac{1^2}{2}\right] = \frac{3\pi}{12} = \frac{\pi}{4}.$$

I think it is fairly clear from this example that we should use Green's Theorem when possible.

8.2 Independence of Path

The theorems we cover in this section should all be familar from your study of multivariate calculus. That said, we do introduce some new constructions allowing for complex-valued components. I'll say more about the correspondence between what we do here and the usual multivariate calculus at the conclusion of this section.

The total differential of a complex function $u + iv$ is defined by $du + idv$ where du and dv are the usual total differentials from multivariate calculus. This is equivalent to the definition below:

Definition 8.2.1. *If* h *is a complex-valued function which has continuous real partial derivative functions* h_x, h_y *then the* **differential** dh *of* h *is*

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy.$$

A differential form $Pdx + Qdy$ *is said to be* **exact** *on* $U \subseteq \mathbb{C}$ *if there exists a function* h *for which* $dh = Pdx + Qdy$ *for each point in* U.

Recall from calculus I, if $F'(t) = f(t)$ for all $t \in [a, b]$ then the FTC states $\int_a^b f(t)dt = F(b) - F(a)$. If we write this in a slightly different notation then the analogy to what follows is even more clear. In particular $F' = dF/dt$ so $F'(t) = f(t)$ means $dF = f(t)\, dt$ hence by an F-substitution,

$$\int_a^b f(t)dt = \int_{F(a)}^{F(b)} dF = F(b) - F(a).$$

You can see that $f(t)dt$ is to F as $Pdx + Qdy$ is to h.

Theorem 8.2.2. *If γ is a piecewise smooth curve from A to B and h is continuously (real) differentiable near γ with $dh = Pdx + Qdy$ then $\displaystyle\int_\gamma Pdx + Qdy = \int_\gamma dh = h(B) - h(A)$.*

Proof: the only thing we need to show is $\int_\gamma dh = h(B) - h(A)$. The key point is that if $\gamma(t) = x(t) + iy(t)$ then by the chain-rule:

$$\frac{d}{dt}h(\gamma(t)) = \frac{d}{dt}h(x(t), y(t)) = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}.$$

We assume $\gamma : [a, b] \to \mathbb{C}$ has $\gamma(a) = A$ and $\gamma(b) = B$. Thus,

$$\begin{aligned}
\int_\gamma dh = \int_\gamma \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy &= \int_a^b \left(\frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}\right) dt \\
&= \int_a^b \frac{d}{dt}\big(\ h(\gamma(t))\ \big)\, dt \\
&= h(\gamma(b)) - h(\gamma(a)) \\
&= h(B) - h(A). \qquad \square
\end{aligned}$$

When we integrate $\int_\gamma Pdx + Qdy$ for an exact differential form $Pdx + Qdy = dh$ then there is no need to work out the details of the integration. Thankfully we can simply evaluate h at the end-points.

Example 8.2.3. *Let γ be some path from i to $1 + i$.*

$$\begin{aligned}
\int_\gamma (y + 3x^2)dx + (x + 4y^3)dy &= \int_i^{1+2i} d\left(xy + x^3 + y^4\right) \\
&= \left(1(2i) + 1^3 + (2i)^4\right) - i^4 \\
&= 2i + 1 + 16 - 1 \\
&= 16 + 2i.
\end{aligned}$$

The replacement of the notation $\int_\gamma$ with $\int_i^{1+2i}$ is only reasonable if the integral dependends only on the endpoints. Theorem 8.2.2 shows this is true whenever $Pdx + Qdy$ is exact near the integration.

To make it official, let me state the definition clearly:

Definition 8.2.4. *The differential form $Pdx + Qdy$ is* **independent of path** *in $U \subseteq \mathbb{C}$ if for every pair of curves γ_1, γ_2 in U with matching starting and ending points have $\int_{\gamma_1} Pdx + Qdy = \int_{\gamma_2} Pdx + Qdy$.*

An equivalent condition to independence of path is given by the vanishing of all integrals around loops; a loop is just a simple closed curve.

Theorem 8.2.5. *$Pdx+Qdy$ is independent of path in U iff $\int_\gamma Pdx+Qdy = 0$ for all simple closed curves γ in U.*

Proof: Suppose $Pdx + Qdy$ is path-independent. Let γ be a loop in U. Pick any two distinct points on the loop, say A and B. Let γ_1 be the part of the loop from A to B. Let γ_2 be the part of the loop from B to A. Then the reversal of γ_2 is γ_{-2} which goes from A to B. Hence γ_1 and γ_{-2} are two paths in U from A to B hence:

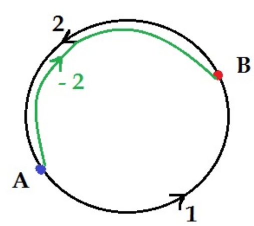
$$\int_{\gamma_1} Pdx + Qdy = \int_{\gamma_{-2}} Pdx + Qdy = -\int_{\gamma_2} Pdx + Qdy.$$

Therfore,

$$0 = \int_{\gamma_1} Pdx + Qdy + \int_{\gamma_2} Pdx + Qdy = \int_{\gamma} Pdx + Qdy.$$

Conversely, if we assume $\int_\gamma Pdx+Qdy = 0$ for all loops then by almost the same argument we can obtain the integrals along two different paths with matching terminal points agree. $\square$

Words are very uncessary, they can only do harm. Ok, maybe that's a bit much, but the proof of the Theorem above is really just contained in the diagram below:



You might suspect that exact differential forms and path-independent differential forms are one and the same: if so, good thinking:

Lemma 8.2.6. *Let P and Q be continuous complex-valued functions on a domain D. Then $Pdx+Qdy$ is path-independent if and only if $Pdx + Qdy$ is exact on D. Furthermore, the h for which $dh = Pdx + Qdy$ is unique up to an additive constant.*

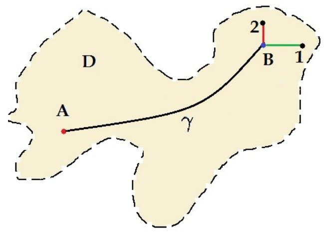
Proof: the reverse implication is a trivial consequence of Theorem 8.2.2. Assume $dh = Pdx+Qdy$ and γ_1, γ_2 are two curves from A to B. Then $\int_{\gamma_1} Pdx+Qdy = h(B)-h(A) = \int_{\gamma_2} Pdx+Qdy$ hence path-independence of $Pdx + Qdy$ on D is established.

The other direction of the proof is perhap a bit more interesting. Beyond just being a proof for this Lemma, the formula we study here closely analogus to the construction of the potential energy function by integration of the force field.

Assume $Pdx + Qdy$ is path-independent on the open connected set D. Pick some reference point $A \in D$ and let $z \in D$ we define

$$h(z) = \int_A^z Pdx + Qdy.$$

Fix a point $(x_o, y_o) = B \in D$, we wish to study the partial derivatives of h at (x_o, y_o). Let γ be a path from A to B in D. Since D is open we can construct paths γ_1 the horizontal path $[x_o + iy_o, x + iy_o]$ and γ_2 be the red vertical path $[x_o + iy_o, x_o + iy]$ both inside D.



Notice the point at the end of γ_1 is $x + iy_o$ and:

$$h(x+iy_o) = \int_{\gamma} Pdx + Qdy + \int_{\gamma_1} Pdx + Qdy$$

However, γ_1 has $x = t$ for $x_o \leq t \leq x$ and $y = y_o$ hence $dx = dt$ and $dy = 0$,

$$h(x+iy_o) = \int_{\gamma} Pdx + Qdy + \int_{x_o}^{x} P(t, y_o)dt.$$

Notice γ has no dependence on x thus:

$$\frac{\partial}{\partial x} h(x+iy_o) = \frac{\partial}{\partial x} \int_{x_o}^{x} P(t, y_o)dt = P(x, y_o).$$

where we have used the FTC in the last equality. Next, note γ_1 ends at $x_o + iy$ thus:

$$h(x_o+iy) = \int_{\gamma} Pdx + Qdy + \int_{\gamma_2} Pdx + Qdy$$

But, γ_2 has $x = x_o$ and $y = t$ for $y_o \leq t \leq y$ thus $dx = 0$ and $dy = dt$. We find

$$h(x_o+iy) = \int_{\gamma} Pdx + Qdy + \int_{y_o}^{y} Q(x_o, t)dt$$

Notice γ has no dependence on y thus:

$$\frac{\partial}{\partial y} h(x_o+iy) = \frac{\partial}{\partial y} \int_{y_o}^{y} Q(x_o, t)dt = Q(x_o, y).$$

In total we have shown $h_x(x, y_o) = P(x, y_o)$ and $h_y(x_o, y) = Q(x_o, y)$. By continuity of P and Q we find $dh = Pdx + Qdy$ at (x_o, y_o). However, (x_o, y_o) is an arbitrary point of D and it follows $Pdx + Qdy$ is exact on D with potential $h(z) = \int_A^z Pdx + Qdy$.

Finally, to study uniqueness, suppose h_1 is another function on D for which $dh_1 = Pdx + Qdy$. Notice $dh = dh_1$ thus $d(h - h_1) = 0$ but this implies $\nabla \mathbf{Re}(h - h_1) = 0$ and $\nabla \mathbf{Im}(h - h_1) = 0$ thus both the real and imaginary components of $h - h_1$ are constant and we find $h = h_1 + c$. The function h is uniquely associated to $Pdx + Qdy$ on a domain up to an additive constant. $\square$

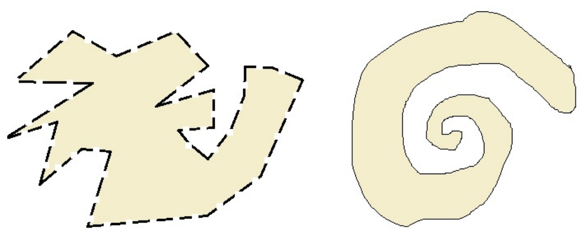
I usually say that h is the *potential* for $Pdx + Qdy$ modulo an additive constant. If all differential forms were exact then integration would be much easier and life would not be so interesting. Fortunately, only some forms are exact. The following definition is a natural criteria to investigate since $P = \frac{\partial h}{\partial x}$ and $Q = \frac{\partial h}{\partial y}$ suggest that P and Q are related by differentiation due to Clairaut's theorem on commuting partial derivatives. We expect $\frac{\partial}{\partial y}\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\frac{\partial h}{\partial y}$ hence $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Thus define:

Definition 8.2.7. *A differential form $Pdx + Qdy$ is* **closed** *on D if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ for all points in D.*

The argument just above the Definition already proves the Lemma below:

Lemma 8.2.8. *If $Pdx + Qdy$ is an exact form on D then $Pdx + Qdy$ is a closed form on D.*

The converse of the Lemma above requires we place a topological restriction on the domain D. It is not enough that D be connected, we need the stricter requirement that D be **simply connected**. Qualitatively, simple connectivity means we can take loops in D an continuously deform them to points without getting stuck on missing points or holes in D. The deformation discussed on pages 80-81 of Gamelin give you a better sense of the technical details involved in such deformations. To be entirely honest, the proper study of simply connected spaces belongs to topology. But, ignoring topology is a luxury we cannot afford. We do need a suitable description of spaces without loop catching holes. A good criteria is **star-shaped**. A star-shaped space is simply connected and the vast majority of all the examples which cross our path will fit the criteria of star-shaped; rectangles, disks, half-disks, sectors, even slit-planes are all star-shaped. There are spaces which are simply connected, yet, not star-shaped:



You can see that while these shapes are not star-shaped, we could subdivide them into a finite number of star-shaped regions.

Example 8.2.9. *Consider $\theta(z) = Arg(z) = \tan^{-1}(y/x) + c$ for $x > 0$. The differential is calculated as follows:*

$$d\theta = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)\tan^{-1}(y/x = dx\frac{-y/x^2}{1+(y/x)^2} + dy\frac{1/x}{1+(y/x)^2} = \frac{-ydx + xdy}{x^2+y^2}$$

Notice that the principal argument for $x \leq 0$ is obtained by addition of a constant hence the same derivatives hold for $x < 0$. Let

$$\omega = \frac{-ydx + xdy}{x^2+y^2}$$

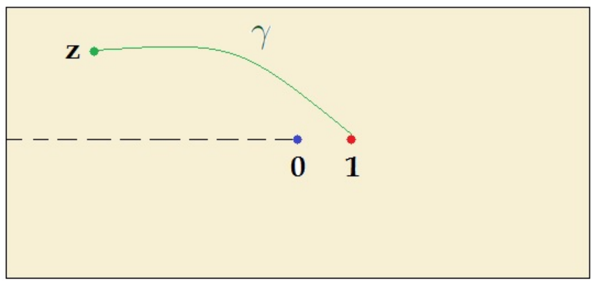
then $\omega = Pdx + Qdy$ where $P = \frac{-y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$. I invite the reader to verify that $\partial_y P = \partial_x Q$ for all points in the punctured plane $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Thus ω is **closed** *on $\mathbb{C}^\times$. However, ω is not exact on the punctured plane as we may easily calculate the integral of ω around the CCW-oriented unit-circle as follows: $\gamma(t) = e^{it}$ has $x = \cos t$ and $y = \sin t$ hence $-ydx + xdy = -\sin t(-\sin t dt) + \cos t(\cos t dt) = dt$ and $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ hence:*

$$\int_\gamma \omega = \int_\gamma \frac{-ydx + xdy}{x^2+y^2} = \int_0^{2\pi} \frac{dt}{1} = 2\pi.$$

Hence, by Theorem 8.2.5 combined with Lemma 8.2.6 we see ω cannot be exact. However, if we consider ω with domain restricted to a slit-complex plane then we can argue that Arg_α is the potential function for ω meaning $d(Arg_\alpha(z)) = \frac{-ydx+xdy}{x^2+y^2}$. In the slit-complex plane there is no path *which encircles the origin and the nontrivial loop integral is removed. If we use 1 as a reference point for the potential construction then we find the following natural integral presentation of the principal argument:*

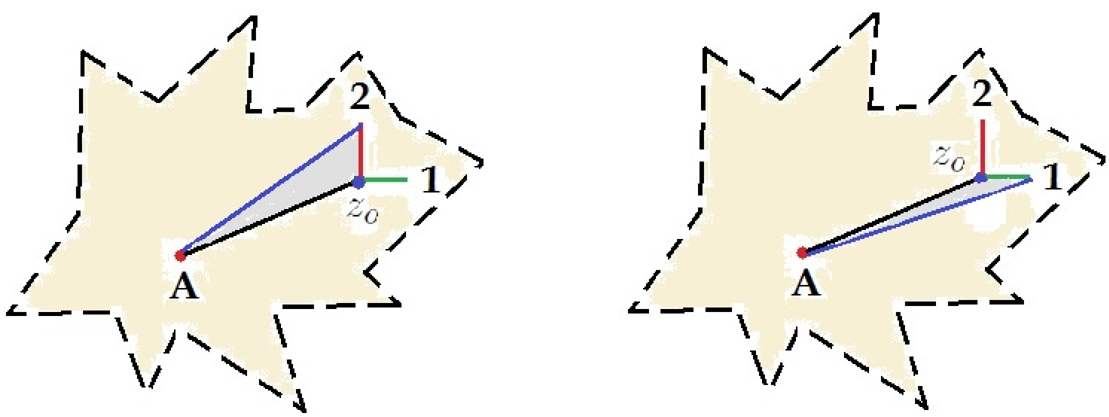
$$Arg(z) = \int_1^z \frac{-ydx + xdy}{x^2 + y^2}$$

The example above is the quintessential example of a form which is closed but not exact. Poincare's Lemma solves this puzzle in more generality. In advanced calculus, I usually share a calculation which shows that in any contractible subset of $\mathbb{R}^n$ every closed differential p-form is the **exterior derivative** of a potential $(p-1)$-form. What we study here is almost the most basic case[3]. What follows is a weakened converse of Lemma 8.2.8; we find if a form is closed on a star-shaped domain then the form must be exact.



Theorem 8.2.10. *If domain D is star-shaped. Then $Pdx + Qdy$ closed in D implies there exists function h on D for which $dh = Pdx + Qdy$ in D.*

Proof: assume D is a star-shaped domain and $Pdx + Qdy$ is a closed form on D. This means we assume $\partial_x Q = \partial_y P$ on D. Let A be a star-center for D and define $h(z) = \int_{[A,z]} Pdx + Qdy$



Fix a point z_o in D and note $[A, z_o]$ is in D. Furthermore, γ_1 is given by $x = t$ for $x_o \leq t \leq x$ and $y = y_o$. Likewise, γ_2 is the line-segment $[z_o, x_o + iy]$ where $x = x_o$ and $y = t$ for $y_o \leq t \leq y$. Note $[A, x_o + iy]$ and $[A, x + iy_o]$ are in D. Apply Green's Theore on the triangle T_2 with vertices $A, z_o, x_o + iy$:

$$\int_{\partial T_2} Pdx + Qdy = \iint_{T_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{T_2} (0) dA = 0.$$

[3]in one-dimension all smooth forms fdx are both closed and exact

Thus,

$$\int_{[A,z_o]} Pdx + Qdy + \int_{[z_o,x_o+iy]} Pdx + Qdy + \int_{[x_o+iy,A]} Pdx + Qdy = 0$$

but, we defined $h(z) = \int_{[A,z]} Pdx + Qdy$ thus

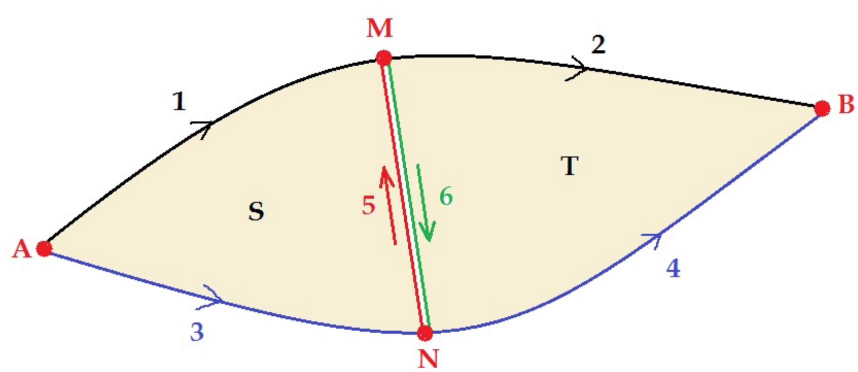
$$h(x_o, y_o) - h(x_o, y) + \int_{y_o}^{y} Q(x_o, t)dt = 0 \quad \Rightarrow \quad \frac{\partial h}{\partial y} = Q(x_o, y).$$

Examine triangle T_1 formed by $A, z_o, x + iy_o$ to derive by a very similar argument $\frac{\partial h}{\partial x} = P(x, y_o)$. Thus, $dh = Pdx + Qdy$ at z_o and it follows $h(z) = \int_{[A,z]} Pdx + Qdy$ serves to define a potential for $Pdx + Qdy$ on D. Thus $Pdx + Qdy$ is exact on D. $\square$

Given that we have shown the closed form $Pdx + Qdy$ on star-shaped domain is exact we have by Lemma 8.2.6 that $Pdx + Qdy$ is path-independent. It follows we can calculate $h(z)$ along any path in D, not just $[A, z]$ which was our starting point for the proof above.

The remainder of Gamelin's section 3.2 is devoted to discussing deformation theorems for closed forms. I give a simplified proof of the deformation. Actually, at the moment, I'm not certain if Gamelin's proof is more or less general than the one I offer below. There may be a pedagogical reason for his development I don't yet appreciate[4].

Suppose γ_{up} and γ_{down} are two curves from A to B. For simplicity of exposition, let us suppose these curves only intersect at their endpoints. Suppose $Pdx + Qdy$ is a closed form on the region between the curves. We may inquire, does $\int_{\gamma_{up}} Pdx + Qdy = \int_{\gamma_{down}} Pdx + Qdy$? To understand the resolution of this question we should consider the picture below:



Here I denote $\gamma_{up} = \gamma_1 \cup \gamma_2$ and $\gamma_{down} = \gamma_3 \cup \gamma_4$. The middle points M, N are joined by the **cross-cuts** γ_5 and $\gamma_6 = \gamma_{-5}$. Notice that $\partial S = \gamma_5 \cup \gamma_{-1} \cup \gamma_3$ whereas $\partial T = \gamma_6 \cup \gamma_4 \cup \gamma_{-2}$. Now, apply Green's Theorem to the given closed form on S and T to obtain:

$$\int_{\partial S} Pdx + Qdy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0 \quad \& \quad \int_{\partial T} Pdx + Qdy = \iint_T \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0.$$

The double integrals above are zero because we know $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. To complete the argument we break the line-integrals around the boundary into pieces taking into account the sign-rule for curve

[4] I think the rolling wave argument is essentially the same as I give here, but I should compare Gamelin's proof to mine when time permits

reversals: for $\partial S = \gamma_5 \cup \gamma_{-1} \cup \gamma_3$ we obtain:

$$\int_{\gamma_5} Pdx + Qdy - \int_{\gamma_1} Pdx + Qdy + \int_{\gamma_3} Pdx + Qdy = 0$$

for $\partial T = \gamma_6 \cup \gamma_4 \cup \gamma_{-2}$ we obtain:

$$\int_{\gamma_6} Pdx + Qdy + \int_{\gamma_4} Pdx + Qdy - \int_{\gamma_2} Pdx + Qdy = 0.$$

Summing the two equations above and noting that $\int_{\gamma_6} Pdx + Qdy + \int_{\gamma_5} Pdx + Qdy = 0$ we obtain: (using Gamelin's slick notation)

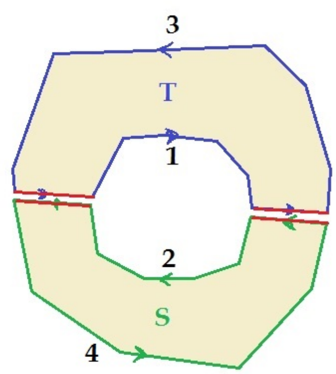
$$\left[-\int_{\gamma_1} + \int_{\gamma_3} + \int_{\gamma_4} - \int_{\gamma_2}\right](Pdx+Qdy) = 0 \ \Rightarrow\ \left[\int_{\gamma_1} + \int_{\gamma_2}\right](Pdx+Qdy) = \left[\int_{\gamma_3} + \int_{\gamma_4}\right](Pdx+Qdy).$$

Consequently, as $\gamma_{up} = \gamma_1 \cup \gamma_2$ and $\gamma_{down} = \gamma_3 \cup \gamma_4$ we conclude

$$\int_{\gamma_{up}} Pdx + Qdy = \int_{\gamma_{down}} Pdx + Qdy.$$

We have shown that the integral of a differential form $Pdx + Qdy$ is unchanged if we deform the curve of integration over a region on which the form $Pdx + Qdy$ is closed[5]

The construction of the previous page is easily extended to deformations over regions which are not simply connected. For example, we can argue that if $Pdx + Qdy$ is closed on the crooked annulus then the integral of $Pdx + Qdy$ on the inner and outer boundaries must conicide.



The argument again centers on the application of Green's Theorem to simply connected domains on which the area integral vanishes hence leaving the integral around the boundary trivial. When we add the integral around ∂T and ∂S the red cross-cuts vanish. Define $\gamma_{out} = \gamma_3 \cup \gamma_4$ and $\gamma_{in} = \gamma_{-1} \cup \gamma_{-2}$ (used the reversals to make the curve have a postive orientation). In view of these defitions, we find:

$$\int_{\gamma_{out}} Pdx + Qdy = \int_{\gamma_{in}} Pdx + Qdy.$$

The deformation of the inner annulus boundary to the outer boundary leaves the integral unchanged because the differential form $Pdx + Qdy$ was closed on the intermediate curves of the deformation.

[5] in the language of exterior calculus; $d(Pdx + Qdy) = (Q_x - P_y)dx \wedge dy = 0$.

Example 8.2.11. *In Example 8.2.9 we learned that the form* $\frac{-ydx+xdy}{x^2+y^2}$ *is closed on the punctured plane* $\mathbb{C}^\times$. *We also showed that* $\int_\gamma \frac{-ydx+xdy}{x^2+y^2} = 2\pi$ *where* γ *is the CCW-oriented unit-circle. Let* C *be any positively oriented loop which encircles the origin we may deform the unit-circle to the loop through a region on which the form is closed hence* $\int_C \frac{-ydx+xdy}{x^2+y^2} = 2\pi$.

There are additional modifications of Green's Theorem for regions with finitely many holes. If you'd like to see my additional thoughts on this topic as well as an attempt at an intuitive justification of Green's Theorem you can look at my 2014 Multivariate Calculus notes §7.5. Finally, I will collect our results as does Gamelin at this point:

Theorem 8.2.12. *Let* D *be a domain and* $Pdx + Qdy$ *a complex-valued differential form. The following are equivalent:*

1. *path-independence of* $Pdx + Qdy$ *in* D
2. $\oint_\gamma Pdx + Qdy = 0$ *for all loops* γ *in* D,
3. $Pdx + Qdy$ *is exact in* D; *there exists* h *such that* $dh = Pdx + Qdy$ *on* D,
4. *(given the additional criteria* D *is star-shaped)* $Pdx + Qdy$ *is closed on* D; $\partial_y P = \partial_x Q$ *on* D.

In addition, if $Pdx + Qdy$ *is closed on a region where* γ_1 *may be continuously deformed to* γ_2 *then* $\int_{\gamma_1} Pdx + Qdy = \int_{\gamma_2} Pdx + Qdy$.

The last sentence of the Theorem above is often used as it was in Example 8.2.11.

8.3 Harmonic Conjugates

In this section we assume u is a real-valued smooth function.

Lemma 8.3.1. *If* $u(x,y)$ *is harmonic then the differential* $-\dfrac{\partial u}{\partial y}dx + \dfrac{\partial u}{\partial x}dy$ *is closed.*

Proof: assume $u_{xx} + u_{yy} = 0$. Consider $Pdx + Qdy$ with $P = -\frac{\partial u}{\partial y}$ and $Q = \frac{\partial u}{\partial x}$. Note:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left[-\frac{\partial u}{\partial y}\right] = -u_{yy} = u_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial Q}{\partial x}.$$

Thus $Pdx + Qdy$ is closed. $\square$

For the sake of discussion let $\omega = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$. By our work in the previous section (see part (4.) of Theorem 8.2.12) if D is a star-shaped domain then there exists some smooth function v such that $dv = \omega$. Explicitly, this gives:

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

But, equating coefficients[6] of dx and dy yields:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \& \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$

Which means $f = u + iv$ has $u_x = v_y$ and $u_y = -v_x$ for all points in the domain D where u, v are smooth hence part (3.) of Theorem 7.2.3 we find $f = u + iv$ is holomorphic on D; $u + iv \in \mathcal{O}(D)$. To summarize we have proved the following:

[6] hey, uh, why can we do that here?

Theorem 8.3.2. *If $u(x,y)$ is harmonic on a star-shaped domain then there exists a function $v(x,y)$ on D such that $u+iv$ is holomorphic on D.*

Let D be star-shaped. Following the proof of (1.) in Theorem 8.2.12 we know the potential function for $-\dfrac{\partial u}{\partial y}dx + \dfrac{\partial u}{\partial x}dy$ can be constructed by integration. In particular, we choose a reference point $A \in D$ and let B be another point in D:

$$v(B) = \int_A^B -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Gamelin points us back to page 56-57 to see this formula was derived in a special case for a disk with a particular path. The Example in Gamelin shows that the harmonic conjugate of $\log|z|$ is given by $Arg(z)$ on the slit complex plane $\mathbb{C}^-$. I will attempt one of the problems I assigned for homework.

Example 8.3.3. *Suppose $u(x,y) = e^{x^2-y^2}\cos(2xy)$. It can be shown that $u_{xx}+u_{yy}=0$ hence u is harmonic on $\mathbb{C}$. Choose reference point $A=0$ and consider:*

$$v(B) = \int_0^B -\frac{\partial(e^{x^2-y^2}\cos(2xy))}{\partial y}dx + \frac{\partial(e^{x^2-y^2}\cos(2xy))}{\partial x}dy$$

Differentiating, we obtain,

$$v(B) = \int_0^B \underbrace{-e^{x^2-y^2}[-2y\cos(2xy) - 2x\sin(2xy)]}_{P}\,dx + \underbrace{e^{x^2-y^2}[2x\cos(2xy) - 2y\sin(2xy)]}_{Q}\,dy$$

Let us calculate the integral from 0 to $B = x_o + iy_o$ by following the horizontal path γ_1 defined by $x=t$ and $y=0$ for $0 \leq t \leq x_o$ for which $dx = dt$ and $dy = 0$

$$\int_{\gamma_1} Pdx + Qdy = \int_0^{x_o} P(t,0)dt = \int_0^{x_o} -e^{x^2}[0]dt = 0.$$

Define γ_2 by $x = x_o$ and $y = t$ for $0 \leq t \leq y_o$ hence $dx = 0$ and $dy = dt$. Thus calculate:

$$\begin{aligned}
v(x_o+iy_o) = \int_{\gamma_2} Pdx + Qdy &= \int_0^{y_o} Q(x_o,t)dt \\
&= \int_0^{y_o} e^{x_o^2-t^2}[2x_o\cos(2x_ot) - 2t\sin(2x_ot)]dt \\
&= e^{x_o^2}\int_0^{y_o}[2x_o\cos(2x_ot)e^{-t^2} - 2t\sin(2x_ot)e^{-t^2}]dt \\
&= e^{x_o^2}\left(e^{-t^2}\sin(2x_ot)\right)\Big|_0^{y_o} \qquad \textit{(integral not too bad)} \\
&= e^{x_o^2}e^{-y_o^2}\sin(2x_oy_o) \\
&= e^{x_o^2-y_o^2}\sin(2x_oy_o).
\end{aligned}$$

Therefore, $e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy) = e^{x^2-y^2+2ixy} = e^{z^2}$ is a holomorphic function on $\mathbb{C}$.

When one of you asked me about this problem, my approach was quite different than the example above. These integrals are generally a sticking point. So, a simple approach is to attempt to see how the given u appears as $\mathbf{Re}f$ for some $f = f(z)$. Theoretically, the integral approach is superior.

8.4 The Mean Value Property

In our usual conversation, this is the *average value property*[7]

Definition 8.4.1. *Let $h : D \to \mathbb{R}$ be a continuous function on a domain D. Let $z_o \in D$ such that the disk $\{z \in \mathbb{C} \mid |z - z_o| < \rho\} \subseteq D$. The* **average value** *of $h(z)$ on the circle $\{z \in \mathbb{C} \mid |z - z_o| = r\}$ is*

$$A(r) = \int_0^{2\pi} h(z_o + re^{i\theta}) \frac{d\theta}{2\pi}$$

for $0 < r < \rho$.

Basically, this says we parametrize the circle around z_o and integrate $h(z)$ around that circle (provided the circle fits within the domain D). We may argue that $A(r) \to A(z_o)$ for small values of r. Notice $\int_0^{2\pi} h(z_o) d\theta = 2\pi h(z_o)$ thus $h(z_o) = \int_0^{2\pi} \frac{d\theta}{2\pi}$. We use this little identity below:

$$|A(r) - h(z_o)| = \left| \int_0^{2\pi} [h(z_o + re^{i\theta}) - h(z_o)] \frac{d\theta}{2\pi} \right| \leq \int_0^{2\pi} \left| h(z_o + re^{i\theta}) - h(z_o) \right| \frac{d\theta}{2\pi}$$

Now, continuity of $h(z)$ at z_o gives us that the integrand tends to zero as $r \to 0$ thus it follows $A(r) \to h(z_o)$ as $r \to 0$. The theorem below was suprising to me when I first saw it. In short, the theorem says the average of the values of a harmonic function on a disk is the value of the function at the center of the disk.

Theorem 8.4.2. *If $u(z)$ is a harmonic function on a domain D, and if the disk $\{z \in \mathbb{C} \mid |z - z_o| < \rho\}$ is contained in D, then*

$$u(z_o) = \int_0^{2\pi} u(z_o + re^{i\theta}) \frac{d\theta}{2\pi} \qquad 0 < r < \rho.$$

Proof: The proof is given on page 86. I'll run through it here: let u be harmonic on the domain D then Lemma 8.3.1 tells us that the differential $-\dfrac{\partial u}{\partial y} dx + \dfrac{\partial u}{\partial x} dy$ is closed. Hence, it is exact and so the integral around a loop is zero:

$$0 = \oint_{|z - z_o| = r} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

The theorem essentially follows from the identity above, we just need to write the integral in detail. Let $z = z_o + re^{i\theta}$ parametrize the circle so $x = x_o + r\cos\theta$ and $y = y_o + r\sin\theta$ thus $dx = -r\sin\theta d\theta$ and $dy = r\cos\theta d\theta$ hence:

$$0 = r \int_0^{2\pi} \left[\frac{\partial u}{\partial y} \sin\theta + \frac{\partial u}{\partial x} \cos\theta \right] d\theta = r \int_0^{2\pi} \left[\frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} \right] d\theta = r \int_0^{2\pi} \frac{\partial u}{\partial r} d\theta.$$

Understand that $\frac{\partial u}{\partial r}$ is evaluated at $z_o + re^{i\theta}$. We find (dividing by $2\pi r$)

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r} \left[u(z_o + re^{i\theta}) \right] d\theta = \frac{\partial}{\partial r} \left[\int_0^{2\pi} u(z_o + re^{i\theta}) \frac{d\theta}{2\pi} \right]$$

[7] notice the average is taken with respect to the angular parameter around the circle. One might also think about the average taken w.r.t. arclength. In an arclength-based average we would divide by $2\pi r$ and we would also integrate from $s = 0$ to $s = 2\pi r$. A $u = s/r$ substitution yields the θ-based integral here. It follows this average is the same as the usual average over a space curve discussed in multivariate calculus.

where we have used a theorem of analysis that allows us to exchange the order of integration and differentiation[8]. We find $\displaystyle\int_0^{2\pi} u(z_o+re^{i\theta})\,\frac{d\theta}{2\pi}$ is constant for $0 < r < \rho$ where ρ is determined by the size of the domain D. More to the point, as we allow $r \to 0$ it is clear that $\int_0^{2\pi} u(z_o+re^{i\theta})\,\frac{d\theta}{2\pi} \to u(z_o)$ hence the the constant value of the integral is just $u(z_o)$ and this completes the proof. $\square$

Definition 8.4.3. *We say a continuous real-valued function $h(z)$ on a domain $D \subseteq \mathbb{C}$ has the* **mean value property** *if for each point $z_o \in D$ the value $h(z_o)$ is the average of $h(z)$ over any small circle centered at z_o.*

The point of this section is that harmonic functions have the mean value property. It is interesting to note that later in the course we the converse is also true; a function with the mean value property must also be harmonic.

8.5 The Maximum Principle

Theorem 8.5.1. Strict Maximum Principle (Real Version). *Let $u(z)$ be a real-valued harmonic function on a domain D such that $u(z) \leq M$ for all $z \in D$. If $u(z_o) = M$ for some $z_o \in D$, then $u(z) = M$ for all $z \in D$.*

The proof is given on Gamelin page 87. In short, we can show the set of points S_M for which $u(z) = M$ is open. However, the set of points $S_{<M}$ for which $u(z) < M$ is also open by continuity of $u(z)$. Note $D = S_M \cup S_{<M}$ hence either $S_M = D$ or $S_{<M} = D$ as D is connected. This proves the theorem.

When a set D is connected it does not allow a **separation**. A separation is a pair of non-empty subsets $U, V \subset D$ for which $U \cap V =$ and $U \cup V = D$. We characterized connectedness in terms of paths in this course, but, there are spaces which path-connected and connected are distinct concepts. See pages 40-43 of [R91] for a fairly nuanced discussion of path-connectedness.

Theorem 8.5.2. Strict Maximum Principle (Complex Version) *Let $h(z)$ be a bounded, complex-valued, harmonic function on a domain D. If $|h(z)| \leq M$ for all $z \in D$, and $|h(z_o)| = M$ for some $z_o \in D$, then $h(z)$ is constant on D.*

The proof is given on page 88 of Gamelin. I will summarize here: because we have a point z_o for which $|h(z_o)| = M$ it follows there exists $c \in \mathbb{C}$ such that $|c| = 1$ and $ch(z_o) = M$. But, $\mathbf{Re}(ch(z))$ is a real-valued harmonic function on a domain hence Theorem 8.5.1 applies to $u(z) = \mathbf{Re}(ch(z)) = M$ for all $z \in D$. Thus $\mathbf{Re}(h(z)) = M/c$ for all $z \in D$. It follows that $\mathbf{Im}(h(z)) = 0$ for all $z \in D$. But, you may recall we showed all real-valued holomorphic functions are constant in Theorem 7.2.8.

Theorem 8.5.3. Maximum Principle *Let $h(z)$ be a complex-valued harmonic function on a bounded domain D such that $h(z)$ extends continuously to the boundary ∂D of D. If $|h(z)| \leq M$ for all $z \in \partial D$ then $|h(z)| \leq M$ for all $z \in D$.*

This theorem means that to bound a harmonic function on some domain it suffices to bound it on the edge of the domain. Well, some fine print is required. We need that there exists a continuous extension of the harmonic function to an open set which is just a little bigger than $D \cup \partial D$. The proof is outlined on page 88. In short, this theorem is a consequence of the big theorem of analysis:

[8] this is not always possible, certain conditions on the function are needed, since u is assumed smooth here that suffices

The continuous image of a compact domain attains extreme values.

In other words, if your domain fits inside some ball (or disk here) of finite radius and the real-valued function of that domain is continuous then there is some point(s) $p, q \in D$ for which $f(p) \leq f(z) \leq f(q)$. Very well, so if the maximum modulus is attained in the interior of D we have $|h(D)| = \{M\}$ for some $M \in \mathbb{R}$ hence by continuity the extension of h to the boundary the modulus of the boundary is also at constant value M. Therefore, the maximum modulus of $h(z)$ is always attained on the boundary given the conditions of the theorem.

The results of this section and the last are important parts of the standard canon of complex analysis. That said, we don't use them all the time. Half the reason I cover them is to assign $III.5\#3$. I want all my students to experience the joy of proving the Fundamental Theorem of Algebra.

8.6 Applications to Fluid Dynamics

The foundation of the applications discussed in the text is the identification that line-integrals in the plane permit two interpretations:

$$\int_\gamma Pdx + Qdy = \text{circulation of vector field } \langle P, Q\rangle \text{ along } \gamma.$$

whereas

$$\int_\gamma Qdx - Pdy = \text{flux of vector field } \langle P, Q\rangle \text{ through } \gamma.$$

Green's Theorem has an interesting meaning with respect to both concepts: let γ be a loop and D a domain for which $\partial D = \gamma$ and consider: for $V = \langle P, Q\rangle$ and $T = (dx/ds, dy/ds)$ for arclength s,

$$\oint_\gamma (V \bullet T)ds = \oint_\gamma Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \nabla \times \langle P, Q, 0\rangle \bullet d\vec{A}$$

Thus Green's Theorem is a special case of Stokes' Theorem. On the other hand, for normal[9] $n = (dy/ds. - dx/ds)$

$$\oint_\gamma (V \bullet n)ds = \oint_\gamma Qdx - Pdy = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iiint_{D\times[0,1]} \nabla \bullet \langle P, Q, 0\rangle dV$$

hence Green's Theorem is a special case of Gauss' Theorem. If the form $Pdx + Qdy$ is closed then we find the vector field $\langle P, Q\rangle$ is **irrotational** (has vanishing curl). If the form $Qdx - Pdy$ is closed then we find the vector field $\langle P, Q\rangle$ has vanishing divergence.

The starting point for our study of fluid physics is to make a few assumptions about the flow and how we will describe it. First, we use $V(z) = P + iQ$ to denote the **velocity field** of the liquid and D is the domain on which we study the flow. If $V(z_o) = A + iB$ then the liquid at $z = z_o$ has velocity $A + iB$. Of course, we make the identification $A + iB = \langle A, B\rangle$ throughout this section. For the math to be reasonable and the flow not worthy of prize winning mathematics:

1. $V(z)$ is time-independent,

[9] if you have studied the Frenet-Serret T, N, B frame, I should caution that n need not coincide with N. Here n is designed to point away from the interior of the loop

2. There are no sources or sinks of liquid in D. Fluid is neither created nor destroyed,

3. The flow is incompressible, the density (mass per unit area) is the same throughout D.

4. The flow is irrotational in the sense that around any little circle in D there is no circulation.

Apply Green's Theorem to condition (4.) to see that $\partial_x Q = \partial_y P$ is a necessary condition for $V = P + iQ$ on D. But, this is just the condition that $Pdx + Qdy$ is closed. Thus, for simply connected subset S of D we may select a function ϕ such that $d\phi = Pdx + Qdy$. which means $\nabla\phi = \langle P, Q\rangle$ in the usual language of multivariate calculus. The function ϕ such that $d\phi = Pdx + Qdy$ is called the **potential function** of V on S.

We argue next that (2.) implies ϕ is harmonic on S. Consider the flux through any little loop γ in S with D_γ the interior of γ; $\partial D_\gamma = \gamma$. If we calculate the flux of V through γ we find it is zero as the fluid is neither created nor destroyed in D. But, Green's Theorem gives us the following:

$$\text{flux of } V \text{ through } \gamma = \oint_\gamma (V \bullet n)ds = \oint_\gamma Qdx - Pdy = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = 0.$$

Hence $P_x + Q_y = 0$ on S. But, $P = \phi_x$ and $Q = \phi_y$ hence $\phi_{xx} + \phi_{yy} = 0$. Thus ϕ is harmonic on S

As we have shown ϕ is harmonic on S, the theory of harmonic conjugates allows us construction of ψ on S for which $f(z) = \phi + i\psi$ is holomorphic on S. We say $f(z)$ so constructed is the **complex velocity potential of the flow**. Note:

$$V(z) = \phi_x + i\phi_y = \phi_x - i\psi_x = \overline{\phi_x + i\psi_x} = \overline{f'(z)}.$$

Recall from our work on conformal mappings we learned the level curves of $\mathbf{Re}f(z)$ and $\mathbf{Im}f(z)$ are orthogonal if $f(z)$ is a non-constant holomorphic mapping. Therefore, if the velocity field is nonzero then we have such a situation where $\mathbf{Re}(f(z)) = \phi(z)$ and $\mathbf{Im}(f(z)) = \psi(z)$. We find the geometric meaning of ψ and ϕ is:

1. level curves of ψ are **streamlines** of the flow. The tangents to the stream lines line up with $V(z)$. In other words, $\psi(z) = c$ describes a path along which the fluid particles travel.

2. level curves of ϕ are orthogonal to the stream lines.

It follows we call ψ the **stream function** of V. At this point we have all the toys we need to look at a few examples.

Example 8.6.1. *Study the constant horizontal flow $V(z) = 1$ on $\mathbb{C}$. We expect stream lines of the form $y = y_o$ hence $\psi(z) = y$. But, the harmonic function $\phi(z)$ for which $f(z) = \phi(z) + i\psi(z)$ with $\psi(z) = y$ is clearly just $\psi(z) = x$. Hence the complex velocity potential is $f(z) = z$. Of course we could also have seen this geometrically, as the orthogonal trajectories of the streamlines $y = y_o$ are just $x = x_o$.*

Is every holomorphic function a flow? **NO.** This is the trap I walked into today (9-12-14). Consider:

Example 8.6.2. *Study the possible flow $V(z) = x + iy$. The potential of the flow ϕ must solve $d\phi = xdx + ydy$. This implies $\phi = \frac{1}{2}(x^2 + y^2)$. However, $\phi_{xx} + \phi_{yy} = 2 \neq 0$ hence the potential potential is not a potential as it is not harmonic ! This "flow" $V(z) = x + iy$ violates are base assumptions since $\nabla \bullet V = 2$. Fluid is created* **everywhere** *in this flow so the technology of this section is totally off base here!*

There is a special subclass of harmonic functions which can be viewed as flows. We need $V = P+iQ$ where the form $Pdx + Qdy$ is closed ($\nabla \times V = 0$) and the form $Pdy - Qdx$ is closed ($\nabla \bullet V = 0$).

Example 8.6.3. *Consider* $V(z) = \frac{x+iy}{x^2+y^2} = \frac{1}{r}\langle \cos\theta, \sin\theta\rangle = \frac{1}{r}\hat{r}$. *We wish to solve* $d\phi = \frac{xdx+idy}{x^2+y^2}$ *hence* $\phi_x = \frac{x}{x^2+y^2}$ *and* $\phi_y = \frac{y}{x^2+y^2}$ *which has solution*

$$\phi(x+iy) = \ln\sqrt{x^2+y^2} \qquad \Rightarrow \qquad \phi(z) = \ln|z|$$

However, we know $\log(z) = \ln|z| + i\arg(z)$ *is holomorphic (locally speaking of course) hence* $f(z) = \log(z)$ *is the complex velocity potential and we identify the stream function is* $\arg(z)$. *If we calculate the circulation of the flow around a circle centered at the origin we obtain* 2π. *Of course,* $z = 0$ *is not in the domain of the flow and in fact we can deduce the origin serves as a source for this flow. The speed of the fluid approaches infinity as we get close to* $z = 0$ *then it slows to zero as it spreads out to* ∞. *The streamlines are rays in this problem.*

The examples and discussion on pages 94-96 of Gamelin about how to morph a standard flow to another via a holomorphic map is very interesting. I will help us appreciate it in the homework.

8.7 Other Applications to Physics

The heat equation is given by $u_t = u_{xx} + u_{yy}$ in two dimensions. Therefore, if we solve the **steady-state** or **time-independent** problem of heat-flow then we must face the problem of solving $u_{xx} + u_{yy} = 0$. Of course, this is just Laplace's equation and an analog of the last section exists here as we study the flow of heat. There are two standard problems:

1. **Dirichlet Problem:** given a prescribed function v on the boundary of D which represents the temperature-distribution on the boundary, find a harmonic function u on D for which $u = v$ on ∂D.

2. **Neumann Problem:** given a prescribed function v on the boundary of D which represents the flow of heat through the boundary, find a harmonic function u on D for which $\frac{\partial u}{\partial n} = v$ where n is the normal direction on ∂D.

The notation $\frac{\partial u}{\partial n} = v$ simply indicates the directional derivative of u in the normal direction n.

We introduce $Q = \nabla u = u_x + iu_y$ as the **flow of heat**. It points in the direction of increasing levels of temperature u. The condition $\nabla \bullet Q = 0$ expresses the lack of heat sources or sinks. The condition $\nabla \times Q = 0$ assumes the heat flow is irrotational. Given both these assumptions we face the same mathematical problem as we studied for fluids. Perhaps you can appreciate why the old theory of heat literally thought of heat as being a liquid or gas which flowed. Only somewhat recently have we understood heat from the perspective of statistical thermodynamics which says temperature and heat flow are simply macroscopic manifestations of the kinetics of atoms. If you want to know more, perhaps you should study thermodynamics[10]

Example 8.7.1. *Suppose* $u(z) = u_o$ *for all* $z \in \mathbb{C}$. *Then,* $Q = \nabla u = 0$. *There is zero flow of heat.*

Example 8.7.2. Problem: *Find the steady-state heat distribution for a circular plate of radius* 1 *(*$|z| \leq 1$*) for which the upper edge (*$y > 0$, $|z| = 1$*) is held at constant temperature* $u = 1$ *and the lower edge (*$y < 0$, $|z| = 1$*)is held at constant temperature* $u = -1$.

[10] coming soon to a university near you

Solution: we assume that there are no heat sources within the disk and the flow of heat is irrotational. Thus, we seek a harmonic function on the disk which fits the presribed boundary conditions. At this point we make a creative leap: this problem reminds us of the upper-half plane and the behaviour of $Arg(w)$. Recall: for $w = t \in \mathbb{R}$ with $t > 0$

$$Arg(t) = 0 \qquad \& \qquad Arg(-t) = -\pi$$

Furthermore, recall Example 7.6.9 where we studied $h(z) = \frac{z-i}{z+i}$. This Cayley map mapped $(0, \infty)$ to the lower-half of the unit-circle. I argue that $(-\infty, 0)$ maps to the upper-half of the circle. In particular, we consider the point $-1 \in (-\infty, 0)$. Observe:

$$h(-1) = \frac{-1-i}{-1+i} = \frac{(-1-i)(-1-i)}{2} = \frac{(1+i)^2}{2} = \frac{1+2i+i^2}{2} = i.$$

I'm just checking to be sure here. Logically, since we know fractional linear transformations map lines to circles or lines and we already know $(0, \infty)$ maps to half of the circle the fact the other half of the line must map to the other half of the circle would seem to be a logically inevitable.

The temperature distribution $u(z) = Arg(z)$ for $z \in \mathbb{H}$ sets $u = 0$ for $z \in (0, \infty)$ and $u = -\pi$ for $z \in (-\infty, 0)$. We shift the temperatures to -1 to 1 by some simple algebra: to send $(-\pi, 0)$ to $(-1, 1)$ we need to stretch by $m = \frac{2}{\pi}$ and shift by 1. The new u:

$$u(z) = 1 + \frac{2}{\pi} Arg(z)$$

Let us check my algebra: for $t > 0$, $u(-t) = 1 + \frac{2(-\pi)}{\pi} = -1$ whereas $u(t) = 1 + \frac{2(0)}{\pi} = 1$.

Next, we wish to transfer the temperature distribution above to the disk via the Cayley map. We wish to pull-back the temperature function in z given by $u(z) = 1 + \dfrac{2}{\pi} Arg(z)$ to a corresponding function $U(w)$ for the disk $|w| \leq 1$. We accomplish the pull-back by setting $U(w) = u(h^{-1}(w))$. What is the inverse of the Cayley map ? We can find this by solving $\frac{z-i}{z+i} = w$ for z:

$$\frac{z-i}{z+i} = w \quad \Rightarrow \quad z - i = zw + iw \quad \Rightarrow \quad z - zw = i + iw \quad \Rightarrow \quad z = i\frac{1+w}{1-w}.$$

Hence $h^{-1}(w) = i\frac{1+w}{1-w}$. And we find the temperature distribution on the disk as desired:

$$\boxed{U(w) = 1 + \frac{2}{\pi} \text{Arg}\left(i\frac{1+w}{1-w}\right)}$$

We can check the answer here. Suppose $w = e^{it}$ then

$$\frac{1+e^{it}}{1-e^{it}} = \frac{e^{-it/2}+e^{it/2}}{e^{-it/2}-e^{it/2}} = \frac{\cos(t/2)}{-i\sin(t/2)} \quad \Rightarrow \quad \text{Arg}\left(i\frac{1+w}{1-w}\right) = \text{Arg}\left(-\frac{\cos(t/2)}{\sin(t/2)}\right)$$

Notice, for $0 < t < \pi$ we have $0 < t/2 < \pi/2$ hence $\cos(t/2) > 0$ and $\sin(t/2) > 0$ hence $\text{Arg}\left(-\frac{\cos(t/2)}{\sin(t/2)}\right) = -\pi$ and so $U(e^{it}) = -1$. On the other hand, if $-\pi < t < 0$ then $-\pi/2 < t/2 < 0$ and $\cos(t/2) > 0$ whereas $\sin(t/2) < 0$ hence $\text{Arg}\left(-\frac{\cos(t/2)}{\sin(t/2)}\right) = 0$ and so $U(e^{it}) = 1$.

Happily we have uncovered another bonus opportunity in the example above. It would seem I have a sign error somewhere, or, a misinterpretation. The solution above is exactly backwards. We have the top edge at $U = -1$ whereas the lower edge is at $U = 1$. Pragmatically, $-U(w)$ is the solution. But, I will award 5 or more bonus points to the student who explains this enigma.

Finally, a word or two about electrostatics. $E = E_1 + iE_2$ being the electric field is related to the **physical potential** V by $E = -dV$. This means $E = -\nabla V$ where V is the potential energy per unit charge or simply the potential. Notice Gamelin has $\phi = -V$ which means the relation between level curves of ϕ and E will not follow the standard commentary in physics. Note the field lines of E point towards higher levels of level curves for ϕ. In the usual story in physics, the field lines of E flow to lower voltage regions. In contrast, Just something to be aware of if you read Gamelin carefully and try to match it to the standard lexicon in physics. Of course, most multivariate calculus treatments share the same lack of insight in their treatment of "potential" functions. The reason for the sign in physics is simply that the choice causes the sum of kinetic and potential energy to be conserved. If we applied Gamelin's choice to physics we would find it necessary to conserve the difference of kinetic and potential energy. Which students might find odd. Setting aside this unfortunate difference in conventions, the example shown by Gamelin on pages 99-100 are pretty. You might constrast against my treatment of two-dimensional electrostatics in my 2014 Multivariable Calculus notes pages 368-371.

Chapter 9

Complex Integration and Analyticity

In this chapter we discover many surprising theorems which connect a holomorphic function and its integrals and derivatives. In part, the results here are merely a continuation of the complex-valued multivariate analysis studied in the previous chapter. However, the Theorem of Goursat and Cauchy's integral formula lead to striking results which are not analogus to the real theory. In particular, if a function is complex differentiable on a domain then Goursat's Theorem provides that $z \mapsto f'(z)$ is automatically a continuous mapping. There is no distinction between complex differentiable and continuously complex differentiable in the function theory on a complex domain. Moreover, if a function is once complex differentiable then it is twice complex differentiable. Continuing this thought, there is no distinction between the complex smooth functions and the complex once-differentiable functions on a complex domain. These distinctions are made in the real case and the distinctions are certainly aspects of the more subtle side of real analysis. These truths and more we discover in this chapter.

Before going into the future, let us pause to enjoy a quote by Gauss from 1811 to a letter to Bessel:

> What should we make of $\int \phi x \cdot dx$ for $x = a + bi$? Obviously, if we're to proceed from clear concepts, we have to assume that x passes, via infinitely small increments (each of the form $\alpha + i\beta$), from that value at which the integral is supposed to be 0, to $x = a + bi$ and that then all the $\phi x \cdot dx$ are summed up. In this way the meaning is made precise. But the progression of x values can take place in infinitely many ways: Just as we think of the realm of all real magnitudes as an infinite straight line, so we can envision the realm of all magnitudes, real and imaginary, as an infinite plane wherein every point which is determined by an abscissa a and ordinate b represents as well the magnitude $a + bi$. The continuous passage from one value of x to another $a + bi$ accordingly occurs along a curve and is consequently possible in infinitely many ways. But I maintain that the integral $\int \phi x \cdot dx$ computed via two different such passages always gets the same value as long as $\phi x = \infty$ never occurs in the region of the plane enclosed by the curves describing these two passages. This is a very beautiful theorem, whose not-so-difficult proof I will give when an appropriate occassion comes up. It is closedly related to other beautiful truths having to do with developing functions in series. The passage from point to point can always be carried out without touching one where $\phi x = \infty$. However, I demand that those points be avoided lest the original basic conception $\int \phi x \cdot dx$ lose its clarity and lead to contradictions. Moreover, it is also clear how a function generated by $\int \phi x \cdot dx$ could have several values for the same values of x depending on whether a

point where $\phi x = \infty$ is gone around not at all, once, or several times. If, for example, we define $\log x$ having gone around $x = 0$ one of more times or not at all, every circuit adds the constant $2\pi i$ or $-2\pi i$; thus the fact that every number has multiple logarithms becomes quite clear" (*Werke* 8, 90-92 according to [R91] page 167-168)

This quote shows Gauss knew complex function theory before Cauchy published the original monumental works on the subject in 1814 and 1825. Apparently, Poisson also published an early work on complex integration in 1813. See [R91] page 175.

9.1 Complex Line Integral

The definition of the complex integral is naturally analogus to the usual Riemann sum in $\mathbb{R}$. In the real integral one considers a partition of $x_o, x_1, \dots, x_n$ which divides $[a, b]$ into n-subintervals. In the complex integral, to integrate along a path γ we consider points $z_o, z_1, \dots, z_n$ along the path. In both cases, as $n \to \infty$ we obtain the integral.

Definition 9.1.1. *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path and $f(z)$ a complex-valued function which is continuous on and near γ. Let $z_o, z_1, \dots, z_n \in trace(\gamma)$ where $a \leq t_o < t_1 < \cdots < t_n \leq b$ and $\gamma(t_j) = z_j$ for $j = 0, 1, \dots, n$. We define:*

$$\int_\gamma f(z)\, dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1}).$$

Equivalently, as a complex-valued integral over the real parameter of the path:

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t)) \frac{d\gamma}{dt}\, dt.$$

Or, as a complex combination of real line-integrals:

$$\int_\gamma f(z)\, dz = \int_\gamma u dx - v dy + i \int_\gamma u dy + v dx.$$

And finally, in terms set in the previous chapter,

$$\int_\gamma f(z)\, dz = \int_\gamma f(z)\, dx + i \int_\gamma f(z)\, dy$$

The initial definition above is not our typical method of calculation! In fact, the boxed formulas we find in the next page or so are equivalent to the initial, Riemann sum definition given above. I thought I should start with this so you better appreciate the boxed-definitions which we uncover below. Consider,

$$z_j - z_{j-1} = \gamma(t_j) - \gamma(t_{j-1}) = \frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}}(t_j - t_{j-1})$$

Applying the mean value theorem we select $t_j^* \in [t_{j-1}, t_j]$ for which $\gamma'(t_j^*) = \frac{\gamma(t_j)-\gamma(t_{j-1})}{t_j - t_{j-1}}$. Returning to the integral, and using $\triangle t_j = t_j - t_{j-1}$ we obtain

$$\int_\gamma f(z)\, dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(\gamma(t_j)) \frac{d\gamma}{dt}(t_j^*) \triangle t_j = \boxed{\int_a^b f(\gamma(t)) \frac{d\gamma}{dt}\, dt.}$$

I sometimes use the boxed formula above as the definition of the complex integral. Moreover, in practice, we set $z = \gamma(t)$ as to symbolically replace dz with $\frac{dz}{dt}dt$. See Example 9.1.3 for an example of this notational convenience. That said, the expression above can be expressed as a complex-linear combination of two real integrals. If we denote $\gamma = x + iy$ and $f = u + iv$ then (I omit some t-dependence to make it fit in second line)

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \lim_{n\to\infty}\sum_{j=1}^{n}(u(\gamma(t_j)) + iv(\gamma(t_j)))\left(\frac{dx}{dt}(t_j^*) + i\frac{dy}{dt}(t_j^*)\right)\triangle t_j \\
&= \lim_{n\to\infty}\sum_{j=1}^{n}\left(u\circ\gamma\frac{dx}{dt} - v\circ\gamma)\frac{dy}{dt}\right)\triangle t_j + i\lim_{n\to\infty}\sum_{j=1}^{n}\left(u\circ\gamma\frac{dy}{dt} + v\circ\gamma\frac{dx}{dt}\right)\triangle t_j \\
&= \int_a^b\left(u(\gamma(t))\frac{dx}{dt} - v(\gamma(t))\frac{dy}{dt}\right)dt + i\int_a^b\left(u(\gamma(t))\frac{dy}{dt} + v(\gamma(t))\frac{dx}{dt}\right)dt \\
&= \boxed{\int_\gamma udx - vdy + i\int_\gamma udy + vdx.}
\end{aligned}$$

Thus, in view of the integrals of complex-valued differential forms defined in the previous chapter we can express the complex integral elegantly as $dz = dx + idy$ where this indicates

$$\int_\gamma f(z)\,dz = \boxed{\int_\gamma f(z)\,dx + i\int_\gamma f(z)\,dy.}$$

To summarize, we could reasonably use any of the boxed formulas to define $\int_\gamma f(z)\,dz$. In view of this comment, let us agree that we call all of these the *definition* of the complex integral. We will use the formulation which is most appropriate for the task at hand.

To integrate over a curve we follow the method laid out in Definition 8.1.7. To calculate the integral over a curve we calculate the integral over each path comprising the curve then we sum all the path integrals.

Definition 9.1.2. *In particular, if γ is a curve formed by joining the smooth paths $\gamma_1, \gamma_2, \dots, \gamma_n$. In terms of the trace denoted trace$(\gamma) = [\gamma]$ we have $[\gamma] = [\gamma_1] \cup [\gamma_2] \cup \cdots \cup [\gamma_n]$. Let $f(z)$ be complex valued and continuous near the trace of γ. Define:*

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{n}\int_{\gamma_j} f(z)\,dz.$$

Example 9.1.3. *Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the unit-circle $\gamma(t) = e^{it}$. Calculate $\int_\gamma \frac{dz}{z}$. Note, if $z = e^{it}$ then $dz = ie^{it}dt$ hence:*

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi}\frac{ie^{it}dt}{e^{it}} = i\int_0^{2\pi} dt = 2\pi i.$$

Example 9.1.4. *Let C be the line-segment from p to q parametrized by $t \in [0, 1]$; $z = p + t(q - p)$ hence $dz = (q - p)dt$. We calculate, for $n \in \mathbb{Z}$ with $n \neq -1$,*

$$\int_C z^n dz = \int_0^1 (p + t(q - p))^n (q - p)dt = \frac{(p + t(q - p))^{n+1}}{n + 1}\bigg|_0^1 = \frac{q^{n+1}}{n + 1} - \frac{p^{n+1}}{n + 1}.$$

Calculational Comment: For your convenience, let us pause to note some basic properties of an integral of a complex-valued function of a real variable. In particular, suppose $f(t), g(t)$ are continuous complex-valued functions of $t \in \mathbb{R}$ and $c \in \mathbb{C}$ and $a, b \in \mathbb{R}$ then

$$\int (f(t) + g(t))\, dt = \int f(t)dt + \int g(t)dt \qquad \& \qquad \int cf(t)dt = c\int f(t)dt$$

More importantly, the FTC naturally extends; if $\frac{dF}{dt} = f$ then

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Notice, this is not quite the same as first semester calculus. Yes, the formulas look the same, but, there is an important distinction. In the last example $p = 4 + 3i$ and $q = 13e^{i\pi/3}$ are possible. I don't think you had that in first semester calculus. You should notice the chain-rule you proved in Problem 27 is immensely useful in what follows from here on out. Often, as we calculate dz by $\frac{d\gamma}{dt}dt$ we have $\gamma(t)$ written as the composition of a holomorphic function of z and some simple function of t. I already used this in Examples 9.1.3 and 9.1.4. Did you notice?

Example 9.1.5. *Let $\gamma = [p, q]$ and let $c \in \mathbb{C}$ with $c \neq -1$. Recall $f(z) = z^c$ is generally a multiply-valued function whose set of values is given by $z^c = exp(c\log(z))$. Suppose p, q fall in a subset of $\mathbb{C}$ on which a single-value of z^c is defined and let z^c denote that* **function** *of z. Let $\gamma(t) = p + tv$ where $v = q - p$ for $0 \leq t \leq 1$ thus $dz = vdt$ and:*

$$\int_\gamma z^c dz = \int_0^1 (p + tv)^c v dt$$

notice $\frac{d}{dt}\left[\frac{(p+tv)^{c+1}}{c+1}\right] = (p+tv)^c v$ *as we know $f(z) = z^{c+1}$ has $f'(z) = (c+1)z^c$ and $\frac{d}{dt}(p+tv) = v$. The chain rule (proved by you in Problem 27) completes the thought. Consequently, by FTC for complex-valued integrals of a real variable,*

$$\int_\gamma z^c dz = \frac{(p+tv)^{c+1}}{c+1}\bigg|_0^1 = \frac{p^{c+1}}{c+1} - \frac{q^{c+1}}{c+1}.$$

The deformation theorem we discussed in the previous chapter is still of great utility here. We continue Example 9.1.3 to consider arbitrary loops.

Example 9.1.6. *The differential form $\omega = dz/z$ is closed on the punctured plane $\mathbb{C}^\times$. In particular,*

$$\omega = \frac{dx + idy}{x + iy} \quad \Rightarrow \quad P = \frac{1}{x+iy} \quad \& \quad Q = \frac{i}{x+iy}$$

Observe $\partial_x Q = \partial_y P$ for $z \neq 0$ thus ω is closed on $\mathbb{C}^\times$ as claimed. Let γ be a, postively oriented, simple, closed, curve containg the origin in its interior. Then by the deformation theorem we argue

$$\int_\gamma \frac{dz}{z} = 2\pi i.$$

since a simple closed loop which encircles the origin can be continuously deformed to the unit-circle.

Notice, in $\mathbb{C}^\times$, any loop not containing the origin can be smoothly deformed to a point in and thus it is true that $\int_\gamma \frac{dz}{z} = 0$ if 0 is not within the interior of the loop.

Example 9.1.7. *Let $R > 0$ and z_o a fixed point in the complex plane. Assume the integration is taken over a positively oriented parametrization of the pointset indicated: for $m \in \mathbb{Z}$,*

$$\int_{|z-z_o|=R} (z-z_o)^m \, dz = \begin{cases} 2\pi i & \text{for } m=-1 \\ 0 & \text{for } m \neq -1. \end{cases}$$

Let $z = z_o + Re^{it}$ for $0 \leq t \leq 2\pi$ parametrize $|z - z_o| = R$. Note $dz = iRe^{it}dt$ hence

$$\begin{aligned} \int_{|z-z_o|=R} (z-z_o)^m \, dz &= \int_0^{2\pi} (Re^{it})^m iRe^{it} dt \\ &= iR^{m+1} \int_0^{2\pi} e^{i(m+1)t} \, dt \\ &= iR^{m+1} \int_0^{2\pi} \big(\cos[(m+1)t] + i\sin[(m+1)t]\big) dt. \end{aligned}$$

The integral of any integer multiple of periods of trigonometric functions is trivial. However, in the case $m = -1$ the calculation reduces to $\int_{|z-z_o|=R}(z-z_o)^{-1}\,dz = i\int_0^{2\pi}\cos(0)dt = 2\pi i$. I encourage the reader to extend this calculation to arbitrary loops by showing the form $(z - z_o)^m dz$ is closed on at least the punctured plane.

Let γ be a loop containing z_o in its interior. An interesting aspect of the example above is the contrast of $\int_\gamma \frac{dz}{z-z_o} = 2\pi i$ and $\int_\gamma \frac{dz}{(z-z_o)^2} = 0$. One might be tempted to think that divergence at a point necessitates a non-trivial loop integral after seeing the $m = -1$ result. However, it is not the case. At least, not at this naive level of investigation. Later we will see the quadratic divergence generates nontrivial integrals for $f'(z)$. Cauchy's Integral formula studied in §4.4 will make this clear. Next, we consider less exact methods. Often, what follows it the only way to calculate something. In contrast to the usual presentation of real-valued calculus, the inequality theorem below is a weapon we will wield to conquer formiddable enemies later in this course. So, sharpen your blade now as to prepare for war.

Following Gamelin, denote the infinitesimal arclength $ds = |dz|$ and define the integral with respect to arclength of a complex-valued function by:

Definition 9.1.8. *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path and $f(z)$ a complex-valued function which is continuous on and near γ. Let $z_o, z_1, \dots, z_n \in trace(\gamma)$ where $a \leq t_o < t_1 < \cdots < t_n \leq b$ and $\gamma(t_j) = z_j$ for $j = 0, 1, \dots, n$. We define:*

$$\int_\gamma f(z)\,|dz| = \lim_{n\to\infty} \sum_{j=1}^{n} f(z_j)|z_j - z_{j-1}|.$$

Equivalently, as a complex-valued integral over the real parameter of the path:

$$\int_\gamma f(z)\,|dz| = \int_a^b f(\gamma(t)) \left|\frac{d\gamma}{dt}\right| dt.$$

We could express this as a complex-linear combination of the standard real-arclength integrals of multivariate calculus, but, I will abstain. It is customary in Gamelin to denote the length of the path γ by L. We may calculate L by integration of $|dz|$ along $\gamma = x + iy : [a, b] \to \mathbb{C}$:

$$L = \int_\gamma |dz| = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt.$$

Of course, this is just the usual formula for arclength of a parametrized curve in the plane. The Theorem below is often called the **ML-estimate** or **ML-theorem** throughout the remainder of this course.

Theorem 9.1.9. *Let $h(z)$ be a continuous near a smooth path γ with length L. Then*

1. $\left|\int_\gamma h(z)\,dz\right| \leq \int_\gamma |h(z)|\,|dz|$.

2. *If* $|h(z)| \leq M$ *for all* $z \in [\gamma]$ *then* $\left|\int_\gamma h(z)\,dz\right| \leq ML$.

Proof: in terms of the Riemann sum formulation of the complex integral and arclength integral the identities above are merely consequences of the triangle inequality applied to a particular approximating sum. Note:

$$\left|\sum_{j=1}^{n} h(z_j)(z_j - z_{j-1})\right| \leq \sum_{j=1}^{n} |h(z_j)(z_j - z_{j-1})| = \sum_{j=1}^{n} |h(z_j)||z_j - z_{j-1}|$$

where we used **multiplicativity of the norm**[1] in the last equality and the triangle inequality in the first inequality. Now, as $n \to \infty$ we obtain (1.). The proof of (2.) is one more step:

$$\left|\sum_{j=1}^{n} h(z_j)(z_j - z_{j-1})\right| \leq= \sum_{j=1}^{n} |h(z_j)||z_j - z_{j-1}| \leq \sum_{j=1}^{n} M|z_j - z_{j-1}| = M\sum_{j=1}^{n} |z_j - z_{j-1}| = ML. \quad \square$$

I should mention, last time I taught this course I tried to prove this on the fly directly from the definition written as $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\frac{d\gamma}{dt}\,dt$. It went badly. There are proofs which are not at the level of the Riemann sum and it's probably worthwhile to share a second proof. I saw this proof in my complex analysis course given by my advisor Dr. R.O. Fulp in 2005 at NCSU.

Alternate Proof: we begin by developing a theorem for complex-valued functions of a real-variable. We claim **Lemma:** $\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|\,dt$. Notice that $w(t)$ denotes the modulus of the complex value $w(t)$. If $w(t) = 0$ on $[a, b]$ then the claim is true. Hence, suppose $w(t)$ is continuous and hence the integral of $w(t)$ exists and we set $R > 0$ and $\theta \in \mathbb{R}$ such that $\int_a^b w(t)dt = Re^{i\theta}$. Let's get real: in particular $R = e^{-i\theta}\int_a^b w(t)dt = \int_a^b e^{-i\theta}w(t)dt$ hence:

$$\begin{aligned}
R &= \int_a^b e^{-i\theta}w(t)dt \\
&= \textbf{Re}\left(\int_a^b e^{-i\theta}w(t)dt\right) \\
&= \int_a^b \textbf{Re}(e^{-i\theta}w(t))\,dt \\
&\leq \int_a^b \left|e^{-i\theta}w(t)\right|dt \qquad \text{due to a property of modulus; } \textbf{Re}(z) \leq |z| \\
&= \int_a^b |w(t)|\,dt
\end{aligned}$$

[1]Bailu, here is a spot we need sub-multiplicativity over $\mathcal{A}$. We will get a modified ML-theorem accordinng to the size of the structure constants.Note, the alternate proof would not go well in $\mathcal{A}$ since we do not have a polar representation of an arbtrary $\mathcal{A}$-number.

Thus, the Lemma follows as: $|\int_a^b w(t)\,dt| = |Re^{i\theta}| \leq \int_a^b |w(t)|\,dt$. Now, suppose $h(z)$ is complex-valued and continuous near $\gamma : [a,b] \to \mathbb{C}$. We calculate, using the Lemma, then multiplicative property of the modulus:

$$\left|\int_\gamma h(z)\,dz\right| = \left|\int_a^b h(\gamma(t))\frac{d\gamma}{dt}\,dt\right| \leq \int_a^b \left|h(\gamma(t))\frac{d\gamma}{dt}\right|\,dt = \int_a^b |h(\gamma(t))|\,\left|\frac{d\gamma}{dt}\right|\,dt = \int_\gamma |h(z)|\,|dz|.$$

This proves (1.) and the proof of (2.) is essentially the same as we discussed in the first proof. □

Example 9.1.10. *Consider $h(z) = 1/z$ on the unit-circle γ. Clearly, $|z| = 1$ for $z \in [\gamma]$ hence $|h(z)| = 1$ which means this estimate is* **sharp***, it cannot be improved. Furthermore, $L = 2\pi$ and the ML-estimate shows $\left|\int_\gamma \frac{dz}{z}\right| \leq 2\pi$. Indeed, in Example 9.1.3 $\int_\gamma \frac{dz}{z} = 2\pi i$ so the estimate is not too shabby.*

Typically, the slightly cumbersome part of applying the ML-estimate is fiinding M. Helpful techniques include: using the polar form of a number, $\mathbf{Re}(z) \leq |z|$ and $\mathbf{Im}(z) \leq |z|$ and naturally $|z + w| \leq |z| + |w|$ as well as $|z - w| \geq ||z| - |w||$ which is useful for manipulating denomiinators.

Example 9.1.11. *Let γ_R be the half-circle of radius R going from R to $-R$ on the real-axis. Find an bound on the modulus of $\int_{\gamma_R} \frac{dz}{z^2+6}$. Notice, on the circle we have $|z| = R$. Furthermore,*

$$\frac{1}{|z^2+6|} \leq \frac{1}{||z^2| - |6||} = \frac{1}{||z|^2 - 6|} = \frac{1}{|R^2 - 6|}$$

If $R > \sqrt{6}$ then we have bound $M = \frac{1}{R^2-6}$ for which $|h(z)| \leq M$ for all $z \in \mathbb{C}$ with $|z| = R$. Note, $L = \pi R$ for the half-circle and the ML-estimate gives:

$$\left|\int_{\gamma_R} \frac{dz}{z^2+6}\right| \leq \frac{\pi R}{R^2 - 6}.$$

Notice, if we consider $R \to \infty$ then we find from the estimate above and the squeeze theorem that $\left|\int_{\gamma_R} \frac{dz}{z^2+6}\right| \to 0$. It follows that the integral of $\frac{dz}{z^2+6}$ over an infinite half-circle is zero.

A similar calculation shows any rational function $f(z) = p(z)/q(z)$ with $deg(p(z)) + 2 \leq deg(q(z))$ has an integral which vanishes over sections of a cricle which has an infinite radius.

9.2 Fundamental Theorem of Calculus for Analytic Functions

The term **primitive** means antiderivative. In particular:

Definition 9.2.1. *We say $F(z)$ is a* **primitive** *of $f(z)$ on D iff $F'(z) = f(z)$ for each $z \in D$.*

The fundamental theorem of calculus part II[2] has a natural analog in our context.

Theorem 9.2.2. Complex FTC II: *Let $f(z)$ be continuous with primitive $F(z)$ on D then if γ is a path from A to B in D then*

$$\int_\gamma f(z)\,dz = F(b) - F(a).$$

[2]following the usual American textbook ordering

Proof: recall the complex derivative can be cast as a partial derivative with respect to x or y in the following sense: $\frac{dF}{dz} = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$. Thus:

$$\begin{aligned} \int_\gamma f(z)\,dz = \int_\gamma \frac{dF}{dz}\,dz &= \int_\gamma \frac{dF}{dz}\,dx + i\int_\gamma \frac{dF}{dz}\,dy \\ &= \int_\gamma \frac{\partial F}{\partial x}\,dx + i\int_\gamma -i\frac{\partial F}{\partial y}\,dy \\ &= \int_\gamma \left(\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy\right) \\ &= \int_\gamma dF \\ &= F(B) - F(A). \end{aligned}$$

where we used Theorem 8.2.2 in the last step[3]. $\square$

In the context of the above theorem we sometimes use the notation $\int_\gamma f(z)\,dz = \int_A^B f(z)\,dz$. This notation should be used with care.

Example 9.2.3.

$$\int_0^{1+3i} z^3\,dz = \frac{1}{4}z^4\bigg|_0^{1+3i} = \frac{(1+3i)^4}{4}.$$

The example below is essentially given on page 108 of Gamelin. I make the ϵ in Gamelin's example explicit.

Example 9.2.4. *The function $f(z) = 1/z$ has primitive $Log(z) = \ln|z| + iArg(z)$ on $\mathbb{C}^-$. We can capture the integral around the unit-circle by a limitiing process. Consider the unit-circle, positively oriented, with an ϵ-sector deleted just below the negative x-axis; $\gamma_\epsilon : [-\pi+\epsilon, \pi] \to \mathbb{C}$ with $\gamma(t) = e^{it}$. The path has starting point $\gamma(\pi) = e^{i\pi}$ and ending point $\gamma(-\pi+\epsilon) = e^{i(-\pi+\epsilon)}$. Note $[\gamma_\epsilon] \subset \mathbb{C}^-$ hence for each $\epsilon > 0$ we are free to apply the complex FTC:*

$$\int_{\gamma_\epsilon} \frac{dz}{z} = Log(e^{i\pi}) - Log(e^{i(-\pi+\epsilon)}) = 2\pi i + i\epsilon.$$

Thus, as $\epsilon \to 0$ we find $2\pi i + i\epsilon \to 2\pi i$ and $\gamma_\epsilon \to \gamma_0$ where γ_0 denotes the positively oriented unit-circle. Therefore, we find: $\int_{\gamma_0} \frac{dz}{z} = 2\pi i$.

The example above gives us another manner to understand Example 9.1.3. It all goes back to the $2\pi\mathbb{Z}$ degeneracy of the standard angle. Let us continue to what Gamelin calls **Fundamental Theorem of Calculus (II)**. Which, I find funny, since the American text books tend to have I and II reversed from Gamelin's usage.

Theorem 9.2.5. Complex FTC I: *let D be star-shaped and let $f(z)$ be holomorphic on D. Then $f(z)$ has a primitive on D and the primitive is unique up to an additive constant. A primitive for $f(z)$ is given by*[4]

$$F(z) = \int_{z_o}^{z} f(\zeta)\,d\zeta$$

where z_o is a star-center of D and the integral is taken along some path in D from z_o to z.

[3]alternative proof: try to derive it via $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$ and Problem 27.

[4]the symbol ζ is used here since z has another set meaning, this is the Greek letter "zeta"

Proof: the basic idea is simply to use Theorem 8.2.10. We need to show $f(z)dz$ is a closed form. Let $f = u + iv$ then:

$$f(z)\,dz = (u+iv)(dx+idy) = \underbrace{(u+iv)}_{P}\,dx + \underbrace{(iu-v)}_{Q}\,dy$$

We wish to show $Q_x = P_y$. Remember, $u_x = v_y$ and $v_x = -u_y$ since $f = u + iv \in \mathcal{O}(D)$,

$$Q_x = iu_x - v_x = iv_y + u_y = P_y.$$

Therefore, the form $f(z)dz$ is closed on the star-shaped domain D hence by the proof of Theorem 8.2.10 the form $f(z)dz$ is exact with potential given by:

$$F(z) = \int_{z_o}^{z} f(\zeta)d\zeta$$

where we identify in our current lexicon $F(z)$ is a primitive of $f(z)$. $\square$.

The assumption of star-shaped (or simply connected to be a bit more general) is needed since there are closed forms on domains with holes which are not exact. The standard example is $\mathbb{C}^\times$ where $\frac{dz}{z}$ is closed, but $\int_{|z|=1} \frac{dz}{z} = 2\pi i$ shows we cannot hope for a primitive to exist on all of $\mathbb{C}^\times$. If such a primitive did exist then the integral around $|z| = 1$ would necessarily be zero which contradicts the always important Example 9.1.3.

9.3 Cauchy's Theorem

It seems we accidentally proved the theorem below in the proof of Theorem 9.2.5.

Theorem 9.3.1. original form of Morera's Theorem: *a continuously differentiable function $f(z)$ on D is holomorphic on D if and only if the differential $f(z)dz$ is closed.*

Proof: If $f(z)$ is holomorphic on D then

$$f(z)\,dz = (u+iv)(dx+idy) = \underbrace{(u+iv)}_{P}\,dx + \underbrace{(iu-v)}_{Q}\,dy$$

and the Cauchy Riemann equations for u, v yield:

$$Q_x = iu_x - v_x = iv_y + u_y = P_y.$$

Conversely, let $f = u + iv$ and note if $f(z)\,dz = Pdx + Qdy$ is closed then this forces u, v to solve the CR-equations by the algebra in the forward direction of the proof. However, we also are given $f(z)$ is continuously differentiable hence $f(z)$ is holomorphic by part (3.) of Theorem 7.2.3. $\square$

Apply Green's Theorem to obtain Cauchy's Theorem:

Theorem 9.3.2. Cauchy's Theorem: *let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is holomorphic and continuously differentiable on D and extends continuously to ∂D then $\displaystyle\int_{\partial D} f(z)\,dz = 0$.*

Proof: assume $f(z)$ is holomorphic on D then Theorem 9.3.1 tells us $f(z)\,dz = Pdx + Qdy$ is closed. Apply Green's Theorem 8.1.10 to obtain $\int_{\partial D} f(z)\,dz = \iint_D (Q_x - P_y)$ but as $Pdx + Qdy$ is closed we know $Q_x = P_y$ hence $\int_{\partial D} f(z)\,dz = 0$ $\square$

Notice, Green's Theorem extends to regions with interior holes in a natural manner: the boundary of interior holes is given a CW-orientation whereas the exterior boundary is given CCW-orientation. It follows that a holomorphic function on an annulus must have integrals on the inner and outer boundaries which cancel. See the discussion before Theorem 8.2.12 for a simple case with one hole. Notice how the CW-orientation of the inner curve allows us to chop the space into two positively oriented simple curves. That construction can be generalized, perhaps you will explore it in homework.

I am pleased with the integration of the theory of exact and closed forms which was initiated in the previous chapter. But, it's probably wise for us to pause on a theorem as important as this and see the proof in a self-contained fashion.

Stand Alone Proof: If $f(z)$ is holomorphic with continuous $f'(z)$ on D and extends continuously to ∂D. Let $f(z) = u + iv$ and use Green's Theorem for complex-valued forms:

$$\begin{aligned}
\int_{\partial D} f(z)\,dz &= \int_{\partial D} (u+iv)(dx+idy) \\
&= \int_{\partial D} (u+iv)dx + (iu-v)dy \\
&= \iint_D \left(\frac{\partial(iu-v)}{\partial x} - \frac{\partial(u+iv)}{\partial y} \right) dA \\
&= \iint_D (iu_x - v_x - u_y - iv_y)\,dA \\
&= \iint_D (0)\,dA \\
&= 0.
\end{aligned}$$

where we used the CR-equations $u_x = v_y$ and $v_x = -u_y$ to cancel terms. $\square$

Technically, the assumption in both proofs above of the continuity of $f'(z)$ throughout D is needed in order that Green's Theorem apply. That said, we shall soon study Goursat's Theorem and gain an appreciation for why this detail is superfluous[5]

Example 9.3.3. *The function $f(z) = \frac{2z}{1+z^2}$ has natural domain of $\mathbb{C} - \{i, -i\}$. Moreover, partial fractions decomposition provides the following identity:*

$$f(z) = \frac{z+i+z-i}{(z-i)(z+i)} = \frac{1}{z+i} + \frac{1}{z-i}$$

If $\epsilon < 1$ and $\gamma_\epsilon(p)$ denotes the circle centered at p with positive orientation and radius ϵ then I invite the student to verify that:

$$\int_{\gamma_\epsilon(-i)} \frac{dz}{z+i} = 2\pi i \qquad \& \qquad \int_{\gamma_\epsilon(-i)} \frac{dz}{z-i} = 0$$

[5]you may recall Gamelin's definition of analytic assumes the continuity of $z \mapsto f'(z)$. This is Gamelin's way of saying,"this detail need not concern the beginning student" Remember, I have replaced analytic with holomorphic throughout this guide. Although, the time for the term "analytic" arises in the next chapter.

whereas

$$\int_{\gamma_\epsilon(i)} \frac{dz}{z+i} = 0 \qquad \& \qquad \int_{\gamma_\epsilon(i)} \frac{dz}{z-i} = 2\pi i.$$

Suppose D is a domain which includes $\pm i$. Let $S = D - interior(\gamma_\epsilon(\pm i))$. That is, S is the domain D with the points inside the circles $\gamma_\epsilon(-i)$ and $\gamma_\epsilon(i)$ deleted. Furthermore, we suppose ϵ is small enough so that the circles are interior to D. This is possible as we assumed D is an open connected set when we said D is a domain. All of this said: note f is holmorphic on S since $f(z)$ is cleary complex-differentiable near each point in S and thus we may apply Cauchy's Theorem on S:

$$0 = \int_{\partial S} \frac{2zdz}{z^2+1} = \int_{\partial D} \frac{2zdz}{z^2+1} - \int_{\gamma_\epsilon(-i)} \left(\frac{dz}{z+i} + \frac{dz}{z-i}\right) - \int_{\gamma_\epsilon(i)} \left(\frac{dz}{z+i} + \frac{dz}{z-i}\right)$$

But, we know the integrals around the circles and it follows:

$$\int_{\partial D} \frac{2dz}{z^2+1} = 4\pi i.$$

Notice the nontriviality of the integral above is due to the singular points $\pm i$ in the domain.

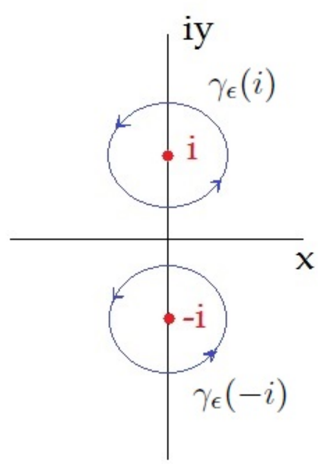
Look back at Example 9.1.7 if you are rusty on how to calculate the integrals around the circles. It is fun to think about the calculation above in terms of what we can and can't do with logarithms:

$$\int_{\gamma_\epsilon(-i)} \left(\frac{dz}{z+i} + \frac{dz}{z-i}\right) = \int_{\gamma_\epsilon(-i)} \left(\frac{dz}{z+i} + d[\log(z-i)]\right) = \int_{\gamma_\epsilon(-i)} \frac{dz}{z+i} = 2\pi i.$$

where the $\log(z-i)$ is taken to be a branch of the logarithm which is holomorphic on the given circle; for example, $\log(z-i) = \mathrm{Log}_{\pi/2}(z-i)$ would be a reasonable choice since the circle is centered at $z=-i$ which falls on $\theta = -\pi/2$. The jump in the $\mathrm{Log}_{\pi/2}(z-i)$ occurs away from where the integration is taken and so long as $\epsilon < 1$ we have that $dz/(z-i)$ is exact with potential $\mathrm{Log}_{\pi/2}(z-i)$. That said, we prefer the notation $\log(z-i)$ when the details are not important to the overall calculation. Notice, see for $dz/(z+i)$ as the differential of a logarithm because the circle of integration necessarily contains the singularity which forbids the existence of the logarithm on the whole punctured plane $\mathbb{C} - \{-i\}$. Similarly,

$$\int_{\gamma_\epsilon(i)} \left(\frac{dz}{z+i} + \frac{dz}{z-i}\right) = \int_{\gamma_\epsilon(i)} \left(d[\log(z+i)] + \frac{dz}{z-i}\right) = \int_{\gamma_\epsilon(i)} \frac{dz}{z-i} = 2\pi i$$

is a slick notation to indicate the use of an appropriate branch of $\log(z+i)$. In particular, $\mathrm{Log}_{-\pi/2}(z+i)$ is appropriate for $\epsilon < 1$.



9.4 The Cauchy Integral Formula

Once again, when we assume holomorphic on a domain we also add the assumption of continuity of $f'(z)$ on the domain. Gamelin assumes continuity of $f'(z)$ when he says $f(z)$ is *analytic* on D. As I have mentioned a few times now, we show in Section 9.7 that $f(z)$ holomorphic on a domain automatically implies that $f'(z)$ is continuous. This means we can safely delete the assumption of continuity of $f'(z)$ once we understand Goursat's Theorem.

The theorem below is rather surprising in my opinion.

Theorem 9.4.1. Cauchy's Integral Formula $(m = 0)$**:** *let D be a bounded domain with piecewise smooth boundary ∂D. If $f(z)$ is holomorphic with continuous $f'(z)$ on D and $f(z), f'(z)$ extend continuously to ∂D then for each $z \in D$,*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw$$

Proof: Assume the preconditions of the theorem. Fix a point $z \in D$. Note D is open hence z is interior thus we are free to choose $\epsilon > 0$ for which $\{w \in \mathbb{C} \mid |w-z| < \epsilon\} \subseteq D$. Define:

$$D_\epsilon = D - \{w \in \mathbb{C} \mid |w-z| \leq \epsilon\}$$

Observe the boundary of D_ϵ consists of the outer boundary ∂D and the circle γ_ϵ^- which is $|w-z| = \epsilon$ given CW-orientation; $\partial D_\epsilon = \partial D \cup \gamma_\epsilon^-$. Further, observe $g(w) = \frac{f(w)}{w-z}$ is holomorphic as

$$g'(w) = \frac{f'(w)}{w-z} - \frac{f(w)}{(w-z)^2}$$

and $g'(w)$ continuous on D_ϵ and $g(w), g'(w)$ both extend continuously to ∂D_ϵ as we have assumed from the outset that $f(w), f'(w)$ extend likewise. We obtain from Cauchy's Theorem 9.3.2 that:

$$\int_{\partial D_\epsilon} \frac{f(w)}{w-z}dw = 0 \quad \Rightarrow \quad \int_{\partial D} \frac{f(w)}{w-z}dw + \int_{\gamma_\epsilon^-} \frac{f(w)}{w-z}dw = 0.$$

However, if γ_ϵ^+ denotes the CCW-oriented circle, we have $\int_{\gamma_\epsilon^-} \frac{f(w)}{w-z}dw = -\int_{\gamma_\epsilon^+} \frac{f(w)}{w-z}dw$ hence:

$$\int_{\partial D} \frac{f(w)}{w-z}dw = \int_{\gamma_\epsilon^+} \frac{f(w)}{w-z}dw$$

The circle γ_ϵ^+ has $w = z + \epsilon e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ thus $dz = i\epsilon e^{i\theta}d\theta$ and we calculate:

$$\int_{\gamma_\epsilon^+} \frac{f(w)}{w-z}dw = \int_0^{2\pi} \frac{f(z+\epsilon e^{i\theta})}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}d\theta = 2\pi i \int_0^{2\pi} f(z+\epsilon e^{i\theta})\frac{d\theta}{2\pi} = 2\pi i f(z).$$

In the last step we used the Mean Value Property given by Theorem 8.4.2. Finally, solve for $f(z)$ to obtain the desired result. $\square$

We can formally derive the higher-order formulae by differentiation:

$$f'(z) = \frac{1}{2\pi i}\frac{d}{dz}\int_{\partial D} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\partial D} \frac{d}{dz}\left[\frac{f(w)}{w-z}\right]\,dw = \frac{1!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^2}\,dw$$

Differentiate once more,

$$f''(z) = \frac{1}{2\pi i}\frac{d}{dz}\int_{\partial D}\frac{f(w)}{(w-z)^2}\,dw = \frac{1}{2\pi i}\int_{\partial D}\frac{d}{dz}\left[\frac{f(w)}{(w-z)^2}\right]dw = \frac{2!}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^3}\,dw$$

continuing, we would arrive at:

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^{m+1}}\,dw$$

which is known as **Cauchy's generalized integral formula**. Note that $0! = 1$ and $f^{(0)}(z) = f(z)$ hence Theorem 9.4.1 naturally fits into the formula above.

It is probably worthwhile to examine a proof of the formulas above which is not based on differentiating under the integral. The arguments below show that our formal derivation above were valid. In the case $m = 1$ the needed algebra is simple enough:

$$\frac{1}{w-(z+\triangle z)} - \frac{1}{w-z} = \frac{\triangle z}{(w-(z+\triangle z))(w-z)}.$$

Then, appealing to the $m = 0$ case to write the functions as integrals:

$$\begin{aligned}\frac{f(z+\triangle z)-f(z)}{\triangle z} &= \frac{1}{2\pi i\triangle z}\int_{\partial D}\frac{1}{w-(z+\triangle z)}dw + \frac{1}{2\pi i\triangle z}\int_{\partial D}\frac{1}{w-z}dw \\ &= \frac{1}{2\pi i\triangle z}\int_{\partial D}\left[\frac{1}{w-(z+\triangle z)} - \frac{1}{w-z}\right]f(w)\,dw \\ &= \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-(z+\triangle z))(w-z)}dw.\end{aligned}$$

Finally, as $\triangle z \to 0$ we find $f'(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^2}\,dw$. We assume that the limiting process $\triangle z \to 0$ can be interchanged with the integration process. Gamelin comments this is acceptable due to the uniform continuity of the integrand.

We now turn to the general case, assume Cauchy's generalized integral formula holds for $m-1$. We need to make use of the binomial theorem:

$$((w-z)+\triangle z)^m = (w-z)^m - m(w-z)^{m-1}\triangle z + \frac{m(m-1)}{2}(w-z)^{m-2}(\triangle z)^2 + \cdots + (\triangle z)^m$$

Clearly, we have $((w-z)+\triangle z)^m = (w-z)^m - m(w-z)^{m-1}\triangle z + g(z,w)(\triangle z)^2$ It follows that:

$$\begin{aligned}\frac{1}{(w-(z+\triangle z))^m} - \frac{1}{(w-z)^m} &= \frac{m(w-z)^{m-1}\triangle z + g(z,w)(\triangle z)^2}{(w-(z+\triangle z))^m(w-z)^m} \\ &= \frac{m\triangle z}{(w-(z+\triangle z))(w-z)^m}. + \frac{g(z,w)(\triangle z)^2}{(w-(z+\triangle z))^m(w-z)^m}\end{aligned}$$

Apply the induction hypothesis to obtain the integrals below: $\dfrac{f^{(m-1)}(z+\triangle z) - f^{(m-1)}(z)}{\triangle z} =$

$$\begin{aligned}&= \frac{(m-1)!}{2\pi i\triangle z}\int_{\partial D}\frac{f(w)}{(w-(z+\triangle z))^m}dw + \frac{(m-1)!}{2\pi i\triangle z}\int_{\partial D}\frac{f(w)}{(w-z)^m}dw \\ &= \frac{(m-1)!}{2\pi i\triangle z}\int_{\partial D}\left[\frac{m\triangle z}{(w-(z+\triangle z))(w-z)^m}. + \frac{g(z,w)(\triangle z)^2}{(w-(z+\triangle z))^m(w-z)^m}\right]f(w)\,dw \\ &= \frac{m!}{2\pi i}\int_{\partial D}\frac{f(w)dw}{(w-(z+\triangle z))(w-z)^m}. + \frac{(m-1)!}{2\pi i}\int_{\partial D}\frac{g(z,w)\triangle z f(w)dw}{(w-(z+\triangle z))^m(w-z)^m}.\end{aligned}$$

As $\triangle z \to 0$ we see the right integral vanishes and the left integral has a denominator which tends to $(w-z)^{m+1}$ hence, by the definition of the m-th derivative,

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)dw}{(w-z)^{m+1}}$$

The arguments just given provide proof of the following theorem:

Theorem 9.4.2. Cauchy's Generalized Integral Formula ($m \in \mathbb{N} \cup \{0\}$)**:** *let D be a bounded domain with piecewise smooth boundary ∂D. If $f(z)$ is holomorphic with continuous $f'(z)$ on D and $f(z), f'(z)$ extend continuously to ∂D then for each $z \in D$,*

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,dw$$

Often we need to use the theorem above with the role of z as the integration variable. For example:

$$f^{(m)}(z_o) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(z)}{(z-z_o)^{m+1}}\,dz$$

from which we obtain the useful identity:

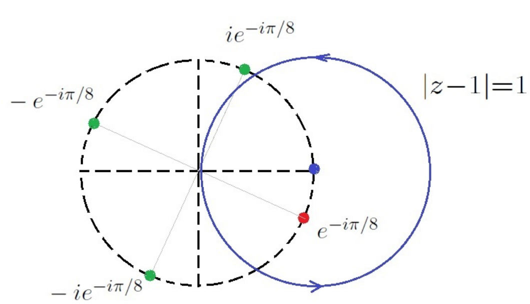
$$\boxed{\int_{\partial D} \frac{f(z)}{(z-z_o)^{m+1}}\,dz = \frac{2\pi i f^{(m)}(z_o)}{m!}}$$

This formula allows us to calculate many difficult integrals by simple evaluation of an approrpriate derivative. That said, we do improve on this result when we uncover the technique of residues later in the course. Think of this as an intermediate step in our calculational maturation.

Example 9.4.3. *Let the integral below be taken over the CCW-oriented curve $|z| = 1$:*

$$\oint_{|z|=2} \frac{\sin(2z)}{(z-i)^6}\,dz = \frac{2\pi i}{5!}\frac{d^5}{dz^5}\bigg|_{z=i}\sin(2z) = \frac{2\pi i}{5\cdot 4\cdot 3\cdot 2}(-32\cos(2i)) = \frac{-8\pi i\cosh(2)}{15}.$$

Example 9.4.4. *Notice that $z^4 + i = 0$ for $z \in (-i)^{1/4} = \left(e^{-i\pi/2}\right)^{1/4} = e^{-i\pi/8}\{1, i, -1, -i\}$ hence $z^4 + i = (z - e^{-i\pi/8})(z - ie^{-i\pi/8})(z + e^{-i\pi/8})(z + ie^{-i\pi/8})$. Consider the circle $|z - 1| = 1$ (blue). The dotted circle is the unit-circle and the intersection near $ie^{-i\pi/8}$ is at $\theta = \pi/3$ which is roughly as illustrated.*



The circle of integration below encloses the principal root (red), but not the other three non-principal fourth roots of $-i$ (green). Consequently, we apply Cauchy's integral formula based on the divergence *of the principal root:*

$$
\begin{aligned}
\oint_{|z-1|=1} \frac{dz}{z^4+i} &= \oint_{|z-1|=1} \frac{dz}{(z-e^{-i\pi/8})(z-ie^{-i\pi/8})(z+e^{-i\pi/8})(z+ie^{-i\pi/8})} \\
&= \frac{2\pi i}{(z-ie^{-i\pi/8})(z+e^{-i\pi/8})(z+ie^{-i\pi/8})}\Bigg|_{z=e^{-i\pi/8}} \\
&= \frac{2\pi i}{(e^{-i\pi/8}-ie^{-i\pi/8})(e^{-i\pi/8}+e^{-i\pi/8})(e^{-i\pi/8}+ie^{-i\pi/8})} \\
&= \frac{2\pi i}{e^{-3i\pi/8}(1-i)(1+1)(1+i)} \\
&= \frac{\pi i}{2}e^{3i\pi/8}.
\end{aligned}
$$

Of course, you could simplify the answer further and present it in Cartesian form.

Some of the most interesting applications involve integrations whose boundaries are allowed to expand to infinity. We saw one such example in Problem 53 which was #1 from $IV.3$ in Gamelin. The key in all of our problems is that we must identify the divergent points for the integrand. Provided they occur either inside or outside the curve we proceed as we have shown in the examples above. We do study divergences on contours later in the course, there are some precise results which are known for improper integrals of that variety[6]

Finally, one last point:

Corollary 9.4.5. *If $f(z)$ is holomorphic with continuous derivative $f'(z)$ on a domain D then $f(z)$ is infinitely complex differentiable. That is, $f', f'', \dots$ all exist and are continuous on D.*

The proof of this is that Cauchy's integral formula gives us an explicit expression (which exists) for any possible derivative of f. There are no just once or twice continuously complex differentiable functions. You get one continuous derivative on a domain, you get infinitely many. Pretty good deal. Moreover, the continuity of the derivative is not even needed as we discover soon.

9.5 Liouville's Theorem

It is our convention to say $f(z)$ is holomorphic on a closed set D iff there exists an open set $\tilde{D}$ containing D on which $f(z) \in \mathcal{O}(\tilde{D})$. Consider a function $f(z)$ for which $f'(z)$ exists and is continuous for $z \in \mathbb{C}$ such that $|z-z_o| \leq \varepsilon$. In such a case Cauchy's integral formula applies: for $\rho < \varepsilon$,

$$f^{(m)}(z_o) = \frac{m!}{2\pi i}\int_{|z-z_o|=\rho} \frac{f(z)}{(z-z_o)^{m+1}}\,dz$$

We parametrize the circle by $z = z_o + \rho e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ where $dz = i\rho e^{i\theta}d\theta$. Therefore,

$$f^{(m)}(z_o) = \frac{m!}{2\pi i}\int_0^{2\pi} \frac{f(z_o+\rho e^{i\theta})}{(\rho e^{i\theta})^{m+1}}\, i\rho e^{i\theta} d\theta = \frac{m!}{2\pi \rho^m}\int_0^{2\pi} f(z_o+\rho e^{i\theta})e^{-im\theta}d\theta$$

If we have $|f(z_o+\rho e^{i\theta})| \leq M$ for $0 \leq \theta \leq 2\pi$ then the we find

$$\left|\int_0^{2\pi} f(z_o+\rho e^{i\theta})e^{-im\theta}d\theta\right| \leq \int_0^{2\pi}\left|f(z_o+\rho e^{i\theta})e^{-im\theta}\right|d\theta = \int_0^{2\pi}\left|f(z_o+\rho e^{i\theta})\right|d\theta \leq 2\pi M.$$

The discussion above serves to justify the bound given below:

[6] in particular, see §$VII.5$ if you wish

Theorem 9.5.1. Cauchy's Estimate: *suppose $f(z)$ is holomorphic with continuous derivative on a domain D then for any closed disk $\{z \in \mathbb{C} \mid |z - z_o| \leq \epsilon\} \subset D$ on which $|f(z)| \leq M$ for all $z \in \mathbb{C}$ with $|z - z_o| = \rho < \varepsilon$ we find*

$$\left| f^{(m)}(z_o) \right| \leq \frac{Mm!}{\rho^m}$$

Many interesting results flow from the estimate above. For example:

Theorem 9.5.2. Liouville's Theorem: *Suppose $f(z)$ is holomorphic with continuous derivative on $\mathbb{C}$. If $|f(z)| \leq M$ for all $z \in \mathbb{C}$ then $f(z)$ is constant.*

Proof: Assume $f(z), f'(z)$ are continuous on $\mathbb{C}$ and $|f(z)| \leq M$ for all $\mathbb{C}$. Let us consider the disk of radius R centered at z_o. From Cauchy's Estimate with $m = 1$ we obtain:

$$\left| f'(z_o) \right| \leq \frac{M}{R}.$$

Observe, as $R \to \infty$ we find $|f'(z_o)| \to 0$ hence $f'(z_o) = 0$. But, z_o was an arbitrary point in $\mathbb{C}$ hence $f'(z) = 0$ for all $z \in \mathbb{C}$ and as $\mathbb{C}$ is connected we find $f(z) = c$ for all $z \in \mathbb{C}$. $\square$

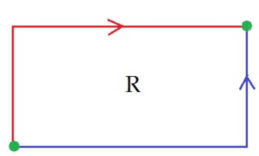
We saw in the homework that this theorem allows a relatively easy proof of the Fundamental Theorem of Algebra. In addition, we were able to show that an entire function whose range misses a disk of values must be constant. As I mentioned in class, the take-away message here is simply this: *every bounded entire function is constant.*

9.6 Morera's Theorem

I think the central result of this section is often attributed to Goursat. More on that in the next section. Let us discuss what is presented in Gamelin. It is important to note that continuous differentiability of $f(z)$ is **not** assumed as a precondition of the theorem.

Theorem 9.6.1. Morera's Theorem: *Let $f(z)$ be a continuous function on a domain U. If $\int_{\partial R} f(z)dz = 0$ for every closed rectangle R contained in U with sides parallel to the coordinate axes then $f(z)$ is holomorphic with continuous $f'(z)$ in U.*

Proof: the vanishing of the rectangular integral allows us to exchange the lower path between two vertices of a rectangle for the upper path:



It suffices to prove the theorem for a disk D with center z_o where $D \subseteq U$[7]. Define:

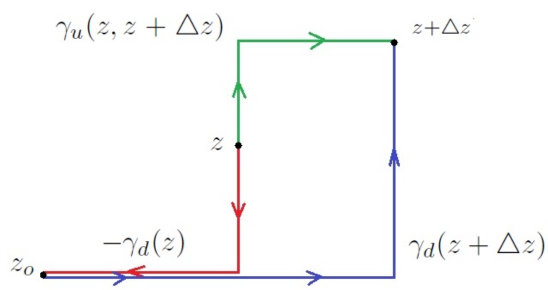
$$F(z) = \int_{\gamma_d(z)} f(w)dw$$

[7] do you understand why this is true and no loss of generality here?

where $\gamma_d(z) = [x_o+iy_o,\ iy_o+x]\cup[iy_o+x,\ x+iy]$ where $z_o = x_o+iy_o$ and $z = x+iy$. To show $F'(z)$ exists we consider the difference: here $\triangle z$ is a small enough displacement as to keep $z+\triangle z \in D$, the calculation below is supported by the diagram which follows after:

$$\begin{aligned} F(z\ +\ \triangle z) - F(z) &= \int_{\gamma_d(z+\triangle z)} f(w)dw - \int_{\gamma_d(z)} f(w)dw \\ &= \int_{\gamma_d(z+\triangle z)} f(w)dw + \int_{-\gamma_d(z)} f(w)dw \\ &= \int_{\gamma_u(z,z+\triangle z)} f(w)dw \qquad \star\,. \end{aligned}$$

Where $-\gamma_d(z)$ denotes the reversal of $\gamma_d(z)$. I plotted it as the red path below. The blue path is $\gamma_d(z+\triangle z)$. By the assumption of the theorem we are able to replace the sum of the blue and red paths by the green path $\gamma_u(z, z+\triangle z)$.



Notice, $f(z)$ is just a constant in the integral below hence:

$$\int_{\gamma_u(z,z+\triangle z)} f(z)dw = f(z)\int_z^{z+\triangle z} dw = f(z)w\Big|_z^{z+\triangle z} = f(z)\triangle z.$$

Return once more to $\star$ and add $f(z) - f(z)$ to the integrand:

$$\begin{aligned} F(z\ +\ \triangle z) - F(z) &= \int_{\gamma_u(z,z+\triangle z)} [f(z) + f(w) - f(z)]dw \\ &= f(z)\triangle z + \int_{\gamma_u(z,z+\triangle z)} \big(f(w) - f(z)\big)dw \quad \star\star \end{aligned}$$

Note $L(\gamma_u(z, z+\triangle z)) < 2|\triangle z|$ and if we set $M = sup\{|f(w) - f(z)| \mid z \in \gamma_u(z, z+\triangle z)\}$ then the ML-estimate provides

$$\left|\int_{\gamma_u(z,z+\triangle z)} \big(f(w) - f(z)\big)dw\right| \leq ML < 2M|\triangle z|$$

Rearranging $\star\star$ we find:

$$\left|\frac{F(z\ +\ \triangle z) - F(z)}{\triangle z} - f(z)\right| \leq 2M.$$

Notice that as $\triangle z \to 0$ we have $2M \to 0$ hence $F'(z) = f(z)$ be the inequality above. Furthermore, we assumed $f(z)$ continuous hence $F'(z)$ is continuous. Consequently $F(z)$ is both holomorphic and possesses continuous derivative $F'(z)$ on D. Apply the Corollary 9.4.5 to Cauchy's Generalized Integral Formula to see that $F''(z) = f'(z)$ exists and is continuous. $\square$

9.7 Goursat's Theorem

Let me begin with presenting Goursat's Theorem as it appears in Gamelin:

Theorem 9.7.1. Goursat's Theorem: (Gamelin Version) *If $f(z)$ is a complex-valued function on a domain D such that*

$$f'(z_o) = \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_o}$$

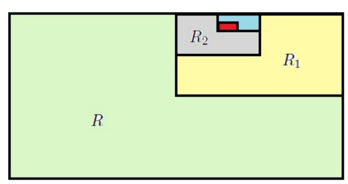
exists at each point z_o of D then $f(z)$ is analytic on D.

Notice, in our language, the theorem above can be stated: *If a function is holomorphic on a domain D then $z \to f'(z)$ is continuous.*

Proof: let R be a closed rectangle in D with sides parallel to the coordinate axes. Divide R into four identical sub-rectangles and let R_1 be the sub-rectangle for which $\left|\int_{\partial R_1} f(z)dz\right|$ is largest (among the 4 sub-rectangles). Observe that $\left|\int_{\partial R_1} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial R} f(z)dz\right|$ or, equivalently, $\left|\int_{\partial R} f(z)dz\right| \leq 4\left|\int_{\partial R_1} f(z)dz\right|$. Then, we subdivide R_1 into 4 sub-rectangles and the rectangle with largest integral R_2. Continuing in this fashion we obtain a sequence of nested rectangles $R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$. It is a simple exercise to verify:

$$\left|\int_{\partial R_n} f(z)dz\right| \leq 4\left|\int_{\partial R_{n+1}} f(z)dz\right| \quad \Rightarrow \quad \left|\int_{\partial R} f(z)dz\right| \leq 4^n\left|\int_{\partial R_n} f(z)dz\right| \quad \star.$$

The subdivision process is illustrated below:



As $n \to \infty$ it is clear that the sequence of nested rectangles converges to a point $z_o \in R$. Furthermore, if L is the length of the perimeter of R then $L/2^n$ is the length of ∂R_n. As $f(z)$ is complex-differentiable at z_o we know for each $z \in R_n$ there must exist an ϵ_n such that

$$\left|\frac{f(z) - f(z_o)}{z - z_o} - f'(z_o)\right| \leq \epsilon_n$$

hence

$$\left|f(z) - f(z_o) - f'(z_o)(z - z_o)\right| \leq \epsilon_n|z - z_o| \leq 2\epsilon_n L/2^n \quad \star\star.$$

The last inequality is very generous since $z_o, z \in R_n$ surely implies they are closer than the perimeter $L/2^n$ apart. Notice, the function $g(z) = f(z_o) + f'(z_o)(z - z_o)$ has primitive $G(z) = f(z_o)z + f'(z_o)(z^2/2 - zz_o)$ on R_n hence[8] $\int_{\partial R_n} g(z)dz = 0$. Subtracting this zero is crucial:

$$\left|\int_{\partial R_n} f(z)dz\right| = \left|\int_{\partial R_n} \left[f(z) - f(z_o) - f'(z_o)(z - z_o)\right]dz\right| \leq (2\epsilon_n L/2^n)(L/2^n) = \frac{2L^2\epsilon_n}{4^n}.$$

[8]this application of Cauchy's Theorem does not beg the question by assuming continuity of $g'(z)$

where we applied the ML-estimate by $\star\star$ and $L(\partial R_n) = L/2^n$. Returning to $\star$,

$$\left|\int_{\partial R} f(z)dz\right| \leq 4^n \left|\int_{\partial R_n} f(z)dz\right| \leq 4^n \cdot \frac{2L^2\epsilon_n}{4^n} = 2L^2\epsilon_n.$$

Finally, as $n \to 0$ we have $\epsilon_n \to 0$ thus it follows $\int_{\partial R} f(z)dz = 0$. But, this shows the integral around an arbitrary rectangle in D is zero hence by Morera's Theorem 9.6.1 we find $f(z)$ is holomorphic with continuous $f'(z)$ on D. $\square$

We now see that holomorphic functions on a domain are indeed analytic (as defined by Gamelin).

9.8 Complex Notation and Pompeiu's Formula

My apologies, it seems I have failed to write much here. I have many things to say, some of them I said in class. Recently, we learned how to generalize the idea of this section to nearly arbitrary associative algebras. More on that somewhere else.

Chapter 10

Power Series

A power series is simply a polynomial without end. But, this begs questions. What does "without end" mean? How can we add, subtract, multiply and divide things which have no end? In this chapter we give a careful account of things which go on without end.

History provides examples of the need for caution[1]. For example, even Cauchy wrongly asserted in 1821 that an infinite series of continuous functions was once more continuous. In 1826 Abel[2] provided a counter-example and in the years to follow the concept of uniform convergence was invented to avoid such blunders. Abel had the following to say about the state of the theory as he saw it: from page 114 of [R91]

> If one examines more closely the reasoning which is usually employed in the treatment of infinite series, he will find that by and large it is unsatisfactory and that the number of propositions about infinite series which can be regarded as rigorously confirmed is small indeed

The concept of uniform convergence is apparently due to the teacher of Weierstrauss. Christoph Gudermann wrote in 1838: "it is a fact worth noting that... the series just found have *all the same convergence rate*". Weierstrauss used the concept of uniform convergence throughout his work. Apparently, Seidel and Stokes independently in 1848 and 1847 also used something akin to uniform convergence of a series, but the emminent British mathematician G.H Hardy gives credit to Weierstrauss:

> Weierstrauss's discovery was the earliest, and he alone fully realized its far-reaching importance as one of the fundamental ideas of analysis

It is fun to note Cauchy's own view of his 1821 oversight. In 1853 in the midst of a work which used and made significant contributions to the theory of uniformly convergent series, he wrote that *it is easy to see how one should modify the statement of the theorem.* See page 102 of [R91] for more details as to be fair to Cauchy.

In this chapter, we study convergence of sequence and series. Ultimately, we find how power series work in the complex domain. The results are surprisingly simple as we shall soon discover. Most importantly, we introduce the term *analytic* and see in what sense it is equivalent to our term *holomorphic*. Obviously, we differ from Gamelin on this point of emphasis.

[1] the facts which follow here are taken from [R91] pages 96-98 primarily
[2] did work on early group theory, we name commutative groups **Abelian** groups in his honor

10.1 Complex Sequences

A function $n \mapsto a_n$ from $\mathbb{N}$ to $\mathbb{C}$ is a **sequence** of complex numbers. Sometimes we think of a sequence as an ordered list; $\{a_n\} = \{a_1, a_2, \dots\}$. We assume the domain of sequences in this section is $\mathbb{N}$ but this is not an essential constraint, we could just as well study sequences with domain $\{k, k+1, \dots\}$ for some $k \in \mathbb{Z}$.

Definition 10.1.1. Sequential Limit: *Let a_n be a complex sequence and $a \in \mathbb{C}$. We say $a_n \to a$ iff for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ whenever $n > N$. In this case we write*

$$\lim_{n\to\infty} a_n = a.$$

Essentially, the idea is that the sequence clusters around L as we go far out in the list.

Definition 10.1.2. Bounded Sequence: *Suppose $R > 0$ and $|a_n| < R$ for all $n \in \mathbb{N}$ then $\{a_n\}$ is a* **bounded sequence**

The condition $|a_n| < R$ implies a_n is in the disk of radius R centered at the origin.

Theorem 10.1.3. Convergent Sequence Properties: *A convergent sequence is bounded. Furthermore, if $s_n \to s$ and $t_n \to t$ then*

(a.) $s_n + t_n \to s + t$

(b.) $s_n t_n \to st$

(c.) $s_n/t_n \to s/t$ *provided* $t \neq 0$.

The proof of the theorem above mirrors the proof you would give for real sequences.

Theorem 10.1.4. in-between theorem: *If $r_n \leq s_n \leq t_n$, and if $r_n \to L$ and $t_n \to L$ then $s_n \to L$.*

The theorem above is for real sequences. We have no[3] **order relations** on $\mathbb{C}$. Recall, by definition, monotonic sequences s_n are either always decreasing ($s_{n+1} \leq s_n$) or always increasing ($s_{n+1} \geq s_n$). The **completeness**, roughly the idea that $\mathbb{R}$ has no *holes*, is captured by the following theorem:

Theorem 10.1.5. *A bounded monotone sequence of real numbers coverges.*

The existence of a limit can be captured by the limit inferior and the limit superior. These are in turn defined in terms of **subsequences**.

Definition 10.1.6. *Let $\{a_n\}$ be a sequence. We define a* **subsequence** *of $\{a_n\}$ to be a sequence of the form $\{a_{n_j}\}$ where $j \mapsto n_j \in \mathbb{N}$ is a strictly increasing function of j.*

Standard examples of subsequences of $\{a_j\}$ are given by $\{a_{2j}\}$ or $\{a_{2j-1}\}$.

Example 10.1.7. *If $a_j = (-1)^j$ then $a_{2j} = 1$ whereas $a_{2j-1} = -1$. In this example, the even subsequence and the odd sequence both converge. However, $\lim a_j$ does not exist.*

Apparently, considering just one subsequence is insufficient to gain much insight. On the other hand, if we consider all possible subsequences then it is possible to say something definitive.

[3] to be fair, you can order $\mathbb{C}$, but the order is not consistent with the algebraic structure. See this answer

Definition 10.1.8. *Let $\{a_n\}$ be a sequence. We define* $\limsup(a_n)$ *to be the upper bound of all possible subsequential limits. That is, if $\{a_{n_j}\}$ is a subsequence which converges to t (we allow $t = \infty$) then $t \leq$* $\limsup(a_n)$*. Likewise, we define* $\liminf(a_n)$ *to be the lower bound (possibly $-\infty$) of all possible subsequential limits of $\{a_n\}$.*

Theorem 10.1.9. *The sequence $a_n \to L \in \mathbb{R}$ if and only iff* $\limsup(a_n) = \liminf(a_n) = L \in \mathbb{R}$.

The concepts above are not available directly on $\mathbb{C}$ as there is no clear definition of an increasing or decreasing complex number. However, we do have many other theorems for complex sequences which we had before for $\mathbb{R}$. In the context of advanced calculus, I call the following the *vector limit theorem*. It says: *the limit of a vector-valued sequence is the vector of the limits of the component sequences*. Here we just have two components, the real part and the imaginary part.

Theorem 10.1.10. *Suppose $z_n = x_n + iy_n \in \mathbb{C}$ for all $n \in \mathbb{N}$ and $z = x + iy \in \mathbb{C}$. The sequence $z_n \to z$ if and only iff both $x_n \to x$ and $y_n \to y$.*

Proof Sketch: Notice that if $x_n \to x$ and $y_n \to y$ then it is an immediate consequence of Theorem 10.1.3 that $x_n + iy_n \to x + iy$. Conversely, suppose $z_n = x_n + iy_n \to z$. We wish to prove that $x_n \to x = \mathbf{Re}(z)$ and $y_n \to y = \mathbf{Im}(z)$. The inequalities below are crucial:

$$|x_n - x| \leq |z_n - z| \qquad \& \qquad |y_n - y| \leq |z_n - z|$$

Let $\varepsilon > 0$. Since $z_n \to z$ we are free to select $N \in \mathbb{N}$ such that for $n \geq N$ we have $|z_n - z| < \varepsilon$. But, then it follows $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ by the crucial inequalities. Hence $x_n \to x$ and $y_n \to y$. $\square$

Definition 10.1.11. *We say a sequence $\{a_n\}$ is* **Cauchy** *if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ for which $N < m < n$ implies $|a_m - a_n| < \varepsilon$.*

As Gamelin explains, a Cauchy sequence is one where the differences $a_m - a_n$ tend to zero in the tail of the sequence. At first glance, this hardly seems like an improvement on the definition of convergence, yet, in practice, so many proofs elegantly filter through the Cauchy criterion. In any space, if a sequence converges then it is Cauchy. However, the converse only holds for special spaces which are called **complete**.

Definition 10.1.12. *A space is complete if every Cauchy sequence converges.*

The content of the theorem below is that $\mathbb{C}$ is complete.

Theorem 10.1.13. *A complex sequence converges iff it is a Cauchy sequence.*

Real numbers as also complete. This is an essential difference between the rational and the real numbers. There are certainly sequences of rational numbers whose limit is irrational. For example, the sequence of partial sums from the $p = 2$ series $\{1, 1 + 1/4, 1 + 1/4 + 1/9, \dots\}$ has rational elements yet limits to $\pi^2/6$. This was shown by Euler in 1734 as is discussed on page 333 of [R91]. The process of adjoining all limits of Cauchy sequences to a space is known as **completing a space**. In particular, the completion of $\mathbb{Q}$ is $\mathbb{R}$. Ideally, you will obtain a deeper appreciation of Cauchy sequences and completion when you study real analysis. That said, if you are willing to accept the truth that $\mathbb{R}$ is complete it is not much more trouble to show $\mathbb{R}^n$ is complete.

10.2 Infinite Series

We discussed and defined complex sequences in Chapter 2. See Definition 10.1.1. We now discuss series of complex numbers. In short, a complex series is formed by adding the terms in some sequence of complex numbers:

$$\sum_{n=0}^{\infty} z_n = z_o + z_1 + z_2 + \cdots$$

If this sum exists as a complex number then the series is convergent whereas if the sum above does not converge then the series is said to be divergent. The convergence (or divergence) of the series is described precisely by the convergence (or divergence) of the sequence of partial sums:

Definition 10.2.1. *Let $a_n \in \mathbb{C}$ for each $n \in \mathbb{N} \cup \{0\}$ then we define*

$$\sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=0}^{n} a_j.$$

If $\lim_{n\to\infty} \sum_{j=0}^{n} a_j = S \in \mathbb{C}$ then the series $a_o + a_1 + \cdots$ is said to **converge** *to S.*

The linearity theorems for sequences induce similar theorems for series. In particular, Theorem 10.1.3 leads us to:

Theorem 10.2.2. *Let $c \in \mathbb{C}$, $\sum a_j = A$ and $\sum b_j = B$ then $\sum(a_j + b_j) = A + B$ and $\sum ca_j = cA$;*

$$\sum(a_j + b_j) = \sum a_j + \sum b_j \qquad \textit{additivity of convergent sums}$$

$$\sum cb_j = c\sum b_j \qquad \textit{homogeneity of convergent sums}$$

Proof: let $S_n = \sum_{j=0}^{n} a_j$ and $T_n = \sum_{j=0}^{n} b_j$. We are given, from the definition of convergent series, that these partial sums converge; $S_n \to A$ and $T_n \to B$ as $n \to \infty$. Consider then,

$$\sum_{j=0}^{n}(a_j + cb_j) = \sum_{j=0}^{n} a_j + c\sum_{j=0}^{n} b_j$$

Thus, the sequence of partial sums for $\sum_{j=0}^{\infty}(a_j + cb_j)$ is found to be $S_n + cT_n$. Apply Theorem 10.1.3 and conclude $S_n + cT_n \to A + cB$ as $n \to \infty$. Therefore,

$$\sum_{j=0}^{\infty}(a_j + cb_j) = \sum_{j=0}^{\infty} a_j + c\sum_{j=0}^{\infty} b_j.$$

If we set $c = 1$ we obtain additivity, if we set $A = 0$ we obtain homogeneity. $\square$

I offered a proof for series which start at $j = 0$, but, it ought to be clear the same holds for series which start at any particular $j \in \mathbb{Z}$.

Let me add a theorem which is a simple consequence of Theorem 10.1.10 applied to partial sums:

Theorem 10.2.3. *Let $x_k, y_k \in \mathbb{R}$ then $\sum x_k + iy_k$ converges iff $\sum x_k$ and $\sum y_k$ converge. Moreover, in the convergent case, $\sum x_k + iy_k = \sum x_k + i\sum y_k$.*

Series of real numbers enjoy a number of results which stem from the ordering of the real numbers. The theory of series with non-negative terms is particularly intuitive. Suppose $a_o, a_1, \dots > 0$ then $\{a_o, a_o + a_1, a_o + a_1 + a_2, \dots\}$ is a monotonically increasing sequence. Recall Theorem 10.1.5 which said that a monotonic sequence converged iff it was **bounded**.

Theorem 10.2.4. *If $0 \leq a_k \leq r_k$, and if $\sum r_k$ converges, then $\sum a_k$ converges, and $\sum a_k \leq \sum r_k$.*

Proof: obviously $a_k, r_k \in \mathbb{R}$ by the condition $0 \leq a_k \leq r_k$. Observe $\sum_{k=0}^{n+1} r_k = r_{n+1} + \sum_{k=0}^{n} r_k$ hence $\sum_{k=0}^{n+1} r_k \geq \sum_{k=0}^{n} r_k$. Thus the sequence of partial sums of $\sum r_k$ is increasing. Since $\sum r_k$ converges it follows that the convergent sequence of partial sums is bounded. That is, there exists $M \geq 0$ such that $\sum_{k=0}^{n} r_k \leq M$ for all $n \in \mathbb{N} \cup \{0\}$. Notice $a_k \leq r_k$ implies $\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{n} r_k$. Therefore, $\sum_{k=0}^{n} a_k \leq M$. Observe $a_k \geq 0$ implies $\sum_{k=0}^{n} a_k$ is increasing by the argument we already offered for $\sum_{k=0}^{n} r_k$. We find $\sum_{k=0}^{n} a_k$ is a bounded, increasing sequence of non-negative real numbers thus $\lim_{n\to\infty} \sum_{k=0}^{n} a_k = A \in \mathbb{R}$ by Theorem 10.1.5. Finally, we appeal to part of the sandwhich theorem for real sequences, if $c_n \leq d_n$ for all n and both c_n and d_n converge then $\lim_{n\to\infty} c_n \leq \lim_{n\to\infty} d_n$. Think of $c_n = \sum_{k=0}^{n} a_k$ and $d_n = \sum_{k=0}^{n} r_k$. Note $\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{n} r_k$ implies $\lim_{n\to\infty} \sum_{k=0}^{n} a_k \leq \lim_{n\to\infty} \sum_{k=0}^{n} r_k$. The theorem follows. $\square$

Can you appreciate the beauty of how Gamelin discusses convergence and proofs ? Compare the proof I give here to his paragraph on page 130-131. His prose captures the essential details of what I wrote above without burying you in details which obscure. In any event, I will continue to add uglified versions of Gamelin's prose in this chapter. I hope that by seeing both your understanding is fortified.

We return to the study of complex series once more. Suppose $a_j \in \mathbb{C}$ in what follows. The definition of a finite sum is made recursively by $\sum_{j=0}^{0} a_j = a_o$ and for $n \geq 1$:

$$\sum_{j=0}^{n} a_j = a_n + \sum_{j=0}^{n-1} a_j.$$

Notice this yields:

$$a_n = \sum_{j=0}^{n} a_j - \sum_{j=0}^{n-1} a_j.$$

Suppose $\sum_{j=0}^{\infty} a_j = S \in \mathbb{C}$. Observe, as $n \to \infty$ we see that $\sum_{j=0}^{n} a_j - \sum_{j=0}^{n-1} a_j \to S - S = 0$. Therefore, the condition $a_n \to 0$ as $n \to \infty$ is a **necessary** condition for convergence of $a_o + a_1 + \cdots$.

Theorem 10.2.5. *If $\sum_{j=0}^{\infty} a_j$ converges then $a_j \to 0$ as $j \to \infty$.*

Of course, you should recall from calculus that the criteria above is not **sufficient** for convergence of the series. For example, $1 + 1/2 + 1/3 + \cdots$ diverges despite the fact $1/n \to 0$ as $n \to \infty$.

I decided to elevate Gamelin's example on page 131 to a proposition.

Proposition 10.2.6. *Let $z_j \in \mathbb{C}$ for $j \in \mathbb{N} \cup \{0\}$.*
If $|z| < 1$ then $\displaystyle\sum_{j=0}^{\infty} z^n = \frac{1}{1-z}$. If $|z| \geq 1$ then $\displaystyle\sum_{j=0}^{\infty} z^n$ diverges.

Proof: if $|z| \geq 1$ then the n-th term test shows the series diverges. Suppose $|z| < 1$. Consider,

$$S_n = 1 + z + z^2 \cdots + z^n \quad \Rightarrow \quad zS_n = z + z^2 + \cdots + z^n + z^{n+1}$$

and we find $S_n - zS_n = 1 - z^{n+1}$ thus $(1-z)S_n = 1 - z^{n+1}$ and derive:

$$S_n = \frac{1 - z^{n+1}}{1 - z}$$

This is a rare and wonderful event that we were able to explicitly calculate the n-th partial sum with such small effort. Note $|z| < 1$ implies $|z|^{n+1} \to 0$ as $n \to \infty$. Therefore,

$$\sum_{j=0}^{\infty} z^n = \lim_{n \to \infty} \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z}. \qquad \square$$

Definition 10.2.7. *A complex series $\sum a_k$ is said to* **converge absolutely** *if $\sum |a_k|$ converges.*

Notice that $|a_k|$ denotes the modulus of a_k. In the case $a_k \in \mathbb{R}$ this reduces to the usual[4] definition of absolute convergence since the modulus is merely the absolute value function in that case. If you'd like to see a proof of absolute convergence in the real case, I recommend page 82 of [J02]. The proof there is based on parsing the real series into non-negative and negative terms. We have no such dichotomy to work with here so something else must be argued.

Theorem 10.2.8. *If $\sum a_k$ is absolutely convergent then $\sum a_k$ converges and $\left|\sum a_k\right| \leq \sum |a_k|$.*

Proof: assume $\sum |a_k|$ converges. Let $a_k = x_k + iy_k$ where $x_k, y_k \in \mathbb{R}$. Observe:

$$|a_k| = \sqrt{x_k^2 + y_k^2} \geq \sqrt{x_k^2} = |x_k| \qquad \& \qquad |a_k| \geq |y_k|.$$

Thus, $|x_k| \leq |a_k|$ hence by comparison test the series $\sum |x_k|$ converges with $\sum |x_k| \leq \sum |a_k|$. Likewise, $|y_k| \leq |a_k|$ hence by comparison test the series $\sum |y_k|$ converges with $\sum |y_k| \leq \sum |a_k|$. Recall that absolute convergence of a real series implies convergence hence $\sum x_k$ and $\sum y_k$ exist. Theorem 10.2.3 allows us to conclude $\sum x_k + iy_k = \sum a_k$ converges. $\square$

Given that I have used the absolute convergence theorem for real series I think it is appropriate to offer the proof of that theorem since many of you may either have never seen it, or at a minimum, have forgotten it. Following page 82 of [J02] consider a real series $\sum_{n=0}^{\infty} x_n$. We define:

$$p_n = \begin{cases} x_n & \text{if } x_n \geq 0 \\ 0 & \text{if } x_n < 0 \end{cases} \qquad \& \qquad q_n = \begin{cases} 0 & \text{if } x_n \geq 0 \\ -x_n & \text{if } x_n < 0 \end{cases}$$

Notice $x_n = p_n - q_n$. Furthermore, notice p_n, q_n are non-negative terms. Observe

$$p_0 + p_1 + \cdots + p_n \leq |x_o| + |x_1| + \cdots + |x_n|$$

Hence $\sum |x_n|$ converging implies $\sum p_n$ converges by Comparison Theorem 10.2.4 and $\sum p_n \leq \sum |x_n|$. Likewise,

$$q_0 + q_1 + \cdots + q_n \leq |x_o| + |x_1| + \cdots + |x_n|$$

[4] in the sense of second semester calculus where you probably first studied series

Hence $\sum |x_n|$ converging implies $\sum q_n$ converges by Comparison Theorem 10.2.4 and $\sum q_n \leq \sum |x_n|$. But, then $\sum x_n = \sum(p_n - q_n) = \sum p_n - \sum q_n$ by Theorem 10.2.2. Finally, notice

$$x_0 + x_1 + \cdots + x_n \leq |x_0| + |x_1| + \cdots + |x_n|$$

thus as $n \to \infty$ we obtain $\sum x_n \leq \sum |x_n|$. This completes the proof that absolute convergence implies convergence for series with real terms.

I challenge you to see that my proof here is really not that different from what Gamelin wrote[5].

Example 10.2.9. *Consider $|z| < 1$. Proposition 10.2.6 applies to show $\sum z_j$ is absolutely convergent by direct calculation and:*

$$\left|\frac{1}{1-z}\right| = \left|\sum_{j=0}^{\infty} z^j\right| \leq \sum_{j=0}^{\infty} |z|^j = \frac{1}{1-|z|}.$$

Following Gamelin,

$$\frac{1}{1-z} - \sum_{k=0}^{n} z^k = \sum_{k=0}^{\infty} z^k - \sum_{k=0}^{n} z^k = \sum_{k=n+1}^{\infty} z^k = z^{n+1}\sum_{k=0}^{\infty} z^k = \frac{z^{n+1}}{1-z}.$$

Therefore,

$$\left|\frac{1}{1-z} - \sum_{k=0}^{n} z^k\right| = \frac{|z|^{n+1}}{|1-z|} \leq \frac{|z|^{n+1}}{1-|z|}.$$

The inequality above gives us a bound on the error for the n-th partial sum of the geometric series.

If you are interested in the history of absolute convergence, you might look at pages 29-30 of [R91] where he describes briefly the influence of Cauchy, Dirichlet and Riemann on the topic. It was Riemann who proved that a series which converges but, does not converge absolutely, could be rearranged to converge to any value in $\mathbb{R}$.

10.3 Sequences and Series of Functions

A sequence of functions on $E \subseteq \mathbb{C}$ is an assignment of a function on E for each $n \in \mathbb{N} \cup \{0\}$. Typically, we denote the sequence by $\{f_n\}$ or simply by f_n. In addition, although we are ultimately interested in the theory of sequences of complex functions, I will give a number of real examples to illustrate the subtle issues which arise in general.

Definition 10.3.1. *A sequence of functions f_n on E is said to* **pointwise converge** *to f if $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in E$.*

You might be tempted to suppose that if each function of the sequence is continuous and the limit exists then surely the limit function is continuous. Well, you'd be wrong:

[5] Bailu, notice the proof I give here easily extends to an associative algebra

Example 10.3.2. *Let $n \in \mathbb{N} \cup \{0\}$ and define $f_n(x) = x^n$ for $x \in [0,1]$. We can calculate the limit function:*

$$f(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Notice, f_n is continuous for each $n \in \mathbb{N}$, but, the limit function f is **not** *continuous. In particular, you can see we cannot switch the order of the limits below:*

$$0 = \lim_{x\to 1^-}\left(\lim_{n\to\infty} x^n\right) \neq \lim_{n\to\infty}\left(\lim_{x\to 1^-} x^n\right) = 1$$

To guarantee the continuity of the limit function we need a stronger mode of convergence. Following Gamelin (and a host of other analysis texts) consider:

Example 10.3.3. *We define a sequence for which each function g_n makes a triangular tent of slope $\pm n^2$ from $x = 0$ to $x = 2/n$. In particular, for $n \in \mathbb{N}$ define:*

$$g_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < 1/n \\ 2n - n^2 x & \text{if } 1/n \le x \le 2/n \\ 0 & \text{if } 2/n \le x \le 1 \end{cases}$$

Notice,

$$\int_0^{1/n} n^2 x dx = n^2 \frac{(1/n)^2}{2} = \frac{1}{2}$$

and

$$\int_{1/n}^{2/n} (2n - n^2 x) dx = 2n(2/n - 1/n) - \frac{n^2}{2}[(2/n)^2 - (1/n)^2] = 2 - \frac{3}{2} = \frac{1}{2}.$$

Therefore, $\int_0^1 g_n(x)\, dx = 1$ for each $n \in \mathbb{N}$. However, as $n \to \infty$ we find $g_n(x) \to 0$ for each $x \in [0,1]$. Observe:

$$1 = \lim_{n\to\infty}\int_0^1 g_n(x)\, dx \neq \int_0^1 \lim_{n\to\infty} g_n(x)\, dx = 0.$$

To guarantee the integral of the limit function is the limit of the integrals of the sequence we need a stronger mode of convergence. Here I break from Gamelin and add one more example.

Example 10.3.4. *For each $n \in \mathbb{N}$ define $f_n(x) = x^n/n$ for $0 \le x \le 1$. Notice that $\lim_{n\to\infty} x^n/n = 0$ for each $x \in [0,1]$. Furthermore, $\lim_{x\to a} x^n/n = a^n/n$ for each $a \in [0,1]$ where we use one-sided limits at $a = 0^+, 1^-$. It follows that:*

$$\lim_{n\to\infty}\lim_{x\to a}\frac{x^n}{n} = \lim_{n\to\infty}\frac{a^n}{n} = 0$$

likewise,

$$\lim_{x\to a}\lim_{n\to\infty}\frac{x^n}{n} = \lim_{x\to a} 0 = 0$$

Thus, the limit $n \to \infty$ and $x \to a$ commute for this sequence of functions.

The example above shows us there is hope for the limit of a sequence of continuous function to be continuous. Perhaps we preserve derivatives under the limit ? Consider:

Example 10.3.5. *Once more study $f_n(x) = x^n/n$ for $0 \leq x \leq 1$. Notice $\frac{df_n}{dx} = x^{n-1}$. However, this is just the sequence we studied in Example 10.3.2,*

$$\lim_{n\to\infty} \frac{df_n}{dx} = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases} \quad \Rightarrow \quad \lim_{x\to 1^-} \lim_{n\to\infty} \frac{df_n}{dx} = \lim_{x\to 1^-} (0) = 0.$$

On the other hand,

$$\lim_{x\to 1^-} \frac{df_n}{dx} = \lim_{x\to 1^-} x^{n-1} = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \lim_{x\to 1^-} \frac{df_n}{dx} = \lim_{n\to\infty} (1) = 1.$$

Therefore, the limit of the sequence of derivatives is not the derivative of the limit function.

The examples above lead us to define a stronger type of convergence which preserves continuity and integrals to the limit. However, in the real case, differentiation is still subtle.

The standard definition of uniform convergence is given below:[6]

Definition 10.3.6. *Let $\{f_n\}$ be a sequence of functions on E. Let f be a function on E. We say $\{f_n\}$* **converges uniformly to** *f if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$.*

This is not quite Gamelin's presentation. Instead, from page 134, Gamelin says:

> We say a sequence of functions $\{f_j\}$ **converges uniformly** to f on E if $|f_j(x) - f(x)| \leq \epsilon_j$ for all $x \in E$ where $\epsilon_j \to 0$ as $j \to \infty$. We call ϵ_j the worst-case estimator of the difference $f_j(x) - f(x)$ and usually take ϵ_j to be the supremum (maximum) of $|f_j(x) - f(x)|$ over $x \in E$,
> $$\epsilon_j = \sup_{x\in E} |f_j(x) - f(x)|.$$

Very well, are these definitions of uniform convergence equivalent? For a moment, let us define the uniform convergence of Gamelin as G-uniform convergence whereas that given in the Definition 10.3.6 defines S-uniform convergence. The question becomes:

> Can we show a sequence of functions $\{f_n\}$ on E is S-uniformly convergent to f on E iff the sequence of functions is G-uniformly convergent to f on E ?

This seems like an excellent homework question, so, I will merely assert it's verity for us here:

Theorem 10.3.7. *Let $\{f_n\}$ be a sequence of functions on E. Then $\{f_n\}$ is S-uniformly convergent to f on E if and only if $\{f_n\}$ is G-uniformly convergent to f on E.*

Proof: by trust in Gamelin, or as is my preference, your homework. $\square$

The beautiful feature of Gamelin's definition is that it gives us a method to calculate the worst-case estimator. We merely need to find the maximum difference between the n-th function in the sequence and the limit function over the given domain of interest (E).

If you think about it, the supremum gives you the *best* worst-case estimator. Let me explain, if ϵ_j has $|f_j(z) - f(z)| \leq \epsilon_j$ for all $z \in E$ then ϵ_j is an upper bound on $|f_j(z) - f(z)|$. But, the

[6] for instance, see page 246 of [J02].

supremum is the **least upper bound** hence $|f_j(w) - f(w)| \leq \sup_{z \in E} |f_j(z) - f(z)| \leq \epsilon_j$ for all $w \in E$. This simple reasoning shows us that when the supremum exists and we may use it as a worst-case estimator **provided** we also know $\sup_{z \in E} |f_j(z) - f(z)| \to 0$ as $j \to \infty$. On the other hand, if no supremum exists or if the supremum does not go to zero as $j \to \infty$ then we have no hope of finding a worst case estimator.

The paragraph above outlines the logic used in the paragraphs to follow.

In Example 10.3.2 we had $f_n(x) = x^n$ for $x \in [0,1]$ pointwise converged to $f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$ from which we may calculate[7] $\sup_{x \in [0,1]} |x^n - f(x)| = 1$. Therefore, it is not possible to find $\epsilon_n \to 0$. In Gamelin's terminology, *the worst-case estimator is* 1 hence this sequence is not uniformly convergent to $f(x)$ on $[0,1]$.

In Example 10.3.3 we had

$$g_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < 1/n \\ 2n - n^2 x & \text{if } 1/n \leq x \leq 2/n \\ 0 & \text{if } 2/n \leq x \leq 1 \end{cases}$$

which is point-wise convergent to $g(x) = 0$ for $x \in [0,1]$. The largest value attained by $g_n(x)$ is found at $x = 1/n$ where

$$g_n(1/n) = n^2(1/n) = n$$

Therefore,

$$\sup_{x \in [0,1]} |g_n(x) - g(x)| = n.$$

Therefore, the convergence of $\{g_n\}$ to g is not uniform on $[0,1]$.

Next, consider Example 10.3.4 where we noted that $f_n(x) = x^n/n$ converges pointwise to $f(x) = 0$ on $[0,1]$. In this case it is clear that $f_n(1) = 1/n$ is the largest value attained by $f_n(x)$ on $[0,1]$ hence:

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = 1/n = \epsilon_n \to 0 \quad \text{as } n \to \infty.$$

Hence $\{x^n/n\}$ converges uniformly to $f(x) = 0$ on $[0,1]$. Apparently, continuity is preserved under uniform convergence. On the other hand, Example 10.3.5 shows us that, for real functions, derivatives need not be preserved in a uniformly convergent limit.

We now present the two major theorems about uniformly convergent sequences of functions.

Theorem 10.3.8. *Let $\{f_j\}$ be a sequence of complex-valued functions on $E \subseteq \mathbb{C}$. If each f_j is continuous on E and if $\{f_j\}$ converges uniformly to f on E then f is continuous on E.*

[7] sometimes the supremum is also known as the **least upper bound**, it is the smallest possible upper bound on the set in question. In this case, 1 is not attained in the set, but numbers arbitrary close to 1 are attained. Technically, this set has **no maximum** which is why the parenthetical comment in Gamelin suggesting supremum and maximum are synonyms is sometimes not helpful.

Proof: let $\epsilon > 0$. By uniform convergence, there exists $N \in \mathbb{N}$ for which

$$|f_N(z) - f(z)| < \frac{\epsilon}{3} \qquad \star$$

for all $z \in E$. However, by continuity of f_N at $z = a$ there exists $\delta > 0$ such that $0 < |z - a| < \delta$ implies

$$|f_N(z) - f_N(a)| < \frac{\epsilon}{3} \qquad \star\star.$$

We claim $f(z)$ is continuous at $z = a$ by the same choice of δ. Consider, for $0 < |z - a| < \delta$,

$$\begin{aligned} |f(z) - f(a)| &= |f(z) - f_N(z) + f_N(z) - f_N(a) + f_N(a) - f(a)| \\ &\leq |f_N(z) - f(z)| + |f_N(z) - f_N(a)| + |f_N(a) - f(a)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

where I have used $\star\star$ for the middle term and $\star$ for the left and rightmost terms. Thus $\lim_{z\to a} f(z) = f(a)$ and as $a \in E$ was arbitrary we have shown f continuous on E. $\square$

I followed the lead of [J02] page 246 where they offer the same proof for an arbitary *metric space*.

Theorem 10.3.9. *Let γ be a piecewise smooth curve in the complex plane. If $\{f_j\}$ is a sequence of continuous complex-valued functions on γ, and if $\{f_j\}$ converges uniformly to f on γ then $\int_\gamma f_j(z)dz$ converges to $\int_\gamma f(z)dz$.*

Proof: let ϵ_j be the worst-case estimator for $f_j - f$ on γ then $|f_j(z) - f(z)| \leq \epsilon_j$ for all $z \in [\gamma]$. Let γ have length L and apply the ML-estimate:

$$\left| \int_\gamma (f_j(z) - f(z))dz \right| \leq \epsilon_j L.$$

Thus, as $j \to \infty$ we find $\left| \int_\gamma f_j(z)dz - \int_\gamma f(z)dz \right| \to 0$. $\square$

This theorem is also true in the real case as you may read on page 249 of [J02]. However, that proof requires we understand the real analysis of integrals which is addressed by our real analysis course. The ML-theorem is the hero here. Furthermore, in the same section of [J02] you'll find what additional conditions are needed to preserve differentiability past the limiting process.

The definitions given for series below are quite natural. As a guiding concept, we say X is a feature of a series if X is a feature of the sequence of partial sums.

Definition 10.3.10. *Let $\sum_{j=0}^{\infty} f_j$ be a sequence of complex-valued functions on E. The partial sums are functions defined by $S_n(z) = \sum_{j=0}^{n} f_j(z) = f_0(z) + f_1(z) + \cdots + f_n(z)$ for each $z \in E$. The series $\sum_{j=0}^{\infty} f_j$ converges pointwise on E iff $\{S_n(z)\}$ converges pointwise on E. The series $\sum_{j=0}^{\infty} f_j$ converges uniformly on E iff $\{S_n(z)\}$ converges uniformly on E.*

The theorem below gives us an analog of the comparison test for series of complex functions.

Theorem 10.3.11. Weierstrauss M-Test: *suppose $M_k \geq 0$ and $\sum M_k$ converges. If g_k are complex-valued functions on a set E such that $|g_k(z)| \leq M_k$ for all $z \in E$ then $\sum g_k$ converges uniformly on E.*

Proof: let $z \in E$ and note that $|g_k(z)| \leq M_k$ implies that $\sum |g_k(z)|$ is convergent by the comparison test Theorem 10.2.4. Moreover, as absolute convergence implies convergence we have $\sum_{k=0}^{\infty} g_k(z) = g(z) \in \mathbb{C}$ with $|g(z)| \leq \sum |g_k(z)| \leq \sum M_k$ by Theorem 10.2.8. The difference between the series and the partial sum is bounded by the tail of the **majorant series**

$$\left| g(z) - \sum_{k=0}^{n} g_k(z) \right| = \left| \sum_{k=n+1}^{\infty} g_k(z) \right| \leq \sum_{k=n+1}^{\infty} M_k.$$

However, this shows a worst-case estimator for $S_n(z) - g(z)$ is given by $\epsilon_n = \sum_{k=n+1}^{\infty} M_k$. We argue $\epsilon_n = \sum_{k=n+1}^{\infty} M_k \to 0$ as $n \to \infty$ for each $z \in E$ hence $\sum g_k$ converges uniformly on E. $\square$
For future reference:

Definition 10.3.12. *A given series of functions $\sum f_j$ on E is dominated by M_j if $|f_j(z)| \leq M_j$. When $\sum M_j$ converges we call M_j a* **majorant** *for $\sum f_j$.*

Just to reiterate: if we can find a majorant for a given series of functions then it serves to show the series is uniformly convergent by Weierstrauss' M-Test. Incidentally, as a historical aside, Weierstrauss gave this M-test as a footnote on page 202 of his 1880 work *Zur Functionenlehre* see [R91] page 103.

Example 10.3.13. *The geometric series $\displaystyle\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ converges for each $z \in \mathbb{C}$ with $|z| < 1$. Consider that in Example 10.2.9 we derived:*

$$\left| \sum_{k=0}^{\infty} z^k - \sum_{k=0}^{n} z^k \right| = \frac{|z|^{n+1}}{|1-z|}.$$

Notice $\sup_{|z|<1} \left(\frac{|z|^{n+1}}{1-|z|} \right)$ is unbounded hence $\sum_{k=0}^{\infty} z^k$ does not converge uniformly on $\mathbb{E} = \{z \in \mathbb{C} \mid |z| < 1\}$. However, if $0 < R < 1$ we consider a disk $D_R = \{z \in \mathbb{C} \mid |z| < R\}$. We can find a majorant for the geometric series $\sum_{k=0}^{\infty} z^k$ as follows: let $M_k = R^k$ for each $z \in D_R$ note $|z^k| = |z|^k \leq R^k$ and $\sum_{k=0}^{\infty} R^k = \frac{1}{1-R}$. Therefore, $\sum_{k=0}^{\infty} z^k$ is uniformly convergent on D_R by Weierstrauss' M-Test.

The example above explains why $\sum_{k=0}^{\infty} z^k$ is pointwise convergent, but not uniformly convergent, on the entire open unit-disk $\mathbb{E}$. On the other hand, we have uniform convergence on any closed disk inside $\mathbb{E}$.

Example 10.3.14. *Consider $\sum_{k=1}^{\infty} \frac{z^k}{k^3}$. If we consider $|z| < 1$ notice we have the inequality $\left|\frac{z^k}{k^3}\right| = \frac{|z|^k}{k^3} \leq \frac{1}{k^3}$. Recall from calculus II that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is the $p = 3$ series which converges. Therefore, by the Weierstrauss M-test, we find $\sum_{k=1}^{\infty} \frac{z^k}{k^3}$ converges uniformly on $|z| < 1$.*

We now turn to complex analysis. In particular, we work to describe how holomorphicity filters through sequential limits. The theorem below is somewhat shocking given what we saw in the real case in Example 10.3.5.

Theorem 10.3.15. *If $\{f_j\}$ is a sequence of holomorphic functions on a domain D that converge uniformly to f on D then f is holomorphic on D.*

Proof: We follow Gamelin and use Morera's Theorem. To begin, We need continuity to apply Morera's Theorem. Notice f_j holomorphic implies f_j converges to f which is continuous on D by the supposed uniform covergence and Theorem 10.3.8.

let R be a rectangle in D with sides parallel to the coordinate axes. Uniform convergence of the sequence and Theorem 10.3.9 shows:

$$\lim_{j\to\infty}\int_{\partial R} f_j(z)dz = \int_{\partial R}\lim_{j\to\infty}\left(f_j(z)\right)dz = \int_{\partial R} f(z)dz.$$

Consider that $f_j \in \mathcal{O}(D)$ allows us to apply Morera's Theorem to deduce $\int_{\partial R} f_j(z)dz = 0$ for each j. Therefore, $\int_{\partial R} f(z)dz = \lim_{j\to\infty}(0) = 0$. However, as R was arbitrary, we have by Morera's Theorem that f is holomorphic on D. $\square$

I suspect the discussion of continuity above is a vestige of our unwillingness to embrace Goursat's result in Gamelin.

Theorem 10.3.16. *Suppose that $\{f_j\}$ is holomorphic for $|z - z_o| \leq R$, and suppose that the sequence $\{f_j\}$ converges uniformly to f for $|z - z_o| \leq R$. Then for each $r < R$ and for each $m \geq 1$, the sequence of m-th derivatives $\{f_j^{(m)}\}$ converges uniformly to $f^{(m)}$ for $|z - z_o| \leq r$.*

Proof: as the convergence of $\{f_j\}$ is uniform we may select ϵ_j such that $|f_j(z) - f(z)| \leq \epsilon_j$ for $|z - z_o| < R$ where $\epsilon_j \to 0$ as $j \to \infty$. Fix s such that $r < s < R$. Apply the Cauchy Integral Formula for the m-th derivative of $f_j(z) - f(z)$ on the disk $|z - z_o| \leq s$:

$$f_j^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i}\oint_{|z-z_o|=s}\frac{f_j(w) - f(w)}{(w-z)^{m+1}}\,dw$$

for $|z - z_o| \leq r$. Consider, if $|w - z_o| = s$ and $|z - z_o| \leq r$ then

$$|w - z| = |w - z_o + z_o - z| \geq ||w - z_o| - |z - z_o|| = |s - |z - z_o|| \geq |s - r|.$$

Thus $|w - z| \geq s - r$ and it follows that

$$\left|\frac{f_j(w) - f(w)}{(w-z)^{m+1}}\right| \leq \frac{\epsilon_j}{(s-r)^{m+1}}$$

Therefore, as $L = 2\pi s$ for $|z - z_o| = s$ the ML-estimate provides:

$$|f_j^{(m)}(z) - f^{(m)}(z)| \leq \frac{m!}{2\pi i}\cdot\frac{\epsilon_j}{(s-r)^{m+1}}\cdot 2\pi s = \rho_j \qquad \text{(this defines } \rho_j)$$

for $|z - z_o| \leq r$. Notice, m is fixed thus $\rho_j \to 0$ as $j \to \infty$. In other words, ρ_j serves as the worst-case estimator for the m-th derivative and we have established the uniform convergence of $\{f_j^{(m)}\}$ for $|z - z_o| \leq r$. $\square$

I believe there are a couple small typos in Gamelin's proof on 136-137. They are corrected in what is given above.

Definition 10.3.17. *A sequence $\{f_j\}$ of holomorphic functions on a domain D* **converges normally** *to an analytic function f on D if it converges uniformly to f on each closed disk contained in D.*

Gamelin points out this leads immediately to our final theorem for this section: (this is really just Theorem 10.3.16 rephrased with our new normal convergence terminology)

Theorem 10.3.18. *Suppose that $\{f_j\}$ is a sequence of holomorphic functions on a domain D that converges normally on D to the holomorphic function f. Then for each $m \geq 1$, the sequence of m-th derivatives $\{f_j^{(m)}\}$ converges normally to $f^{(m)}$ on D.*

We already saw this behaviour with the geometric series. Notice that Example 10.3.13 shows $\sum_{j=0}^{\infty} z^j$ converges normally to $\frac{1}{1-z}$ on $\mathbb{E} = \{z \in \mathbb{C} \mid |z| < 1\}$. Furthermore, we ought to note that the Weierstrauss M-test provides normal convergence. See [R91] page 92-93 for a nuanced discussion of the applicability and purpose of each mode of convergence. In summary, local uniform convergence is a natural mode for sequences of holomorphic functions whereas, normal convergence is the prefered mode of convergence for series of holomorphic functions. If the series are not normally convergent then we face the rearrangement ambiguity just as we did in the real case. Finally, a historical note which is a bit amusing. The term *normally convergent* is due to Baire of the famed Baire Catagory Theorem. From page 107 of [R91]

> Although in my opinion the introduction of new terms must only be made with extreme prudence, it appeared indispensable to me to characterize by a brief phrase the simplest and by far the most prevalent case of uniformly convergent series, that of series whose terms are smaller in modulus than positive numbers forming a convergent series (what one sometimes calls the Weierstrauss criterion). I call these series *normally* convergent, and I hope that people will be willing to excuse this innovation. A great number of demonstrations, be they in theory of series or somewhat further along in the theory of infinite products, are considerably simplified when one advances this notion, which is much more manageable than that of uniform convergence. (1908)

10.4 Power Series

In this section we study series of power functions.

Definition 10.4.1. *A* **power series centered at** z_o *is a series of the form* $\displaystyle\sum_{k=0}^{\infty} a_k(z-z_o)^k$ *where* $a_k, z_o \in \mathbb{C}$ *for all* $k \in \mathbb{N} \cup \{0\}$*. We say* a_k *are the* **coefficients of the series**.

Example 10.4.2. $\sum_{k=0}^{\infty} \frac{2^k}{k!}(z-3i)^k$ *is a power series centered at* $z_o = 3i$ *with coefficient* $a_k = \frac{2^k}{k!}$.

I will diverge from Gamelin slightly here and add some structure from [R91] page 110-111.

Lemma 10.4.3. Abel's Convergence Lemma: *Suppose for the power series $\sum a_k z^k$ there are positive real numbers s and M such that $|a_k|s^k \leq M$ for all k. Then this power series is normally convergent in $\{z \in \mathbb{C} \mid |z| < s\}$.*

Proof: consider r with $0 < r < s$ and let $q = r/s$. Observe, for $z \in \{z \in \mathbb{C} \mid |z| < r\}$,

$$|a_k z^k| < |a_k| r^k = |a_k| s^k \left(\frac{r}{s}\right)^k \leq M q^k$$

The series $\sum M q^k$ is geometric with $q = r/s < 1$ hence $\sum M q^k = \frac{M}{1-q}$. Therefore, by Weierstrauss' criterion we find $\sum a_k z^k$ is normally convergent on $\{z \in \mathbb{C} \mid |z| < s\}$. $\square$
This leads to the insightful result below:

Corollary 10.4.4. *If the series $\sum a_k z^k$ converges at $z_o \neq 0$, then it converges normally in the open disk $\{z \in \mathbb{C} \mid |z| < |z_o|\}$.*

Proof: as $\sum a_k z_o^k$ converges we have $a_k z_o^k \to 0$ as $k \to \infty$. Thus, $|a_k||z_o^k| \to 0$ as $k \to \infty$. Consequently, the sequence $\{|a_k||z_o^k|\}$ of positive terms is convergent and hence bounded. That is, there exists $M > 0$ for which $a_k||z_o^k| \leq M$ for all k. $\square$

The result above is a guiding principle as we search for possible domains of a given power series. If we find even one point at a certain distance from the center of the expansion then the whole disk is included in the domain. On the other hand, if we found the series diverged at a particular point then we can be sure no larger disk is included in the domain of power series. However, there might be points closer to the center which are also divergent. To find the domain of convergence we need to find the closest singularity to the center of the expansion (the center was $z = 0$ in Lemma and Corollary above, but, clearly these results translate naturally to series of the form $\sum a_k (z - z_o)^k$). Indeed, we should make a definition in view of our findings:

Definition 10.4.5. *A power series $\displaystyle\sum_{k=0}^{\infty} a_k (z - z_o)^k$ has* **radius of convergence** *R if the series converges for $|z - z_o| < R$ but diverges for $|z - z_o| > R$. In the case the series converges everywhere we say $R = \infty$ and in the case the series only converges at $z = z_o$ we say $R = 0$.*

It turns out the concept above is meaningful for **all** power series:

Theorem 10.4.6. *Let $\sum a_k (z - z_o)^k$ be a power series. Then there is R, $0 \leq R \leq \infty$ such that $\sum a_k (z - z_o)^k$ converges normally on $\{z \in \mathbb{C} \mid |z - z_o| < R\}$, and $\sum a_k (z - z_o)^k$ does not converge if $|z - z_o| > R$.*

Proof: Let us define (this can be a non-negative real number or ∞)

$$R = \sup\{t \in [0, \infty) \mid |a_k| t^k \text{ is a bounded sequence}\}$$

If $R = 0$ then the series converges only at $z = z_o$. Suppose $R > 0$ and let s be such that $0 < s < R$. By construction of R, the sequence $|a_k| s^k$ is bounded and by Abel's convergence lemma $\sum a_k (z - z_o)^k$ is normally convergent in $\{z \in \mathbb{C} \mid |z - z_o| < s\}$. However, $\{z \in \mathbb{C} \mid |z - z_o| < R\}$ is formed by a union of the open s-disks and thus we find normal convergence on the open R-disk centered at z_o. $\square$

The proof above is from page 111 of [R91]. Note the union argument is similar to $V.2\#10$ of page 138 in Gamelin where you were asked to show uniform convergence extends to finite unions.

Example 10.4.7. *The series $\sum_{k=0}^{\infty} z^k$ is the geometric series. We have shown it converges iff $|z| < 1$ which shows $R = 1$.*

Example 10.4.8. *The series $\sum_{k=1}^{\infty} \frac{z^k}{k^4}$ has majorant $M_k = 1/k^4$ for $|z| < 1$. Recall, by the p-series test, with $p = 4 > 1$ the series $\sum_{k=1}^{\infty} \frac{1}{k^4}$ converges. Thus, the given series in z is normally convergent on $|z| < 1$.*

Example 10.4.9. *Consider $\displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j}{4^j} (z - i)^{2j}$. Notice this is geometric, simply let $w = -(z - i)^2/4$ and note:*

$$w^j = \left(\frac{-(z - i)^2}{4}\right)^j = \frac{(-1)^j (z - i)^{2j}}{4^j} \Rightarrow \sum_{j=0}^{\infty} \frac{(-1)^j}{4^j} (z - i)^{2j} = \sum_{j=0}^{\infty} w^j = \frac{1}{1 - w} = \frac{1}{1 + (z - i)^2/4}.$$

The convergence above is only given if we have $|w| < 1$ which means $|-(z-i)^2/4| < 1$ which yields $|z-i| < 2$. The given series represents the function $f(z) = \frac{1}{1+(z-i)^2/4}$ on the open disk $|z-i| < 2$. The power series $\displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j}{4^j}(z-i)^{2j}$ is centered at $z_o = i$ and has $R = 2$.

It is customary to begin series where the formula is reasonable when the start of the sum is not indicated.

Example 10.4.10. *The series $\sum k^k z^k$ has $R = 0$. Notice this series diverges by the n-th term test whenever $z \neq 0$.*

Example 10.4.11. *The series $\sum k^{-k} z^k$ has $R = \infty$. To see this, apply of Theorem 10.4.17 .*

At times I refer to what follows as Taylor's Theorem. This is probably not a good practice since Taylor's work was in the real domain and we make no mention of an estimate on the remainder term. That said, Cauchy has enough already so I continue this abuse of attribution.

Theorem 10.4.12. *Let $\sum a_k(z-z_o)^k$ be a power series with radius of convergence $R > 0$. Then, the function*

$$f(z) = \sum a_k(z-z_o)^k, \qquad |z-z_o| < R,$$

is holomorphic. The derivatives of $f(z)$ are obtained by term-by-term differentiation ,

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z-z_o)^{k-1}, \qquad f''(z) = \sum_{k=2}^{\infty} k(k-1) a_k (z-z_o)^{k-2},$$

and similarly for higher-order derivatives. The coefficients are given by:

$$a_k = \frac{1}{k!} f^{(k)}(z_o), \qquad k \geq 0.$$

Proof: by Theorem 10.4.6 the given series is normally convergent on $D_R(z_o)$; recall, $D_R(z_o) = \{z \in \mathbb{C} \mid |z-z_o| < R\}$. Notice that, for each $k \in \{0\} \cup \mathbb{N}$, $f_k(z) = a_k(z-z_o)^k$ is holomorphic on $D_R(z_o)$ hence by Theorem 10.3.15 we find $f(z)$ is holomorphic on $D_R(z_o)$. Furthermore, by Theorem 10.3.16, f' and f'' are holomorphic on $D_R(z_o)$ and are formed by the series of derivatives and second derivatives of $f_k(z) = a_k(z-z_o)^k$. We can calculate,

$$\frac{df_k}{dz} = k a_k (z-z_o)^{k-1} \qquad \& \qquad \frac{d^2 f_k}{dz^2} = k(k-1) a_k (z-z_o)^{k-2}.$$

Finally, the k-th coefficients of the series may be selected by evaluation at z_o of the k-th derivative of f. For $k = 0$ notice

$$f(z_o) = a_o + a_1(z_o - z_o) + a_2(z_o - z_o)^2 + \cdots = a_o$$

thus, as $f^{(0)}(z) = f(z)$ we have $f^{(0)}(z_o) = a_o$. Consider $f^{(k)}(z)$, apply the earlier result of this theorem for the k-th derivative,

$$f^{(k)}(z) = \sum_{j=k}^{\infty} j(j-1)(j-2)\cdots(j-k+1) a_j (z-z_o)^{j-k}$$

evaluate the above at $z = z_o$, only $j - k = 0$ gives nonzero term:

$$f^{(k)}(z_o) = k(k-1)(k-2)\cdots(k-k+1)a_k = k!a_k \qquad \Rightarrow \qquad a_k = \frac{f^{(k)}(z_o)}{k!}. \qquad \square$$

The next few examples illustrate an important calculational technique in this course. Basically, the idea is to twist geometric series via the term-by-term calculus to obtain near-geometric series. This allows us a wealth of examples with a minimum of calculation. I begin with a basic algebra trick before moving to the calculus-based slight of hand.

Example 10.4.13.

$$\sum_{k=0}^{\infty} z^{3k+4} = \sum_{k=0}^{\infty} z^4 z^{3k} = z^4 \sum_{k=0}^{\infty} (z^3)^k = \frac{z^4}{1 - z^{3k}}.$$

The series above normally converges to $f(z) = \frac{z^4}{1-z^{3k}}$ for $|z^3| < 1$ which is simply $|z| < 1$.

Example 10.4.14.

$$\sum_{k=0}^{\infty} \left(z^{2k} + (z-1)^{2k} \right) = \sum_{k=0}^{\infty} z^{2k} + \sum_{k=0}^{\infty} (z-1)^{2k} = \frac{1}{1-z^2} + \frac{1}{1-(z-1)^2}$$

where the geometric series both converge only if we have a simultaneous solution of $|z| < 1$ and $|z - 1| < 1$. The open region on which the series above converges is not a disk. Why does this not contradict Theorem 10.4.6 ?

Ok, getting back to the calculus tricks I mentioned previous to the above pair of examples,

Example 10.4.15. *Notice $f(z) = \frac{1}{1-z^2}$ has $\frac{df}{dz} = \frac{2z}{(1-z^2)^2}$. However, for $|z^2| < 1$ which is more naturally presented as $|z| < 1$ we have:*

$$f(z) = \frac{1}{1-z^2} = \sum_{k=0}^{\infty} z^{2k} \quad \Rightarrow \quad \frac{df}{dz} = \sum_{k=1}^{\infty} 2kz^{2k-1}.$$

Therefore, we discover, for $|z| < 1$ the function $g(z) = \frac{2z}{(1-z^2)^2}$ has the following power series representation centered at $z_o = 0$,

$$\frac{2z}{(1-z^2)^2} = \sum_{k=1}^{\infty} 2kz^{2k-1} = 2z + 4z^3 + 6z^5 + \cdots .$$

Example 10.4.16. *The singularity of $f(z) = Log(1 - z)$ is found at $z = 1$ hence we have hope to look for power series representations for this function away from $z_o = 1$. Differentiate $f(z)$ to obtain (note, the -1 is from the chain rule):*

$$\frac{df}{dz} = \frac{-1}{1-z} = -\sum_{k=0}^{\infty} z^k.$$

Integrate both sides of the above to see that there must exist a constant C for which

$$Log(1-z) = C - \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1}$$

But, we have $Log(1-0) = 0 = C$ hence,

$$-Log(1-z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots .$$

The calculation above holds for $|z| < 1$ according to the theorems we have developed about the geometric series and term-by-term calculus. However, in this case, we may also observe $z = -1$ produces the negative of alternating harmonic series which converges. Thus, there is at least one point on which the series for $-Log(1-z)$ converges where the differentiated series did not converge. This is illustrative of a general principle which is worth noticing: differentiation may remove points from the boundary of the disk of convergence whereas integration tends to add points of convergence on the boundary.

Theorem 10.4.17. *If $|a_k/a_{k+1}|$ has a limit as $k \to \infty$, either finite or $+\infty$, then the limit is the radius of convergence R of $\sum a_k(z-z_o)^k$*

Proof: Let $L = \lim_{k\to\infty} |a_k/a_{k+1}|$. If $r < L$ then there must exist $N \in \mathbb{N}$ such that $|a_k/a_{k+1}| > r$ for all $k > N$. Observe $|a_k| > r|a_{k+1}|$ for $k > N$. It follows,

$$|a_N|r^N \geq |a_{N+1}|r^{N+1} \geq |a_{N+2}|r^{N+2} \geq \cdots$$

Let $M = \max\{|a_o|, |a_1|r, \dots, |a_{N-1}|r^{N-1}, |a_N|r^N\}$ and note $|a_k|r^k \leq M$ for all k hence by Abel's Convergence Lemma, the power series $\sum a_k(z-z_o)^k$ is normally convergent for $|z| < r$. Thus, $r \leq R$ as R defines the maximal disk on which $\sum a_k(z-z_o)^k$ is normally convergent. Let $\{r_n\}$ be a sequence of such that $r_n < L$ for each n and $r_n \to L$ as $n \to \infty$. For $r_n < L$ we've shown $r_n \leq R$ hence $\lim_{n\to\infty} r_n \leq \lim_{n\to\infty} R$ by the sandwhich theorem. Thus $L \leq R$.

Suppose $s > L$. We again begin with an observation that there exists an $N \in \mathbb{N}$ such that $|a_k/a_{k+1}| < s$ for $k > N$. It follows,

$$|a_N|s^N \leq |a_{N+1}|s^{N+1} \leq |a_{N+2}|s^{N+2} \leq \cdots$$

and clearly $\sum a_k(z-z_o)^k$ fails the n-th term test for $z \in \mathbb{C}$ with $|z-z_o| > s$. We find the series diverges for $|z-z_o| > s$ and thus we find $s \geq R$. Let $\{s_n\}$ be a sequence of values with $s_n > L$ for each n and $\lim_{n\to\infty} s_n = L$. The argument we gave for s equally well applies to each s_n hence $s_n \geq R$ for all n. Once again, take $n \to \infty$ and apply the sandwhich lemma to obtain $\lim_{n\to\infty} s_n = L \leq R$.

Thus $L \leq R$ and $L \geq R$ and we conclude $L = R$ as desired. $\square$

Theorem 10.4.18. *If $\sqrt[k]{|a_k|}$ has a limit as $k \to \infty$, either finite or $+\infty$, then the radius of convergence R of $\sum a_k(z-z_o)^k$ is given by:*

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}}.$$

Proof: see page 142. Again, you can see Abel's Convergence Lemma at work. $\square$

One serious short-coming of the ratio and root tests is their failure to apply to series with infinitely many terms which are zero. The **Cauchy Hadamard** formula gives a refinement which allows us to capture such examples. In short, the limit superior replaces the limit in Theorem 10.4.18. If you would like to read more, I recommend page 112 of [R91].

10.5 Power Series Expansion of an Analytic Function

In the previous section we studied some of the basic properties of complex power series. Our main result was that a function defined by a power series is holomorphic on the open disk of convergence. We discover a converse in this section: holomorphic functions on a disk admit power series representation on the disk. We finally introduce the term *analytic*

Definition 10.5.1. *A function $f(z)$ is* **analytic** *on $D_R(z_o) = \{z \in \mathbb{C} \mid |z - z_o| < R\}$ if there exist coefficients $a_k \in \mathbb{C}$ such that $f(z) = \sum_{k=0}^{\infty} a_k(z - z_o)^k$ for all $z \in D_R(z_o)$.*

Of course, by Theorem 10.4.12 we immediately know $f(z)$ analytic on some disk about z_o forces the coefficients to follow Taylor's Theorem $a_k = f^{(k)}(z_o)/k!$. Thus, another way of characterizing an analytic function is that an analytic function is one which is generated by its Taylor series[8].

Theorem 10.5.2. *Suppose $f(z)$ is holomorphic for $|z - z_o| < \rho$. Then $f(z)$ is represented by the power series*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_o)^k, \qquad |z - z_o| < \rho,$$

where

$$a_k = \frac{f^{(k)}(z_o)}{k!}, \qquad k \geq 0,$$

and where the power series has radius of convergence[9] $R \geq \rho$. For any fixed r, $0 < r < \rho$, we have

$$a_k = \frac{1}{2\pi i} \oint_{|w - z_o| = r} \frac{f(w)}{(w - z_o)^{k+1}}\, dw, \qquad k \geq 0.$$

Further, if $|f(z)| \leq M$ for $|z - z_o| = r$, then

$$|a_k| \leq \frac{M}{r^k}, \qquad k \geq 0.$$

Proof: assume $f(z)$ is as stated in the theorem. Let $z \in \mathbb{C}$ such that $|z| < r < \rho$. Suppose $|w| = r$ then by the geometric series Proposition 10.2.6

$$\frac{f(w)}{w - z} = \frac{f(w)}{w} \frac{1}{1 - z/w} = \frac{f(w)}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k = \sum_{k=0}^{\infty} f(w) \frac{z^k}{w^{k+1}}.$$

Moreover, we are given the convergence of the above series is uniform for $|w| = r$. This allows us to expand Cauchy's Integral formula into the integral of a series of holomorphic functions which converges uniformly. It follows we are free to apply Theorem 10.3.9 to exchange the order of the integration and the infinite summation in what follows:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w - z}\, dw \\ &= \frac{1}{2\pi i} \int_{|w|=r} \left(\sum_{k=0}^{\infty} f(w) \frac{z^k}{w^{k+1}} \right) dw \\ &= \sum_{k=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w^{k+1}}\, dw \right)}_{a_k} z^k. \end{aligned}$$

[8] again, I feel obligated to mention Taylor's work was in the real domain, so this term is primarily to allow the reader to connect with their experience with real power series

[9] we should remember Theorem 10.4.6 provides the series is normally convergent

This suffices to prove the theorem in the case $z_o = 0$. Notice the result holds whenever $|z| < r$ and as $r < \rho$ is arbitrary, we must have the radius of convergence[10] $R \geq \rho$. Continuing, I reiterate the argument for $z_o \neq 0$ as I think it is healthy to see the argument twice and as the algebra I use in this proof **is relevant** to future work on a multitude of examples.

Suppose $z \in \mathbb{C}$ such that $|z - z_o| < r < \rho$. Suppose $|w - z_o| = r$ hence $|z - z_o|/|w - z_o| < 1$ thus:

$$\begin{aligned}
\frac{f(w)}{w-z} &= \frac{f(w)}{w - z_o - (z - z_o)} \\
&= \frac{f(w)}{w - z_o} \cdot \frac{1}{1 - \left(\frac{z-z_o}{w-z_o}\right)} \\
&= \frac{f(w)}{w - z_o} \sum_{k=0}^{\infty} \left(\frac{z - z_o}{w - z_o}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{f(w)(z - z_o)^k}{(w - z_o)^{k+1}}
\end{aligned}$$

Thus, following the same logic as in the $z_o = 0$ case, but now for $|w - z_o| = r$, we obtain:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|w-z_o|=r} \frac{f(w)}{w-z}\, dw \\
&= \frac{1}{2\pi i} \int_{|w-z_o|=r} \left(\sum_{k=0}^{\infty} \frac{f(w)(z-z_o)^k}{(w-z_o)^{k+1}} \right) dw \\
&= \sum_{k=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{|w-z_o|=r} \frac{f(w)}{(w-z_o)^{k+1}}\, dw \right)}_{a_k} (z - z_o)^k.
\end{aligned}$$

Once again we can argue that as $|z - z_o| < r < \rho$ gives $f(z)$ presented as the power series centered at z_o above for arbitrary r it must be that the radius of convergence $R \geq \rho$.

The derivative identity $a_k = \frac{f^{(k)}(z_o)}{k!}$ is given by Theorem 10.4.12 and certain applies here as we have shown the power series representation of $f(z)$ exists. Finally, if $|f(z)| \leq M$ for $|z - z_o| < r$ then apply Cauchy's Estimate 9.5.1

$$|a_k| = \left| \frac{f^{(k)}(z_o)}{k!} \right| \leq \frac{1}{k!} \frac{Mk!}{r^k} = \frac{M}{r^k} \qquad \square$$

Consider the argument of the theorem above. If you were a carefree early nineteenth century mathematician you might have tried the same calculations. If you look at was derived for a_k and compare the differential to the integral result then you would have **derived** the Generalized Cauchy Integral Formula:

$$a_k = \frac{f^{(k)}(z_o)}{k!} = \frac{1}{2\pi i} \int_{|w-z_o|=r} \frac{f(w)}{(w-z_o)^{k+1}}\, dw.$$

You can contrast our viewpoint now with that which we proved the Generalized Cauchy Integral Formula back in Theorem 9.4.2. The technique of expanding $\frac{1}{w-z}$ into a power series for which

[10] this can be made rigorous with a sequential argument as I offered twice in the proof of Theorem 10.4.17

integration and differentiation term-by-term was to be utilized was known and practiced by Cauchy at least as early as 1831 see page 210 of [R91]. In retrospect, it is easy to see how once one of these theorems was discovered, the discovery of the rest was inevitable to the curious.

What follows is a corollary to Theorem 10.5.2.

Corollary 10.5.3. *Suppose $f(z)$ and $g(z)$ are holomorphic for $|z-z_o|<r$. If $f^{(k)}(z_o)=g^{(k)}(z_o)$ for $k\geq 0$ then $f(z)=g(z)$ for $|z-z_o|<r$.*

Proof: if f,g are holomorphic on $|z-z_o|<r$ then Theorem 10.5.2 said they are also analytic on $|z-z_o|<r$ with coefficients fixed by the values of the function and their derivatives at z_o. Consequently, both functions share identical power series on $|z-z_o|<r$ hence their values match at each point in the disk. □

Theorem 10.4.6 told us that the domain of a power series included an open disk of some maximal radius R. Now, we learn that if $f(z)$ is holomorphic on an open disk centered at z_o then it has a power series representation on the disk. It follows that the function cannot be holomorphic beyond the radius of convergence given to us by Theorem 10.4.6 for if it did then we would find the power series centered at z_o converged beyond the radius of convergence.

Corollary 10.5.4. *Suppose $f(z)$ is analytic at z_o, with power series expansion centered at z_o; $f(z)=\sum_{k=0}^{\infty}a_k(z-z_o)^k$. The radius of convergence of the power series is the largest number R such that $f(z)$ extends to be holomorphic on the disk $\{z\in\mathbb{C}\mid |z-z_o|<R\}$*

Notice that power series converge normally on the disk of their convergence. It seems that Gamelin is unwilling to use the term *normally convergent* except to introduce it. Of course, this is not a big deal, we can either use the term or state it's equivalent in terms of uniform convergence on closed subsets.

Example 10.5.5. *Let $f(z)=\displaystyle\sum_{k=0}^{\infty}\frac{1}{k!}z^k=1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots$. We can show $f(z)f(w)=f(z+w)$ by direct calculation of the Cauchy product. Once that is known and we observe $f(0)=0$ then it is simple to see $f(z)f(-z)=f(z-z)=f(0)=1$ hence $\frac{1}{f(z)}=f(-z)$. Furthermore, we can easily show $\frac{df}{dz}=f$. All of these facts are derived from the arithmetic of power series alone. That said, perhaps you recognize these properties as those of the exponential function. There are two viewpoints to take here:*

1. *define the complex exponential function by the power series here and derive the basic properties by the calculus of series*

2. *define the complex exponential function by $e^{x+iy}=e^x(\cos y+i\sin y)$ and verify the given series represents the complex exponential on $\mathbb{C}$.*

Whichever viewpoint you prefer, we all agree:

$$\boxed{e^z=\sum_{k=0}^{\infty}\frac{1}{k!}z^k=1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots}$$

Notice $a_k=1/k!$ hence $a_k/a_{k+1}=(k+1)!/k!=k+1$ hence $R=\infty$ by ratio test for series.

Example 10.5.6. *Consider $f(z) = \cosh z$ notice $f'(z) = \sinh z$ and $f''(z) = \cosh z$ and in general $f^{(2k)}(z) = \cosh z$ and $f^{(2k+1)}(z) = \sinh z$. We calculate $f^{(2k)}(0) = \cosh 0 = 1$ and $f^{(2k+1)}(0) = \sinh 0 = 0$. Thus,*

$$\boxed{\cosh z = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k} = 1 + \frac{1}{2} z^2 + \frac{1}{4!} z^4 + \cdots}$$

Example 10.5.7. *Following from Definition 2.5.2 we find $e^z = \cosh z + \sinh z$. Thus, $\sinh z = e^z - \cosh z$. Therefore,*

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n - \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k}.$$

However, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^n = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} z^{2k+1}$ hence the even terms cancel and we find the odd series below for hyperbolic sine:

$$\boxed{\sinh z = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} z^{2k+1} = 1 + \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \cdots}$$

Example 10.5.8. *To derive the power series for $\sin z$ and $\cos z$ we use the relations $\cosh(iz) = \cos(z)$ and $\sinh(iz) = i \sin z$ hence*

$$\cos z = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (iz)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

since $i^{2k} = (i^2)^k = (-1)^k$. Likewise, as $i^{2k+1} = i(-1)^k$

$$i \sin z = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (iz)^{2k+1} = i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

Therefore,

$$\boxed{\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = 1 - \frac{1}{2} z^2 + \frac{1}{4!} z^4 + \cdots}$$

and

$$\boxed{\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \cdots .}$$

Once again, I should comment, we could use the boxed formulas above to **define** cosine and sine. It is then straightforward to derive all the usual properties of sine and cosine. A very nice presentation of this is found on pages 274-278 of [J02]. You might be interested to know that π can be carefully defined as twice the smallest positive zero of $\cos z$. Since the series definition of cosine does not implicitly use the definition of π, this gives us a careful, non-geometric, definition of π.

10.6 Power Series Expansion at Infinity

The technique used in this section could have been utilized in earlier discussions of ∞. To study the behaviour of $f(z)$ at $z = \infty$ we simple study the corresponding function $g(w) = f(1/w)$ at $w = 0$.

Example 10.6.1. *Notice* $\lim_{z\to\infty} f(z) = \lim_{w\to 0} f(1/w)$ *allows us to calculate:*

$$\lim_{z\to\infty} \frac{z}{z+1} = \lim_{w\to 0} \frac{1/w}{1/w+1} = \lim_{w\to 0} \frac{1}{1+w} = \frac{1}{1+0} = 1.$$

Definition 10.6.2. *A function $f(z)$ is* **analytic** *at $z = \infty$ if $g(w) = f(1/w)$ is analytic at $w = 0$.*

In particular, we mean that there exist coefficients $b_o, b_1, \dots$ and $\rho > 0$ such that $g(w) = b_o + b_1 w + b_2 w^2 + \cdots$ for all $w \in \mathbb{C}$ such that $0 < |w| < \rho$. Recall, by Theorem 10.5.2 we have $\sum_{k=0} b_k w^k$ converging normally to $g(w)$ on the open disk of convergence. If $|z| > 1/\rho$ then $1/|z| < \rho$ hence

$$f(z) = g(1/z) = b_o + b_1/z + b_2/z^2 + \cdots .$$

The series $b_o + b_1/z + b_2/z^2 + \cdots$ coverges normally to $f(z)$ on the **exterior domain** $\{z \in \mathbb{C} \mid |z| > R\}$ where $R = 1/\rho$. Recall that normal convergence previous meant we had uniform convergence on all closed subdisks, in this context, it means we have uniform convergence for any $S > R$. In particular, for each $S > R$, the series $b_o + b_1/z + b_2/z^2 + \cdots$ converges uniformly to $f(z)$ for $\{z \in \mathbb{C} \mid |z| > S\}$.

Example 10.6.3. *Let $P(z) \in \mathbb{C}[z]$ be a polynomial of order N. Then $P(z) = a_o + a_1 z + \cdots + a_N z^N$ is not analytic at $z = \infty$ as the function $g(w) = a_o + a_1/w + \cdots + a_n/z^N$ is not analytic at $w = 0$.*

Example 10.6.4. *Let $f(z) = \frac{1}{z^2} + \frac{1}{z^{42}}$ is analytic at $z = \infty$ since $g(w) = f(1/w) = w^2 + w^{42}$ is analytic at $w = 0$. In fact, g is entire which goes to show $f(z) = \frac{1}{z^2} + \frac{1}{z^{42}}$ on $\mathbb{C}^\times$. Refering to the terminology just after 10.6.2 we have $\rho = \infty$ hence $R = 0$.*

The example above is a rather silly example of a **Laurent Series**. It is much like being asked to find the Taylor polynomial for $f(z) = z^2 + 3z + 2$ centered at $z = 0$; in the same way, the function is defined by a *Laurent polynomial* centered at $z = 0$, there's nothing to find. The major effort of the next Chapter is to develop theory to understand the structure of these Laurent series.

Example 10.6.5. *Let $f(z) = \frac{z^2}{z^2-1}$ consider $g(w) = f(1/w) = \frac{1/w^2}{1/w^2-1} = \frac{1}{1-w^2} = \sum_{k=0}^{\infty} w^{2k}$. Hence $f(z)$ is analytic at $z = \infty$. Notice, the power series centered at $w = 0$ converges normally on $|w| < 1$ hence the series below converges normally to $f(z)$ for $|z| > 1$*

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^{2k} = 1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots .$$

Example 10.6.6. *Let $f(z) = \sin(1/z^2)$. Notice $g(w) = \sin(w^2) = w^2 - \frac{1}{3!}(w^2)^3 + \cdots$ for $w \in \mathbb{C}$. Thus $f(z)$ is analytic at $z = \infty$ and $f(z)$ is represented normally on the punctured plane by:*

$$f(z) = \frac{1}{z^2} - \frac{1}{3!}\frac{1}{z^6} + \frac{1}{5!}\frac{1}{z^{10}} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\frac{1}{z^{4k+2}}.$$

In summary, we have seen that a function which is analytic at $z = z_o \neq \infty$ allows a power series representation $\sum_{k=0}^{\infty} a_k (z - z_o)^k$ on disk of radius $0 < R \leq \infty$. On the other hand, a function which is analytic at $z = \infty$ has a representation of the form $\sum_{-\infty}^{k=0} a_k z^k = a_o + a_{-1}/z + a_{-2}/z^2 + \cdots$ on an annulus $|z| > R$ where $0 \leq R < \infty$.

Theorem 10.6.7. *If f is analyic at ∞ then there exists $\rho > 0$ such that for $|z - z_o| > \rho$*

$$f(z) = \sum_{k=-\infty}^{0} a_k (z-z_o)^k = a_o + \frac{a_{-1}}{z-z_o} + \frac{a_{-1}}{(z-z_o)^2} + \cdots .$$

I should mention, if you wish a more careful treatment, you might meditate on the arguments offered on page 348 of [R91].

10.7 Manipulation of Power Series

The sum, difference, scalar multiple, product and quotient of power series are discussed in this section.

Theorem 10.7.1. *Suppose $\sum_{k=0}^{\infty} a_k(z-z_o)^k$ and $\sum_{k=0}^{\infty} b_k(z-z_o)^k$ are convergent power series on a domain D then*

$$\sum_{k=0}^{\infty} a_k(z-z_o)^k + c\sum_{k=0}^{\infty} b_k(z-z_o)^k = \sum_{k=0}^{\infty}(a_k + cb_k)(z-z_o)^k$$

for all $z \in D$.

Proof: suppose f, g are analytic on D where $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ and $g(z) = \sum_{k=0}^{\infty} b_k(z-z_o)^k$. Let $c \in \mathbb{C}$ and define $h(z) = f(z) + cg(z)$ for each $z \in D$. Observe,

$$h^{(k)}(z_o) = f^{(k)}(z_o) + cg^{(k)}(z_o) \quad \Rightarrow \quad \frac{h^{(k)}(z_o)}{k!} = \frac{f^{(k)}(z_o)}{k!} + c\frac{g^{(k)}(z_o)}{k!} = a_k + cb_k$$

by Theorem 10.5.2. Thus, $h(z) = \sum_{k=0}^{\infty}(a_k + cb_k)(z-z_o)^k$ by Corollary 10.5.4. $\square$

The method of proof is essentially the same for the product of series theorem. We use Corollary 10.5.4 to obtain equality of functions by comparing derivatives. I suppose we should define the product of series:

Definition 10.7.2. Cauchy Product: *Let $\sum_{k=0}^{\infty} a_k(z-z_o)^k$ and $\sum_{k=0}^{\infty} b_k(z-z_o)^k$ then*

$$\left(\sum_{k=0}^{\infty} a_k(z-z_o)^k\right)\left(\sum_{k=0}^{\infty} b_k(z-z_o)^k\right) = \sum_{k=0}^{\infty} c_k(z-z_o)^k$$

where we define $c_k = \sum_{n=0}^{k} a_n b_{k-n}$ for each $k \geq 0$.

Technically, we ought to wait until we prove the theorem below to make the definition above. I hope you can forgive me.

Theorem 10.7.3. *Suppose $\sum_{k=0}^{\infty} a_k(z-z_o)^k$ and $\sum_{k=0}^{\infty} b_k(z-z_o)^k$ are convergent power series on an open disk D with center $z_o \in D$ then*

$$\left(\sum_{k=0}^{\infty} a_k(z-z_o)^k\right)\left(\sum_{k=0}^{\infty} b_k(z-z_o)^k\right) = \sum_{k=0}^{\infty} c_k(z-z_o)^k$$

for all $z \in D$ where c_k is defined by the Cauchy Product; $c_k = \sum_{n=0}^{k} a_n b_{k-n}$ for each $k \geq 0$.

Proof: I follow the proof on page 217 of [R91]. Let $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ and $g(z) = \sum_{k=0}^{\infty} b_k(z-z_o)^k$ for each $z \in D$. By Theorem 10.4.12 both f and g are holomorphic on D. Therefore, $h = fg$ is holomorphic on D as $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ for each $z \in D$. Theorem 10.5.2 then shows fg is analytic at z_o hence there exist c_k such that $h(z) = f(z)g(z) = \sum_k c_k(z-z_o)^k$. It remains to show that c_k is as given by the Cauchy product. We proceed via Corollary 10.5.4. We need to show $\frac{h^{(k)}(z_o)}{k!} = c_k$ for $k \geq 0$. Begin with $k = 0$,

$$h(z_o) = f(z_o)g(z_o) = a_o b_o = c_o.$$

Continuing, for $k = 1$,

$$h'(z_o) = f'(z_o)g(z_o) + f(z_o)g'(z_o) = a_1 b_0 + a_0 b_1 = c_1.$$

Differentiating once again we find $k = 2$, note $f''(z_o)/2 = a_2$,

$$\begin{aligned} h''(z_o) &= f''(z_o)g(z_o) + f'(z_o)g'(z_o) + g'(z_o)f'(z_o) + f(z_o)g''(z_o) \\ &= 2a_2b_0 + 2a_1b_1 + 2a_0b_2 \\ &= 2c_2. \end{aligned}$$

To treat the k-th coefficient in general it is useful for us to observe the Leibniz k-th derivative rule:

$$(fg)^{(k)}(z) = \sum_{i+j=k} \frac{k!}{i!j!} f^{(i)}(z)g^{(j)}(z) = f^{(k)}(z)g(z) + kf^{(k-1)}(z)g'(z) \cdots + f(z)g^{(k)}(z)$$

Observe, $f^{(i)}(z_o)/i! = a_i$ and $g^{(j)}(z_o)/j! = b_j$ hence:

$$(fg)^{(k)}(z_o) = \sum_{i+j=k} k! a_i b_j = k!(a_o b_k + \cdots + a_k b_o) = k! c_k.$$

Thus, $(fg)^{(k)}(z_o)/k! = c_k$ and the theorem by Corollary 10.5.4. $\square$

I offered the argument for $k = 0, 1$ and 2 explicitly to take the mystery out of the Leibniz rule argument. I leave the proof of the Leibniz rule to the reader. There are other proofs of the product theorem which are just given in terms of the explicit analysis of the series. For example, see Theorem 3.50d of [R76] where the product of a convergent and an absolutely convergent series is shown to converge to an absolutely convergent series defined by the Cauchy Product.

Example 10.7.4. *Find the power series to order 5 centered at $z = 0$ for $2\sin z \cos z$*

$$\begin{aligned} 2\sin z\cos z &= 2\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)\left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + \cdots\right) \\ &= 2\left(z - \left[\frac{1}{2} + \frac{1}{6}\right]z^3 + \left[\frac{1}{24} + \frac{1}{12} + \frac{1}{120}\right]z^5 + \cdots\right) \\ &= 2z - \frac{4}{3}z^3 + \frac{4}{15}z^5 + \cdots \end{aligned}$$

Of course, as $2\sin z\cos z = \sin(2z) = 2z - \frac{1}{3!}(2z)^3 + \frac{1}{5!}(2z)^5 + \cdots$ we can avoid the calculation above. I merely illustrate the consistency.

The example below is typical of the type of calculation we wish to master:

Example 10.7.5. *Calculate the product below to second order in z:*

$$\begin{aligned} e^z\cos(2z+1) &= e^z\left(\cos(2z)\cos(1)-\sin(2z)\sin(1)\right) \\ &= \left(1+z+\frac{1}{2}z^2\right)\left(\cos(1)(1-\frac{1}{2}(2z)^2)-2z\sin(1)\right)+\cdots \\ &= \left(1+z+\frac{1}{2}z^2\right)\left(\cos(1)-2\sin(1)z-2\cos(1)z^2\right)+\cdots \\ &= \cos(1)+[\cos(1)-2\sin(1)]\,z+\left(\frac{\cos(1)}{2}-2\sin(1)-2\cos(1)\right)z^2+\cdots \end{aligned}$$

Stop and ponder why I did not directly expand $\cos(2z+1)$ as $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(2z+1)^{2k+1}$. If you did that, then you would need to gather infinitely many terms together to form the sines and cosines we derived with relative ease from the adding-angles formula for cosine.

The geometric series allows fascinating calculation:

Example 10.7.6. *Multiply* $1+z+z^2+\cdots$ *and* $1-z+z^2+\cdots$.

$$(1+z+z^2+\cdots)(1-z+z^2+\cdots)=\frac{1}{1-z}\cdot\frac{1}{1+z}=\frac{1}{1-z^2}=1+z^2+z^4+\cdots.$$

I probably could add some insight here by merging the calculations I cover in calculus II here, however, I'll stop at this point and turn to the question of division.

Suppose $\sum_{k=0}^{\infty}a_k(z-z_o)^k$ where $a_o\neq 0$. Calculation of $\frac{1}{\sum_{k=0}^{\infty}a_k(z-z_o)^k}$ amounts to calculation of coefficients b_k for $k\geq 0$ such that $\left(\sum_{k=0}^{\infty}a_k(z-z_o)^k\right)\left(\sum_{k=0}^{\infty}b_k(z-z_o)^k\right)=1$. The Cauchy product provides a sequence of equations we must solve:

$$\begin{aligned} a_ob_o=1 &\quad\Rightarrow\quad b_o=1/a_o. \\ a_ob_1+a_1b_o=0, &\quad\Rightarrow\quad b_1=\frac{-a_1b_o}{a_o}=\frac{-a_1}{a_o^2}. \\ a_ob_2+a_1b_1+a_2b_o=0, &\quad\Rightarrow\quad b_2=-\frac{a_1b_1+a_2b_o}{a_o}=\frac{a_1^2}{a_o^3}-\frac{a_2}{a_o^2}. \\ a_ob_3+a_1b_2+a_2b_1+a_3b_0=0 &\quad\Rightarrow\quad b_3=-\frac{a_1b_2+a_2b_1+a_3b_o}{a_o}. \end{aligned}$$

The calculation above can clearly be extended to higher order. Recursively, we have solution:

$$b_k=-\frac{a_1b_{k-1}+a_2b_{k-2}+\cdots+a_{k-1}b_1+a_kb_o}{a_o}$$

for $k\geq 0$.

Example 10.7.7. *Consider* $2-4z+8z^2-16z^3\cdots$ *identify* $a_o=2, a_1=-4, a_2=8$ *and* $a_3=-16$. *Using the general calculation above this example, calculate*

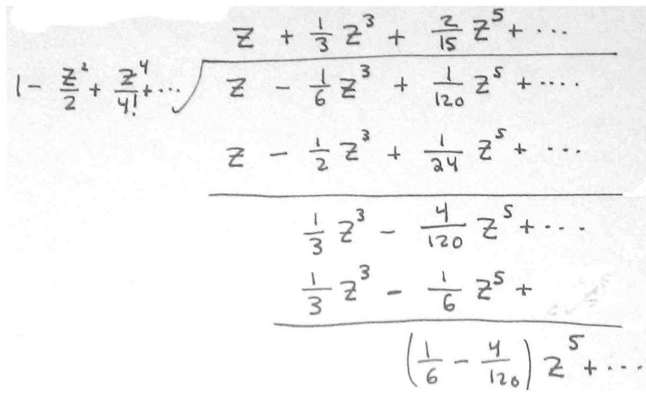
$$b_o=\frac{1}{2},\quad b_1=\frac{4}{4}=1,\quad b_2=\frac{-(-4)(1)-(8)(1/2)}{2}=0,\quad b_3=-\frac{-4(0)+(8)(1)+(-16)(1/2)}{2}=0.$$

Hence,

$$\frac{1}{2-4z+8z^2-16z^3\cdots}=\frac{1}{2}+z+\cdots.$$

I can check our work as $2-4z+8z^2-16z^3\cdots=2(1-2z+(-2z)^2+(-2z)^3\cdots)=\frac{2}{1+2z}$ *hence* $\frac{1}{2-4z+8z^2-16z^3\cdots}=\frac{1+2z}{2}=\frac{1}{2}+z$. *Apparently, we could calculate* $b_k=0$ *for* $k\geq 2$.

We next illustrate how to find the power series for $\tan(z)$ by *long-division*:



The calculation above shows that $\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots$ divided by $\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots$ yields:

$$\tan z = \frac{\sin z}{\cos z} = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots .$$

It should be fairly clear how to obtain higher-order terms by the method of long-division.

We now consider a different method to calculate the power series for $\tan z$ which uses the geometric series to obtain the reciprocal of the cosine series. Consider,

$$\begin{aligned}
\frac{1}{\cos z} &= \frac{1}{1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots} \\
&= \frac{1}{1 - \left(\frac{1}{2}z^2 - \frac{1}{24}z^4 + \cdots\right)} \\
&= 1 + \left(\frac{1}{2}z^2 - \frac{1}{24}z^4 + \cdots\right) + \left(\frac{1}{2}z^2 - \frac{1}{24}z^4 + \cdots\right)^2 + \cdots \\
&= 1 + \frac{1}{2}z^2 + \left(-\frac{1}{24} + \frac{1}{2}\cdot\frac{1}{2}\right)z^4 + \cdots \\
&= 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \cdots .
\end{aligned}$$

Then, to find $\tan(z)$ we simply multiply by the sine series,

$$\begin{aligned}
\sin z \cdot \frac{1}{\cos z} &= \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)\left(1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \cdots\right) \\
&= z + \left(\frac{1}{2} - \frac{1}{6}\right)z^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)z^5 + \cdots \\
&= z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots .
\end{aligned}$$

The recursive technique, long-division and geometric series manipulation are all excellent tools which we use freely throughout the remainder of our study. Some additional techniques are elucidated in §10.9. There I show my standard bag of tricks for recentering series.

10.8 The Zeros of an Analytic Function

Power series are, according to Dr. Monty Kester, *Texas sized polynomials*. With all due respect to Texas, it's not *that* big. That said, power series and polynomials do share much in common. In particular, we find a meaningful and interesting generalization of the *factor theorem*.

Definition 10.8.1. *Let f be an analytic function which is not identically zero near $z = z_o$ then we say f has a zero of order N at z_o if*

$$f(z_o) = 0, \quad f'(z_o) = 0, \ \cdots \ f^{(N-1)}(z_o) = 0, \quad f^{(N)}(z_o) \neq 0.$$

A zero of order $N = 1$ is called a **simple zero***. A zero of order $N = 2$ is called a* **double zero***.*

Suppose $f(z)$ has a zero of order N at z_o. If $f(z) = \sum_{k=0}^{\infty} a_k(z - z_o)^k$ then as $a_k = \frac{f^{(k)}}{k!} = 0$ for $k = 0, 1, \ldots, N - 1$ we have

$$f(z) = \sum_{k=N}^{\infty} a_k(z - z_o)^k = (z - z_o)^N \sum_{k=N}^{\infty} a_k(z - z_o)^{k-N} = (z - z_o)^N \underbrace{\sum_{j=0}^{\infty} a_{j+N}(z - z_o)^j}_{h(z)}$$

Observe that $h(z)$ is also analytic at z_o and $h(z_o) = a_N = \frac{f^{(N)}(z_o)}{N!} \neq 0$. It follows that there exists $\rho > 0$ for which $0 < |z - z_o| < \rho$ implies $f(z) \neq 0$. In other words, the zero of an analytic function is **isolated**.

Definition 10.8.2. *Let $U \subseteq \mathbb{C}$ then $z_o \in U$ is an* **isolated point of** U *if there exists some $\rho > 0$ such that $\{z \in U \mid |z - z_o| < \rho\} = \{z_o\}$.*

We prove that all zeros of an analytic function are isolated a bit later in this section. However, first let me record the content of our calculations thus far:

Theorem 10.8.3. Factor Theorem for Power Series: *If $f(z)$ is an analytic function with a zero of order N at z_o then there exists $h(z)$ analytic at z_o with $h(z_o) \neq 0$ and $f(z) = (z - z_o)^N h(z)$.*

Example 10.8.4. *The prototypical example is simply the monomial $f(z) = (z - z_o)^n$. You can easily check f has a zero $z = z_o$ of order n.*

Example 10.8.5. *Consider $f(z) = \sin(z^2) = z^2 - \frac{1}{6}z^6 + \frac{1}{120}z^{10} + \cdots$. Notice $f(0) = f'(0) = 0$ and $f''(0) = 2$ thus $f(z)$ as a double zero of $z = 0$ and we can factor out z^2 from the power series centered at $z = 0$ for $f(z)$:*

$$f(z) = z^2 \left(1 - \frac{1}{6}z^4 + \frac{1}{120}z^8 + \cdots\right).$$

Example 10.8.6. *Consider $f(z) = \sin(z^2) = z^2 - \frac{1}{6}z^6 + \frac{1}{120}z^{10} + \cdots$ once again. Let us consider the zero for $f(z)$ which is given by $z^2 = n\pi$ for some $n \in \mathbb{Z}$ with $n \neq 0$. This has solutions $z = \pm\sqrt{n\pi}$. In each case, $f(\pm\sqrt{n\pi}) = \sin n\pi = 0$ and $f'(\pm\sqrt{n\pi}) = \pm 2\sqrt{n\pi}\cos \pm\sqrt{n\pi} \neq 0$. Therefore, every other zero of $f(z)$ is simple. Only $z = 0$ is a double zero for $f(z)$. Although the arguments offered* *thus far suffice, I find explicit calculation of the power series centered at $\sqrt{n\pi}$ a worthwhile exercise:*

$$\begin{aligned}
\sin(z^2) &= \sin\left([z-\sqrt{n\pi}+\sqrt{n\pi}\,]^2\right) \\
&= \sin\left((z-\sqrt{n\pi})^2+2\sqrt{n\pi}(z-\sqrt{n\pi})+n\pi\right) \\
&= (-1)^n\sin\left((z-\sqrt{n\pi})^2+2\sqrt{n\pi}(z-\sqrt{n\pi})\right) \\
&= (-1)^n\left((z-\sqrt{n\pi})^2+2\sqrt{n\pi}(z-\sqrt{n\pi})-\frac{1}{6}\left((z-\sqrt{n\pi})^2+2\sqrt{n\pi}(z-\sqrt{n\pi})\right)^3+\cdots\right) \\
&= (z-\sqrt{n\pi})(-1)^n\left(2\sqrt{n\pi}+(z-\sqrt{n\pi})-\frac{4n\pi\sqrt{n\pi}}{3}(z-\sqrt{n\pi})^2+\cdots\right)
\end{aligned}$$

Example 10.8.7. *Consider $f(z)=1-\cosh(z)$ once again $f(0)=1-1=0$ and $f'(0)=\sinh(0)=0$ whereas $f''(0)=-\cosh(0)=-1\neq 0$ hence $f(z)$ has a double zero at $z=0$. The power series for hyperbolic cosine is $\cosh(z)=1+z^2/2+z^4/4!+\cdots$ and thus*

$$f(z)=\frac{1}{2}z^2+\frac{1}{4!}z^4+\cdots=z^2\left(\frac{1}{2}+\frac{1}{4!}z^2+\cdots\right)$$

Definition 10.8.8. *Let f be an analytic function on an exterior domain $|z|>R$ for some $R>0$. If f is not identically zero for $|z|>R$ then we say f has a zero of order N at ∞ if $g(w)=f(1/w)$ has a zero of order N at $w=0$.*

Theorem 10.8.9 translates to the following result for Laurent series[11]:

Theorem 10.8.9. *If $f(z)$ is an analytic function with a zero of order N at ∞ then*

$$f(z)=\frac{a_N}{(z-z_o)^N}+\frac{a_{N+1}}{(z-z_o)^{N+1}}+\frac{a_{N+2}}{(z-z_o)^{N+2}}+\cdots.$$

Example 10.8.10. *Let $f(z)=\frac{1}{1+z^3}$ has*

$$g(w)=\frac{1}{1+1/w^3}=\frac{w^3}{w^3+1}=w^3-w^6+w^9+\cdots$$

hence $g(w)$ has a triple zero at $w=0$ which means $f(z)$ has a triple zero at ∞. We could also have seen this simply by expressing f as a function of $1/z$:

$$f(z)=\frac{1}{1+z^3}=\frac{1/z^3}{1+1/z^3}=\frac{1}{z^3}-\frac{1}{z^6}+\frac{1}{z^9}+\cdots.$$

Example 10.8.11. *Consider $f(z)=e^z$ notice $g(w)=f(1/w)=e^{1/w}=1+\frac{1}{w}+\frac{1}{2}\frac{1}{w^2}+\cdots$ is not analytic at $w=0$ hence we cannot even hope to ask if there is a zero at ∞ for $f(z)$ or what its order is.*

Following Gamelin, we include this nice example.

[11] I will get around to properly defining this term in the next chaper

Example 10.8.12. *Let $f(z) = \frac{1}{(z-z_o)^n}$ then observe*

$$\begin{aligned}
f(z) &= \frac{1}{z^n - nz^{n-1}z_o + \cdots - nzz_o^{n-1} + z_o^n} \\
&= \frac{1}{z^n}\left(\frac{1}{1 - \frac{nz^{n-1}z_o + \cdots + nzz_o^{n-1} - z_o^n}{z^n}}\right) \\
&= \frac{1}{z^n}\left(\frac{1}{1 - \frac{nz_o}{z} + \cdots + \frac{nz_o^{n-1}}{z^{n-1}} - \frac{z_o^n}{z^n}}\right) \\
&= \frac{1}{z^n}\left(1 + \frac{nz_o}{z} + \cdots - \frac{nz_o^{n-1}}{z^{n-1}} + \frac{z_o^n}{z^n} + \cdots\right).
\end{aligned}$$

This serves to show $f(z)$ has $z = \infty$ as a zero of order n.

Statements as above may be understood literally on the extended complex plane $\mathbb{C} \cup \{\infty\}$ or simply as a shorthand for facts about exterior domains in $\mathbb{C}$.

If you survey the examples we have covered so far in this section you might have noticed that when $f(z)$ is analytic at z_o then $f(z)$ has a zero at z_o iff the zero has finite order. If we were to discuss a zero of infinite order then intuitively that would produce the zero function since all the coefficients in the Taylor series would vanish. Intuition is not always the best guide on such matters, therefore, let us establish the result carefully:

Theorem 10.8.13. *If D is a domain and f is an analytic function on D, which is not identically zero, then the zeros of f are isolated points in D.*

Proof: let $U = \{z \in D \mid f^{(m)}(z) = 0 \text{ for all } m \geq 0\}$. Suppose $z_o \in U$ then $f^{(k)}(z_o)/k! = 0$ for all $k \geq 0$ hence $f(z) = \sum_{k=0}^{\infty} a_k(z - z_o)^k = 0$. Thus, $f(z)$ vanishes on an open disk $D(z_o)$ centered at z_o and it follows $f^{(k)}(z) = 0$ for each $z \in D(z_o)$ and $k \geq 0$. Thus $D(z_o) \subseteq U$. Hence z_o is an interior point of U, but, as z_o was arbitrary, it follows U is open.

Next, consider $V = D - U$ and let $z_o \in V$. There must exist $n \geq 0$ such that $f^{(n)}(z_o) \neq 0$ thus $a_n \neq 0$ and consequently $f(z) = \sum_{k=0}^{\infty} a_k(z - z_o)^k \neq 0$. It follows there is a disk $D(z_o)$ centered at z_o on which $f(z) \neq 0$ for each $z \in D(z_o)$. Thus $D(z_o) \subseteq V$ and this shows V is an open set.

Consider then, $D = U \cup (D - U)$ hence as D is connected we can only have $U = \emptyset$ or $U = D$. If $U = D$ then we find $f(z) = 0$ for all $z \in D$ and that is not possible by the preconditions of the theorem. Therefore $U = \emptyset$. In simple terms, we have shown that every zero of an non-indentically-vanishing analytic function must have finite order.

To complete the argument, we must show the zeros are isolated. Notice that z_o a zero of $f(z)$ has finite order N hence, by Theorem 10.8.9, $f(z) = (z - z_o)^n h(z)$ where h is analytic at z_o with $h(z_o) \neq 0$. Therefore, there exists $\rho > 0$ for which the series for $h(z)$ centered at z_o represents $h(z)$ for each $|z - z_o| < \rho$. Moreover, observe $h(z) \neq 0$ for all $|z - z_o| < \rho$. Consider $|f(z)| = |z - z_o|^N |h(z)|$, this cannot be zero except at the point $z = z_o$ hence there is no other zero for $f(z)$ on $|z - z_o| < \rho$ hence z_o is isolated. $\square$.

The theorem above has interesting consequences.

Theorem 10.8.14. *If f and g are analytic on a domain D, and if $f(z) = g(z)$ for each z belonging to a set with a nonisolated point, then $f(z) = g(z)$ for all $z \in D$.*

Proof: let $C = \{z \in D \mid f(z) = g(z)\}$ and suppose the **coincidence** set C has a nonisolated point. Consider $h(z) = f(z) - g(z)$ for $z \in D$. If $h(z)$ is not identically zero on D then the existence of C contradicts Theorem 10.8.13 since C by its definition is a set with non-isolated zeros for $h(z)$. Consequently, $h(z) = f(z) - g(z) = 0$ for all $z \in D$. $\square$

Gamelin points out that if we apply the theorem above twice we obtain:

Theorem 10.8.15. *Let D be a domain, and let E be a subset of D that has a nonisolated point. Let $F(z, w)$ be a function defined for each $z, w \in D$ which is analytic in z with w-fixed and likewise analytic in w when we fix z. If $F(z, w) = 0$ whenever $z, w \in E$, then $F(z, w) = 0$ for all $z, w \in D$.*

Early in this course I made some transitional definitions which you might argue are somewhat adhoc. For example, we defined e^z, $\sin z$, $\sinh z$, $\cos z$ and $\cosh z$ all by simply extending their real formulas in the natural manner in view of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. The pair of theorems above show us an astonishing fact about complex analysis: there is just one way to define it as a natural extension of real calculus. Once Euler found his formula for real θ, there was only one complex extension which could be found.

Example 10.8.16. *Let $f(z) = e^z$. Let $g(z)$ be another entire function. Suppose $f(z) = g(z)$ for all $z \in \mathbb{R}$. Then, as $\mathbb{R}$ has a nonisolated point we find $f(z) = g(z)$ for all $z \in \mathbb{C}$. In other words, there is only one entire function on $\mathbb{C}$ which restricts to the real exponential on $\mathbb{R} \subset \mathbb{C}$.*

The same argument may be repeated for $\sin z$, $\sinh z$, $\cos z$ and $\cosh z$. Each of these functions is the unique entire extension of the corresponding function on $\mathbb{R}$. So, in complex analysis, we fix an analytic function on a domain if we know its values on some set with a nonisolated point. For example, the values of an analytic function on a domain are uniquely prescribed if we are given the values on a line-segment, open or closed disk, or even a sequence with a *cluster-point* in the domain. For further insight and some history on the topic of the identity theorem you can read pages 227-232 of [R91].

You might constrast this situation to that of linear algebra; if we are given the **finite** set of values to which a given **basis** in the domain must map then there is a **unique** linear transformation which is the extension from the finite set to the infinite set of points which forms the vector space. On the other side, a smooth function on an interval of $\mathbb{R}$ may be extended smoothly in infinitely many ways. Thus, the structure of complex analysis is stronger than that of real analysis and weaker than that of linear algebra.

One last thought, I have discussed extensions of functions to entire functions on $\mathbb{C}$. However, there may not exist an entire function to which we may extend. For example, $\ln(x)$ for $x \in (0, \infty)$ does not permit an extension to an entire function. Worse yet, we know this extends most naturally to $\log(z)$ which is a *multiply-valued* function. Remmert explains that 18-th century mathematicians wrestled with this issue. The temptation to assume by the *principle of permanence* there was a unique extension for the natural log led to considerable confusion. Euler wrote this in 1749 (page 159 [R91])

> We see therefore that is is essential to the nature of logarithms that each number have an infinity of logarithms and that all these logarithms be different, not only from one another, but also[different] from all the logarithms of every other number.

Ok, to be fair, this is a translation.

10.9 Analytic Continuation

Suppose we have a function $f(z)$ which is holomorphic on a domain D. If we consider $z_o \in D$ then there exist a_k for $k \geq 0$ such that $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ for all $z \in D(z_o) \subseteq D$. However, if we define $g(z)$ by the power series for $f(z)$ at z_o then the natural domain of $g(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ is the disk of convergence $D_R(z_o)$ where generally $D(z_o) \subseteq D_R(z_o)$. The function g is an **analytic continuation** of f.

Example 10.9.1. *Consider $f(z) = e^z$ for $z \in A = \{z \in \mathbb{C} \mid 1/2 < |z| < 2\}$. If we note $f(z) = e^{z-1+1} = ee^{z-1} = \sum_{k=0}^{\infty} \frac{e}{k!}(z-1)^k$ for all $z \in A$. However, $D_R(1) = \mathbb{C}$ thus the function defined by the series is an analytic continuation of the exponential from the given annulus to the entire plane.*

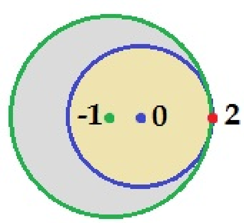
Analytic continuation is most interesting when there are singular points to work around. We can also begin with a function defined by a power series as in the next example.

Example 10.9.2. *Let $f(z) = \displaystyle\sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k$ for $|z| < 2$. Notice that $f(z) = \frac{1}{1-z/2} = \frac{2}{2-z}$ and we can expand the function as a power series centered at $z = -1$,*

$$f(z) = \frac{2}{2-(z+1-1)} = \frac{2}{3-(z+1)} = \frac{2}{3} \cdot \frac{1}{1-(z+1)/3} = \frac{2}{3}\sum_{k=0}^{\infty} \frac{1}{3^k}(z+1)^k.$$

for each z with $|z+1|/3 < 1$ or $|z+1| < 3$. In this case, the power series centered at $z = -1$ extends past $|z| < 2$. If we define $g(z) = \displaystyle\sum_{k=0}^{\infty} \frac{2}{3^{k+1}}(z+1)^k$ then $R = 3$ and the natural domain is $|z+1| < 3$.

The example above is easy to understand in the picture below:



Recentering the given series moves the center further from the singularity of the underlying function $z \mapsto \frac{2}{2-z}$ for $z \neq 2$. We know what will happen if we move the center somewhere else, the new radius of convergence will simply be the distance from the new center to $z = 2$.

In Gamelin §V.8 problem 2 you will see that the analytic continuation of a given holomorphic function need not match the function. It is possible to continue from one branch of a multiply-valued function to another branch. This is also shown on page 160 of Gamelin where he continues the principal branch of the squareroot mapping to the negative branch.

If we study the analytic continuation of a function defined by a series the main question which we face is the nature of the function on the boundary of the disk of convergence. There must be at least one point of divergence. See our Corollary 10.4.4 or look at page 234 of [R91] for a careful argument. Given $f(z) = \sum a_k(z - z_o)^k$ with disk $D_R(z_o)$ of convergence, a point $z_1 \in \partial D_R(z_o)$ is a **singular point** of f if there does not exist a holomorphic function $g(z)$ on $D_s(z_1)$ for which $f(z) = g(z)$ for all $z \in D_R(z_o) \cap D_s(z_1)$. The set of all singular points for f is called the **natural boundary of** f and the disk $D_R(z_o)$ is called the **region of holomorphy for** f. On page 150 of [R91] the following example is offered:

Example 10.9.3. *Set $g(z) = z + z^2 + z^4 + z^8 + \cdots$. The radius of convergence is found to be $R = 1$. Furthermore, we can argue that $g(z) \to \infty$ as z approaches any even root of unity. Remmert shows on the page before that the even (or odd) roots of unity are* **dense** *on the unit circle hence the function $g(z)$ is unbounded at each point on $|z| = 1$ and it follows that the unit-circle is the natural boundary of this series.*

Certainly, many other things can happen on the boundary.

Example 10.9.4. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k = z - \frac{z}{2} + \frac{z}{3} + \cdots$ *converges for each z with $|z| = 1$* **except** *the single singular point $z = -1$.*

Remmert informs that Lusin in 1911 found a series with coefficients $c_k \to 0$ yet $\sum c_k z^k$ diverges at each $|z| = 1$. Then Sierpinski in 1912 produced a series which diverges at every point on the unit-circle **except** $z = 1$. See pages 120-121 [R91] for further details.

In summary, the problem of analytic continuation is subtle. When given a series presentation of an analytic function it may not be immediately obvious where the natural boundary of the given function resides. On the other hand, when the given function is captured by an algebraic expression or a formula in terms of sine, cosine etc. then through essentially precalculus-type domain considerations we can find see the natural boundary arise from the nature of the formula. Any series which represents the function will face the same natural boundaries. Well, I have tried not to overstate anything here, I hope I was successful. The full appreciation of analytic continuation is far beyond this course. For an attack similar to what I have done in examples here, see this MSE question. For a still bigger picture, see Wikipedia article on analytic continuation where it is mentioned that trouble with analytic continuation for functions of several complex variables prompted the invention of sheaf cohomology.

Let me collect a few main points from Gamelin. If D is a disk and f is analytic on D and $R(z_1)$ is the radius of convergence of the power series at $z_1 \in D$ and $R(z_2)$ is the radius of convergence of the power series at $z_2 \in D$, then $|R(z_1) - R(z_2)| \leq |z_1 - z_2|$. This inequality shows the radius of convergence is a continuous function on the domain of an analytic function.

Definition 10.9.5. *We say that f is* **analytically continuable along** γ *if for each t there is a convergent power series*

$$f_t(z) = \sum_{n=0}^{\infty} a_n(t)(z - \gamma(t))^n, \qquad |z - \gamma(t)| < r(t),$$

such that $f_a(z)$ is the power series representing $f(z)$ at z_o, and such that when s is near t, then $f_s(z) = f_t(z)$ for all z in the intersection of the disks of convergence for $f_s(z)$ and $f_t(z)$.

It turns out that when we analytically continue a given function from one initial point to a final point it could be the continuations do not match. However, there is a simple condition which assures the continuations do coincide. The idea here is quite like our deformation theorem for closed forms.

Theorem 10.9.6. Monodromy Theorem: *Let $f(z)$ be analytic at z_o. Let $\gamma_o(t)$ and $\gamma_1(t)$ for $a \leq t \leq b$ be paths from z_o to z_1 along which $f(z)$ can be continued analytically. Suppose γ_o can be continuously deformed to γ_1 along paths γ_s which begin at z_o and end at z_1 and allow $f(z)$ to be continued analytically. Then the analytic continuations of $f(z)$ along γ_o and γ_1 coincide at z_1.*

If there is a singularity, that is a point near the domain where the function cannot be analytically extended, then the curves of continuation might not be able to be smoothly deformed. The deformation could get snagged on a singularity. Of course, there is more to learn from Gamelin on this point. I will not attempt to add to his treatment further here.

Chapter 11

Laurent Series and Isolated Singularities

Laurent was a French engineer who lived from 1813 to 1854. He extended Cauchy's work on disks to annuli by introducing reciprocal terms centered about the center of the annulus. His original work was not published. However, Cauchy was aware of the result and has this to say about Laurent's work in his report to the French Academy of 1843:

> the extension given by M. Laurent $\cdots$ seems to us worthy of note

In this chapter we extend Cauchy's theorems on power series for analytic functions. In particular, we study how we can reproduce any analytic function on an annulus by simply adjoining reciprocal powers to the power series. A series built, in general, from both positive and negative power functions centered about some point z_o is called a Laurent series centered at z_o. The annulus we consider can reduce to a deleted disk or extend to ∞. Most of these results are fairly clean extensions of what we have done in previous chapters. Excitingly, we shall see the generalized Cauchy integral formula naturally extends. The extended theorem naturally ties coefficients of a given Laurent series to integrals around a circle in the annulus of convergence. That simple connection lays the foundation for the residue calculus of the next chapter. In terms of explicit calculation, we continue to use the same techniques as in our previous work. However, the domain of consideration is markedly different. We must keep in mind our study is about some annulus.

Laurent's proof of the Laurent series development can be found in a publication which his widow published in his honor in 1863. Apparently both Cauchy and Weierstrauss also has similar results in terms of mean values around 1840-1841. As Remmert explains (page 350-355 [R91]), all known proofs of the Laurent decomposition involve integration. Well, apparently, Pringsheim wrote a 1223 page work which avoided integration and instead did everything in terms of mean values. So, we should say, no efficient proof without integrals is known. Also of note, Laurent's Theorem can be derived from the Cauchy-Taylor theorem by direct calculational attack; this difficult proof due to Scheffer in 1884 (which also implicitly uses integral theory) is reproduced on p. 352-355 of [R91].

We could have made the definition some time ago, but, I give it here since I found myself using the term at various points in my exposition of this chapter.

Definition 11.0.1. *If $f \in \mathcal{O}(z_o)$ then there exists some $r > 0$ such that f is holomorphic on $|z - z_o| < r$. In other words, $\mathcal{O}(z_o)$ is the set of holomorphic functions at z_o.*

11.1 The Laurent Decomposition

If a function f is analytic on an annulus then the function can be written as the sum of two analytic functions f_o, f_1 on the annulus. Where, f_o is analytic from the outer circle of the annulus to the center and f_1 is analytic from the inner circle of the annulus to ∞.

Theorem 11.1.1. Laurent Decomposition: *Suppose $0 \leq \rho < \sigma \leq \infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_o| < \sigma$. Then $f(z)$ can be decomposed as a sum*

$$f(z) = f_o(z) + f_1(z),$$

where f_o is analytic for $|z - z_o| < \sigma$ and f_1 is analytic for $|z - z_o| > \rho$ and at ∞. If we normalize the decomposition such that $f_1(\infty) = 0$ then the decomposition is unique.

Let us examine a few examples and then we will offer a proof of the general assertion.

Example 11.1.2. *Let $f(z) = \dfrac{z^3 + z + 1}{z} = z^2 + 1 + \dfrac{1}{z}$ for $z \neq 0$. In this example $\rho = 0$ and $\sigma = \infty$ and $f_o(z) = z^2 + 1$ whereas $f_1(z) = 1/z$.*

Example 11.1.3. *Let $f(z)$ be an entire function. For example, $e^z, \sin z, \sinh z, \cos z$ or $\cosh z$. Then $f(z) = f_o(z)$ and $f_1(z) = 0$. The function f_o is analytic on any disk, but, we do not assume it is analytic at ∞. On the other hand, notice that $f_1 = 0$ is analytic at ∞ as claimed.*

Example 11.1.4. *Suppose $f(z)$ is analytic at $z_o = \infty$ then there exists some exterior domain $|z - z_o| > \rho$ for which $f(z)$ is analytic. In this case, $f(z) = f_1(z)$ and $f_o(z) = 0$ for all $z \in \mathbb{C} \cup \{\infty\}$.*

Proof: Suppose $0 \leq \rho < \sigma \leq \infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_o| < \sigma$. Furthermore, suppose $f(z) = f_o(z) + f_1(z)$ where f_o is analyic for $|z - z_o| < \sigma$ and f_1 is analytic for $|z - z_o| > \rho$ and at ∞. Suppose g_o, g_1 form another Laurent decomposition with $f(z) = g_o(z) + g_1(z)$. Notice,

$$g_o(z) - f_o(z) = g_1(z) - f_1(z)$$

for $\rho < |z - z_o| < \sigma$. In view of the above overlap condition we are free to define:

$$h(z) = \begin{cases} g_o(z) - f_o(z), & \text{for } |z - z_o| < \sigma \\ g_1(z) - f_1(z), & \text{for } |z - z_o| > \rho \end{cases}$$

Notice h is entire and $h(z) \to 0$ as $z \to \infty$. Thus h is bounded and entire and we apply Liouville's Theorem to conclude $h(z) = c$ for all $z \in \mathbb{C}$. In particular, $h(z) = 0$ on the annulus $\rho < |z - z_o| < \sigma$ and we conclude that if a Laurent decomposition exists then it must be unique.

The existence of the Laurent Decomposition is due to Cauchy's Integral formula on an annulus. Technically, we have not shown this result explicitly[1], to derive it we simply need to use the cross-cut idea which is illustrated in the discussion preceding Theorem 8.2.12. Once more, suppose $0 \leq \rho < \sigma \leq \infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_o| < \sigma$. Consider some subannulus $\rho < r < |z - z_o| < s < \sigma$. Cauchy's Integral formula gives

$$f(z) = \underbrace{\frac{1}{2\pi i} \oint_{|w - z_o| = s} \frac{f(w)}{w - z} dw}_{f_o(z)} - \underbrace{\frac{1}{2\pi i} \oint_{|w - z_o| = r} \frac{f(w)}{w - z} dw}_{-f_1(z)}.$$

[1]see pages 344-346 of [R91] for careful proofs of these results

Notice f_o is analytic for $|z-z_o|<s$ and f_1 is analytic for $|z-z_o|>r$ and $f_1(z)\to 0$ as $z\to\infty$. As Gameline points out here, our current formulation would seem to depend on r,s but we already showed the decomposition is unique if it exists thus f_o and f_1 must be defined for $\rho<|z-z_o|<\sigma$. $\square$

If you wish to read a different formulation of essentially the same proof, I recommend page 347 of [R91].

Example 11.1.5. *Consider* $f(z)=\frac{2z-i}{z(z-i)}$. *This function is analytic on* $\mathbb{C}-\{0,i\}$. *A simple calculation reveals:*

$$f(z)=\frac{1}{z}+\frac{1}{z-i}$$

With respect to the annulus $0<|z|<1$ *we have* $f_o(z)=\frac{1}{z-i}$ *and* $f_1(z)=\frac{1}{z}$. *On the other hand, for the annulus* $0<|z-i|<1$ *we have* $f_1(z)=\frac{1}{z-i}$ *and* $f_0(z)=\frac{1}{z}$. *If we study disks centered at any point in* $\mathbb{C}-\{0,i\}$ *then* $f_o(z)=f(z)$ *and* $f_1(z)=0$.

We sometimes call the set such as $0<|z-i|<1$ an *annulus*, but, we would do equally well to call it a *punctured disk* centered at $i=1$.

Example 11.1.6. *Consider* $f(z)=\frac{1}{\sin z}$ *this has a Laurent decomposition on the annuli which fit between the successive zeros of* $\sin z$. *That is, on* $n\pi<|z|<(n+1)\pi$. *For example, when* $n=0$ *we have* $\sin z=z-\frac{1}{6}z^3+\cdots$ *hence, using our geometric series reciprocal technique,*

$$f(z)=\frac{1}{\sin z}=\frac{1}{z-\frac{1}{6}z^3+\cdots}=\frac{1}{z(1-\frac{1}{6}z^2+\cdots)}=\frac{1}{z}\bigg(1+(z^2/6+\cdots)^2+\cdots\bigg)=\frac{1}{z}+\frac{1}{36}z^3+\cdots$$

Hence $f_1(z)=1/z$ *whereas* $f_o(z)=z^3/36+\cdots$ *for the punctured disk of raduis* π *centered about* $z=0$.

Suppose $f(z)=f_o(z)+f_1(z)$ is the Laurent decomposition on $\rho<|z-z_o|<\sigma$. By Theorem 10.5.2 there exists a power series representation of f_o

$$f_o(z)=\sum_{k=0}^{\infty}a_k(z-z_o)^k$$

for $|z-z_o|<\sigma$. Next, by Theorem 10.6.7, noting that $a_o=f_1(\infty)=0$ gives

$$f_1(z)=\sum_{k=-\infty}^{-1}a_k(z-z_o)^k$$

for $|z-z_o|>\rho$. Notice both the series for f_o and f_1 converge normally and summing both together gives:

$$f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_o)^k$$

which is normally convergen on $\rho<|z-z_o|<\sigma$. In this context, normally convergent means we have uniform convergence for each $s\le|z-z_o|\le t$ where $\rho<s<t<\sigma$.

Given a function $f(z)$ defined by a Laurent series centered at z_o:

$$f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_o)^k \qquad \star$$

for $\rho < |z - z_o| < \sigma$. We naturally wish to characterize the meaning of the coefficients[2] a_k. This is accomplished by integration. In particular, we begin by integration over the circle $|z - z_o| = r$ where $\rho < r < \sigma$:

$$\begin{aligned}
\int_{|z-z_o|=r} f(z)\,dz &= \int_{|z-z_o|=r} \left(\sum_{k=-\infty}^{\infty} a_k (z - z_o)^k \right) dz \\
&= \sum_{k=-\infty}^{\infty} a_k \left(\int_{|z-z_o|=r} (z - z_o)^k dz \right) \\
&= \sum_{k=-\infty}^{\infty} a_k \left(2\pi i \delta_{k,-1} \right) \\
&= 2\pi i a_{-1}
\end{aligned}$$

We have used the uniform convergence of the given series which allows term-by-term integration. In addition, the integration was before discussed in Example 9.1.7. In summary, we find the $k = -1$ coefficient has a rather beautiful significance:

$$\boxed{a_{-1} = \frac{1}{2\pi i} \int_{|z-z_o|=r} f(z)\,dz}$$

where the circle of integration can be taken as any circle in the annulus of convergence for the Laurent series. What does this formula mean?

We can integrate by finding a Laurent expansion of the integrand!

Example 11.1.7. *Let* $f(z) = \frac{\sin z}{1-z}$. *Observe,*

$$\frac{\sin z}{1-z} = \frac{\sin(z-1+1)}{1-z} = \frac{\cos(1)\sin(z-1) + \sin(1)\cos(z-1)}{z-1} = \frac{\sin 1}{z-1} + \cos(1) - \frac{\sin 1}{2}(z-1) + \cdots$$

thus $a_{-1} = \sin 1$ *and we find:*

$$\int_{|z-1|=2} \frac{\sin z}{1-z}\,dz = 2\pi i \sin 1.$$

We now continue our derivation of the values for the coefficients in $\star$, we divide by $(z - z_o)^{n+1}$ and once more integrate over the circle $|z - z_o| = r$ where $\rho < r < \sigma$:

$$\begin{aligned}
\int_{|z-z_o|=r} \frac{f(z)}{(z-z_o)^{n+1}}\,dz &= \int_{|z-z_o|=r} \left(\sum_{k=-\infty}^{\infty} a_k (z - z_o)^{k-n-1} \right) dz \\
&= \sum_{k=-\infty}^{\infty} a_k \left(\int_{|z-z_o|=r} (z - z_o)^{k-n-1} dz \right) \\
&= \sum_{k=-\infty}^{\infty} a_k \left(2\pi i \delta_{k-n-1,-1} \right) \\
&= 2\pi i a_n
\end{aligned}$$

[2] We already know for power series on a disk the coefficients are tied to the derivatives of the function at the center of the expansion. However, in the case of the Laurent expansion we only have knowledge about the function on the annulus centered at z_o and z_o may not even be in the domain of the function.

Once again, we have used the uniform convergence of the given series which allows term-by-term integration and the integral identity shown in Example 9.1.7. Notice the Kronecker delta $\delta_{k-n-1,-1} = \begin{cases} 1 & \text{if } k-n-1=-1 \\ 0 & \text{if } k-n-1 \neq -1 \end{cases}$ which means the only nonzero term occurs when $k-n-1=-1$ which is simply $k=n$. Of course, the integral is familar to us. We saw this identity for $k \geq 0$ in our previous study of power series. In particular, Theorem 9.4.2 where we proved the generalized Cauchy integral formula: adapted to our current notation

$$\frac{1}{2\pi i}\int_{|z-z_o|=r} \frac{f(z)}{(z-z_o)^{n+1}}\,dz = \frac{f^{(n)}(z_o)}{n!}.$$

For the Laurent series we study on $\rho < |z-z_o| < \sigma$ we cannot in general calculate $f^{(n)}(z_o)$. However, in the case $\rho = 0$, we have $f(z)$ analytic on the disk $|z-z_o| < \sigma$ and then we are able to either calculate, for $n \geq 0$ a_n by differentiation or integration. Let us collect our results for future reference:

Theorem 11.1.8. Laurent Series Decomposition: *Suppose $0 \leq \rho < \sigma \leq \infty$, and suppose $f(z)$ is analytic for $\rho < |z-z_o| < \sigma$. Then $f(z)$ can be decomposed as a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_o)^n$$

where the coefficients a_n are given by:

$$a_n = \frac{1}{2\pi i}\int_{|z-z_o|=r} \frac{f(z)}{(z-z_o)^{n+1}}\,dz$$

for $r > 0$ with $\rho < r < \sigma$.

Notice the deformation theorem goes to show there is no hidden dependence on r in the formulation of the coefficient a_n. The function f is assumed holomorphic between the inner and outer circles of the annulus of convergence hence $\frac{f(z)}{(z-z_o)^{n+1}}$ is holomorpic on the annulus as well and the complex integral is unchanged as we alter the value of r on (ρ, σ).

11.2 Isolated Singularities of an Analytic Function

A singularity of a function is some point which is nearly in the domain, and yet, is not. An isolated singularity is a singular point which is also isolated. A careful definition is given below:

Definition 11.2.1. *A function f has an* **isolated singularity at** *z_o if there exists $r > 0$ such that f is analytic on the punctured disk $0 < |z-z_o| < r$.*

We describe in this section how isolated singularity fall into three classes where each class has a particular type of Laurent series about the singular point. Let me define these now and we will explain the terms as the section continues. Notice Theorem 11.1.8 implies $f(z)$ has a Laurent series in a punctured disk about singularity hence the definition below covers all possible isolated singularities.

Definition 11.2.2. *Suppose f has an isolated singularity at z_o.*

(i.) *If* $f(z) = \sum_{k=0}^{\infty} a_k(z-z_o)^k$ *then* z_o *is a* **removable singularity**.

(ii.) *Let* $N \in \mathbb{N}$. *If* $f(z) = \sum_{k=-N}^{\infty} a_k(z-z_o)^k$ *with* $a_{-N} \neq 0$ *then* z_o *is a* **pole of order** N.

(iii.) *If* $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_o)^k$ *where* $a_k \neq 0$ *for infinitely many* $k < 0$ *then* z_o *is an* **essential singularity**.

We begin by studying the case of removable singularity. This is essentially the generalization of a hole in the graph you studied a few years ago.

Theorem 11.2.3. Riemann's Theorem on Removable Singularities: *let* z_o *be an isolated singularity of* $f(z)$. *If* $f(z)$ *is bounded near* z_o *then* $f(z)$ *has a removable singularity.*

Proof: expand $f(z)$ in a Laurent series about the punctured disk at z_o:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_o)^n$$

for $0 < |z-z_o| < \sigma$. If $|f(z)| < M$ for $0 < |z-z_o| < r$ then for $r < min(\sigma, r)$ we may apply the ML-theorem to the formula for the n-th coefficient of the Laurent series as given by Theorem 11.1.8

$$|a_n| = \left| \frac{1}{2\pi i} \int_{|z-z_o|=r} \frac{f(z)}{(z-z_o)^{n+1}}\, dz \right| \leq \frac{M(2\pi r)}{2\pi r^{n+1}} = \frac{M}{r^n}.$$

As $r \to 0$ we find $|a_n| \to 0$ for $n < 0$. Thus $a_n = 0$ for all $n = -1, -2, \dots$. Thus, the Laurent series for $f(z)$ reduces to a power series for $f(z)$ on the deleted disk $0 < |z-z_o| < \sigma$ and it follows we may extend $f(z)$ to the disk $|z-z_o| < \sigma$ by simply defining $f(z_o) = a_o$. $\square$

Example 11.2.4. *Let* $f(z) = \frac{\sin z}{z}$ *on the punctured plane* $\mathbb{C}^{\times}$. *Notice,*

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j+1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j} = 1 - \frac{1}{3!}z^2 + \cdots.$$

We can extend f *to* $\mathbb{C}$ *by defining* $f(0) = 1$.

To be a bit more pedantic, $\tilde{f}$ is the extension of f defined by $\tilde{f}(z) = f(z)$ for $z \neq 0$ and $\tilde{f}(0) = 1$. The point ? The extension $\tilde{f}$ is a new function which is distinct from f.

We now study poles of order N. Let us begin by making a definition:

Definition 11.2.5. *Suppose* f *has a pole of order* N *at* z_o. *If*

$$f(z) = \frac{a_{-N}}{(z-z_o)^N} + \cdots + \frac{a_{-1}}{z-z_o} + \sum_{k=0}^{\infty} a_k(z-z_o)^k$$

then $P(z) = \frac{a_{-N}}{(z-z_o)^N} + \cdots + \frac{a_{-1}}{z-z_o}$ *is the* **principal part of** $f(z)$ **about** z_o. *When* $N=1$ *then* z_o *is called a* **simple pole**, *when* $N=2$ *then* z_o *is called a* **double pole**.

Notice $f(z) - P(z)$ is analytic.

Theorem 11.2.6. *Let z_o be an isolated singularity of f. Then z_o is a pole of f of order N iff $f(z) = g(z)/(z-z_o)^N$ where g is analytic at z_o with $g(z_o) \neq 0$.*

Proof: suppose f has a pole of order N at z_o then by definition it has a Laurent series which begins at $n = -N$. We calculate, for $|z - z_o| < r$,

$$f(z) = \sum_{k=-N}^{\infty} a_k (z-z_o)^k = \frac{1}{(z-z_o)^N} \sum_{k=-N}^{\infty} a_k (z-z_o)^{k+N} = \frac{1}{(z-z_o)^N} \sum_{j=0}^{\infty} a_{j-N} (z-z_o)^j.$$

Define $g(z) = \sum_{j=0}^{\infty} a_{j-N}(z-z_o)^j$ and note that g is analytic at z_o with $g(z_o) = a_{-N} \neq 0$. We know $a_{-N} \neq 0$ by the definition of a pole of order N. Thus $f(z) = g(z)/(z-z_o)^N$ as claimed.

Conversely, suppose there exists g analytic at z_o with $g(z_o) \neq 0$ and $f(z) = g(z)/(z-z_o)^N$. There exist $b_o, b_1, \dots$ with $g(z_o) = b_o \neq 0$ such that

$$g(z) = \sum_{k=0}^{\infty} b_k (z-z_o)^k$$

divide by $(z-z_o)^N$ to obtain:

$$f(z) = \frac{1}{(z-z_o)^N} \sum_{k=0}^{\infty} b_k (z-z_o)^k = \sum_{k=0}^{\infty} b_k (z-z_o)^{k-N} = \sum_{j=-N}^{\infty} b_{j+N} (z-z_o)^j$$

identify that the coefficient of the Laurent series at order $-N$ is precisely $b_o \neq 0$ and thus we have shown f has a pole of order N at z_o. $\square$

Example 11.2.7. *Consider $f(z) = \dfrac{e^z}{(z-1)^5}$. Notice e^z is analytic on $\mathbb{C}$ hence by Theorem 11.2.6 the function f has a pole of order $N = 5$ at $z_o = 1$.*

Example 11.2.8. *Consider $f(z) = \dfrac{\sin(z+2)^5}{(z+2)^2}$ notice*

$$f(z) = \frac{1}{(z+2)^5}\left((z+2)^3 - \frac{1}{3!}(z+2)^9 + \frac{1}{5!}(z+2)^{15} + \cdots \right) =$$

simplifying yields

$$f(z) = \frac{1}{(z+2)^2} \underbrace{\left(1 - \frac{1}{3!}(z+2)^6 + \frac{1}{5!}(z+2)^{12} + \cdots \right)}_{g(z)}$$

which shows, by Theorem 11.2.6, the function f has a pole of order $N = 2$ at $z_o = -2$.

Theorem 11.2.9. *Let z_o be an isolated singularity of f. Then z_o is a pole of f of order N iff $1/f$ is analytic at z_o with a zero of order N.*

Proof: we know f has pole of order N iff $f(z) = g(z)/(z-z_o)^N$ with $g(z_o) \neq 0$ and $g \in \mathcal{O}(z_o)$. Suppose f has a pole of order N then observe

$$\frac{1}{f(z)} = (z-z_o)^N \cdot \frac{1}{g(z)}.$$

hence $1/f(z)$ has a zero of order N by Theorem 10.8.9. Conversely, if $1/f(z)$ has a zero of order N then by Theorem 10.8.9 we have $\frac{1}{f(z)} = (z-z_o)^N h(z)$ where $h \in \mathcal{O}(z_o)$ and $h(z_o) \neq 0$. Define $g(z) = 1/h(z)$ and note $g \in \mathcal{O}(z_o)$ and $g(z_o) = 1/h(z_o) \neq 0$ moreover,

$$\frac{1}{f(z)} = (z-z_o)^N h(z) \quad \Rightarrow \quad f(z) = \frac{1}{(z-z_o)^N h(z)} = \frac{g(z)}{(z-z_o)^N}$$

and we conclude by Theorem 11.2.6 that f has a pole of order N at z_o. $\Box$

The theorem above can be quite useful for quick calculation.

Example 11.2.10. *$f(z) = 1/\sin z$ has a simple pole at $z_o = n\pi$ for $n \in \mathbb{N} \cup \{0\}$ since*

$$\sin(z) = \sin(z - n\pi + n\pi) = \cos(n\pi)\sin(z-n\pi) = (-1)^n(z-n)\pi - \frac{(-1)^n}{3!}(z-n)^3 + \cdots$$

shows $\sin z$ has a simple zero at $z_o = n\pi$ for $n \in \mathbb{N} \cup \{0\}$.

Example 11.2.11. *You should be sure to study the example given by Gamelin on page 173 to 174 where he derives the Laurent expansion which converges on $|z| = 4$ for $f(z) = 1/\sin z$.*

Example 11.2.12. *Let $f(z) = \dfrac{1}{z^3(z-2-3i)^6}$ then f has a pole of order $N = 3$ at $z_o = 0$ and a pole of order $N = 6$ at $z_1 = 2+3i$*

Definition 11.2.13. *We say a function f is* **meromorphic on a domain** *D if f is analytic on D except possibly at isolated singularities of which each is a pole.*

Example 11.2.14. *An entire function is meromorphic on $\mathbb{C}$. However, an entire function may not be analytic at ∞. For example, $\sin z$ is not analytic at ∞ and it has an essential singularity at ∞ so $f(z) = \sin z$ is not meromorphic on $\mathbb{C} \cup \{\infty\}$.*

Example 11.2.15. *A rational function is formed by the quotient of two polynomials $p(z), q(z) \in \mathbb{C}[z]$ where $q(z)$ is not identically zero; $f(z) = p(z)/q(z)$. We will explain in Example 11.3.3 that $f(z)$ is meromorphic on the extended complex plane $\mathbb{C} \cup \{\infty\}$.*

Theorem 11.2.16. *Let z_o be an isolated singularity of f. Then z_o is a pole of f of order $N \geq 1$ iff $|f(z)| \to \infty$ as $z \to z_o$.*

Proof: if z_o is a pole of order N then $f(z) = g(z)/(z-z_o)^N$ for $g(z_o) \neq 0$ for $0 < |z-z_o| < r$ for some $r > 0$ where g is analytic at z_o. Since g is analytic at z_o it is continuous and hence bounded on the disk; $|g(z)| \leq M$ for $|z-z_o| < r$. Thus,

$$|f(z)| = |g(z)(z-z_o)^{-N}| \leq M(z-z_o)^{-N} \to \infty$$

as $z \to z_o$. Thus $|f(z)| \to \infty$ as $z \to z_o$.

Conversely, suppose $|f(z)| \to \infty$ as $z \to z_o$. Hence, there exists $r > 0$ such that $f(z) \neq 0$ for $0 < |z - z_o| < r$. It follows that $h(z) = 1/f(z)$ is analytic in for $0 < |z - z_o| < r$. Note that $|f(z)| \to \infty$ as $z \to z_o$ implies $h(z) \to 0$ as $z \to z_o$. Thus $h(z)$ is bounded near z_o and we find by Riemann's removable singularity Theorem 11.2.3 there exist a_n for $n = 0, 1, 2, \dots$ such that:

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n$$

However, $h(z) \to 0$ as $z \to z_o$ hence the extension of $h(z)$ is zero at z_o. If the zero has order N then $h(z) = (z - z_o)^N b(z)$ where $b \in \mathcal{O}(z_o)$ and $b(z_o) \neq 0$. Therefore, we obtain $f(z) = g(z)/(z - z_o)^N$ where $g(z) = 1/b(z)$ where $g \in \mathcal{O}(z_o)$ and $g(z_o) \neq 0$. We conclude z_o is a pole of order N by Theorem 11.2.6.

Example 11.2.17. *Let $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$. Clearly $z_o = 0$ is an essential singularity of f. It has different behaviour than a removable singularity or a pole. First, notice for $z = x > 0$ we have $f(z) = e^{1/x} \to \infty$ as $x \to 0^+$ thus f is not bounded at $z_o = 0$. On the other hand, if we study $z = iy$ for $y > 0$ then $|f(z)| = |e^{\frac{1}{iy}}| = 1$ hence $|f(z)|$ does not tend to ∞ along the imaginary axis.*

Theorem 11.2.18. Casorati-Weierstrauss Theorem: *Let z_o be an essential isolated singularity of $f(z)$. Then for every complex number w_o, there is a sequence $z_n \to z_o$ such that $f(z_n) \to w_o$ as $n \to \infty$.*

Proof: by contrapositive argument. Suppose there exists a complex number w_o for which there does not exist a sequence $z_n \to z_o$ such that $f(z_n) \to w_o$ as $n \to \infty$. It follows there exists $\epsilon > 0$ for which $|f(z) - w_o| > \epsilon$ for all z in a small punctured disk about z_o. Thus, $h(z) = 1/(f(z) - w_o)$ is bounded close to z_o. Consequently, z_o is a removable singularity of $h(z)$ and $h(z) = (z - z_o)^N g(z)$ for some $N \geq 0$ and some analytic function g such that $g(z_o) \neq 0$. But, this gives:

$$\frac{1}{f(z) - w_o} = (z - z_o)^N g(z) \quad \Rightarrow \quad f(z) = w_o + \frac{b(z)}{(z - z_o)^N}$$

where $b = 1/g \in \mathcal{O}(z_o)$ and $b(z_o) \neq 0$. If $N = 0$ then f extends to be analytic at z_o. If $N > 0$ then f has a pole of order N at z_o. In all cases we have a contradiction to the given fact that z_o is an essential singularity. The theorem follows. $\square$

Gamelin mentions **Picard's Theorem** which states that for an essential singularity at z_o, for all w_o except possibly one value, there is a sequence $z_n \to z_o$ for which $f(z_n) = w_o$ for each n. In our example $e^{1/z}$ the exceptional value is $w_o = 0$.

11.3 Isolated Singularity at Infinity

As usual, we use the reciprocal function to transfer the definition from zero to infinity.

Definition 11.3.1. *We say f has an isolated singular point at ∞ if there exists $r > 0$ such that f is analytic on $|z| > r$. Equivalently, we say f has an isolated singular point at ∞ if $g(w) = f(1/w)$ has an isolated singularity at $w = 0$. Furthermore, we say that the isolated singular point at ∞ is removable singularity, a pole of order N or an essential singularity if the corresponding singularity* *at $w = 0$ is likewise a removable singularity, pole of order N or an essential singular point of g. In particular, if ∞ is a pole of order N then the Laurent series expansion:*

$$f(z) = b_N z^N + \cdots + b_1 z + b_o + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \cdots$$

has **principal part**

$$P_\infty(z) = b_N z^N + \cdots + b_1 z + b_o$$

hence $f(z) - P_\infty(z)$ is analytic at ∞.

This section is mostly a definition. I now give a few illustrative examples, partly following Gamelin.

Example 11.3.2. *The function $e^z = 1 + z + z^2/2! + z^3/3! + \cdots$ has an essential singularity at ∞. This implies that while e^z is meromorphic on $\mathbb{C}$, it is* **not** *meromorphic on $\mathbb{C} \cup \{\infty\}$ as it has a singularity which is not a pole or removable.*

Example 11.3.3. *Let $p(z), q(z) \in \mathbb{C}[z]$ with $deg(p(z)) = m$ and $deg(q(z)) = n$ such that $m > n$. Notice that long-division gives $d(z), r(z) \in \mathbb{C}[z]$ for which $deg(d(z)) = m - n$ and $deg(r(z)) < m$ such that*

$$f(z) = \frac{p(z)}{q(z)} = d(z) + \frac{r(z)}{q(z)}$$

The function $\frac{r(z)}{q(z)}$ is analytic at ∞ and $d(z)$ serves as the principal part. We identify f has a pole of order $m - n$ at ∞. It follows that any rational function is **meromorphic** *on the extended complex plane $\mathbb{C} \cup \{\infty\}$*

Example 11.3.4. *Following the last example, suppose $m = n$ then $d(z) = 0$ and the singularity at ∞ is seen to be removable. If $p(z) = a_m z^m + \cdots + a_o$ and $q(z) = b_n z^n + \cdots + b_o$ then we can extend f analytically at ∞ by defining $f(\infty) = a_m/b_n$.*

Example 11.3.5. *Consider $f(z) = (e^{1/z} - 1)/z$ for $z > 0$. Observe*

$$f(z) = (e^{1/z} - 1)/z = \left(\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right)$$

hence the singularity at ∞ is removable and we may extend f to be analytic on the extended complex plane by defining $f(\infty) = 0$.

11.4 Partial Fractions Decomposition

In the last section we noticed in Example 11.3.3 that rational functions were meromorphic on the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. Furthermore, it is interesting to notice the algebra of meromorphic functions is very nice: sums, products, quotients where the denominator is not identically zero, all of these are once more meromorphic. In terms of abstract algebra, the set of meromorphic functions on a domain forms a subalgebra of the algebra of holomorphic functions on D. See pages 315-320 of [R91] for a discussion which focuses on the algebraic aspects of meromorphic functions.

It turns out that not only are the rational functions meromorphic on $\mathbb{C}^*$, in fact, they are the only meromorphic functions on $\mathbb{C}^*$.

Theorem 11.4.1. *A meromorphic function on $\mathbb{C}^*$ is a rational function.*

Proof: let $f(z)$ be a meromorphic function on $\mathbb{C}^*$. The number of poles of f must be finite otherwise they would acculumate to give a singularity which was not isolated. If f is analytic at ∞ then we define $P_\infty(z) = f(\infty)$. Otherwise, f has a pole of order N and $P_\infty(z)$ is a polynomial of order N. In both cases, $f(z) - P_\infty(z)$ is analytic at ∞ with $f(z) - P_\infty(z) \to 0$ as $z \to \infty$. Let us label the poles in $\mathbb{C}$ as $z_1, z_2, \dots, z_m$. Furthermore, let $P_k(z)$ be the principal part of $f(z)$ at z_k for $k = 1, 2, \dots, m$. Notice, there exist $\alpha_1, \dots, \alpha_{n_k}$ such that

$$P_k(z) = \frac{\alpha_1}{z - z_k} + \frac{\alpha_2}{(z - z_k)^2} + \cdots + \frac{\alpha_{n_k}}{(z - z_k)^{n_k}}$$

for each k. Notice $P_k(z) \to 0$ as $z \to \infty$ and P_k is analytic at ∞. We define (still following Gamelin)

$$g(z) = f(z) - P_\infty(z) - \sum_{k=1}^{m} P_k(z).$$

Notice g is analytic at each of the poles including ∞. Thus g is an entire function and as $g(z) \to 0$ as $z \to \infty$ it follows g is bounded and by Liouville's Theorem we find $g(z) = 0$ for all $z \in \mathbb{C}$. Therefore,

$$\boxed{f(z) = P_\infty(z) + \sum_{k=1}^{m} P_k(z).}$$

This completes the proof as we already argued the converse direction in Example 11.3.3. $\square$

The boxed formula is the **partial fractions decomposition of** f. In fact, we have shown:

Theorem 11.4.2. *Every rational function has a partial fractions decomposition: in particular, if $z_1, \dots, z_m$ are the poles of f then*

$$f(z) = P_\infty(z) + \sum_{k=1}^{m} P_k(z)$$

where $P_\infty(z)$ is a polynomial and $P_k(z)$ is the principal part of $f(z)$ around the pole z_k.

The method to obtain the partial fractions decomposition of a given rational function is described algorithmically on pages 180-181. Essentially, the first thing to do is to we can use long-division to discover the principal part at ∞. Once that is done, factor the denominator to discover the poles of $f(z)$ and then we can simply write out a generic form of $\sum_{k=1}^{m} P_k(z)$. Then, we determine the unknown coefficients implicit within the generic form by algebra. I will illustrate with a few examples:

Example 11.4.3. *Let $f(z) = \dfrac{z^3 + z + 1}{z^2 + 1}$. Notice that $z^3 + z + 1 = z(z^2 + 1) + 1$ hence $f(z) = z + \dfrac{1}{z^2 + 1}$. We now focus on $\dfrac{1}{z^2 + 1}$ notice $z^2 + 1 = (z - i)(z + i)$ hence each pole is simple and we seek complex constants A, B such that:*

$$\frac{1}{z^2 + 1} = \frac{A}{z + i} + \frac{B}{z - i}.$$

Multiply by $z^2 + 1$ to obtain:

$$1 = A(z - i) + B(z + i)$$

Next, evaluate at $z = -i$ and $z = i$ to obtain $1 = -2iA$ and $1 = 2iB$ hence $A = -1/2i$ and $B = 1/2i$ and we conclude:

$$f(z) = z - \frac{1}{2i}\frac{1}{z+i} + \frac{1}{2i}\frac{1}{z-i}.$$

Example 11.4.4. *Let $f(z) = \dfrac{2z+1}{z^2-3z-4}$ notice $z^2 - 3z - 4 = (z-4)(z+1)$ hence*

$$\frac{2z+1}{z^2-3z-4} = \frac{A}{z-4} + \frac{B}{z+1} \quad \Rightarrow \quad 2z+1 = A(z+1) + B(z-4)$$

Evaluate at $z = -1$ and $z = 4$ to obtain:

$$-1 = -5B \;\; \& \;\; 9 = 5A \quad \Rightarrow \quad A = 9/5, \quad B = 1/5.$$

Thus,

$$f(z) = \frac{1}{5}\left(\frac{5}{z-4} + \frac{1}{z+1}\right)$$

Example 11.4.5. *Suppose $f(z) = \dfrac{1+z}{z^4 - 3z^3 + 3z^2 - z}$. Long division is not needed as this is already a proper rational function. Notice*

$$z^4 - 3z^3 + 3z^2 - z = z(z^3 - 3z^2 + 3z - 1) = z(z-1)^3.$$

Thus we seek: complex constants A, B, C, D for which

$$\frac{1+z}{z^4 - 3z^3 + 3z^2 - z} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3}$$

Multiplying by the denominator yields,

$$1 + z = A(z-1)^3 + Bz(z-1)^2 + Cz(z-1) + Dz, \quad \star$$

which is nice to write as

$$1 + z = A(z^3 - 3z^2 + 3z - 1) + B(z^3 - 2z^2 + z) + C(z^2 - z) + Dz$$

for what follows. Differentiating gives

$$1 = A(3z^2 - 6z + 3) + B(3z^2 - 4z + 1) + C(2z - 1) + D, \quad \frac{d\star}{dz}$$

differentiating once more yields

$$0 = A(6z - 6) + B(6z - 4) + C(2), \quad \frac{d^2\star}{dz^2}$$

differentiating for the third time:

$$0 = 6A + 6B$$

Thus $A = -B$. Set $z = 1$ in $\star$ to obtain $2 = D$. Once again, set $z = 1$ in $\frac{d\star}{dz}$ to obtain $1 = C(2-1)+2$ hence $C = -1$. Finally, set $z = 1$ in $\frac{d^2\star}{dz^2}$ to obtain $0 = 2B - 2$ thus $B = 1$ and we find $A = -1$ as a consequence. In summary:

$$\frac{1+z}{z^4 - 3z^3 + 3z^2 - z} = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}.$$

Perhaps you did not see the technique I used in the example above in your previous work with partial fractions. It is a nice addition to the usual algebraic technique.

Example 11.4.6. On how partial fractions helps us find Laurent Series *in the last example we found:*

$$f(z) = \frac{1+z}{z^4 - 3z^3 + 3z^2 - z} = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}.$$

If we want the explicit Laurent series about $z = 1$ *we simply need to expand the analytic function* $-1/z$ *as a power series:*

$$\frac{-1}{z} = \frac{-1}{1+(z-1)} = \sum_{n=0}^{\infty}(-1)^{n+1}(z-1)^n$$

thus for $0 < |z-1| < 1$

$$f(z) = \frac{2}{(z-1)^3} - \frac{1}{(z-1)^2} + \frac{1}{z-1} + \sum_{n=0}^{\infty}(-1)^{n+1}(z-1)^n.$$

This is the Laurent series of f *about* $z_o = 1$*. The other singular point is* $z_1 = 0$*. To find the Laurent series about* z_1 *we need to expand* $\frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}$ *as a power series about* $z_1 = 0$*. To begin,*

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Let $g(z) = -\frac{1}{(z-1)^2}$ *and notice* $\int g(z)dz = C + \frac{1}{z-1} = C - \sum_{n=0}^{\infty} z^n$ *thus*

$$g(z) = \frac{d}{dz}\left[\int g(z)dz\right] = \frac{d}{dz}\left[C - \sum_{n=0}^{\infty} z^n\right] = -\sum_{n=1}^{\infty} nz^{n-1} = -\sum_{j=0}^{\infty}(j+1)z^j.$$

Let $h(z) = 2/(z-1)^3$ *notice* $\int h(z)dz = -1/(z-1)^2$ *and* $\int(\int h(z)dz)dz = 1/(z-1) = -\sum_{n=0}^{\infty} z^n$*. I have ignored the constants of integration (why is this ok?). Observe,*

$$\begin{aligned} h(z) = \frac{d}{dz}\frac{d}{dz}\left[\int\left(\int h(z)dz\right)dz\right] = \frac{d}{dz}\frac{d}{dz}\left[-\sum_{n=0}^{\infty} z^n\right] &= \frac{d}{dz}\left[-\sum_{n=1}^{\infty} nz^{n-1}\right] \\ &= -\sum_{n=2}^{\infty} n(n-1)z^{n-2} \\ &= -\sum_{j=0}^{\infty}(j+2)(j+1)z^j. \end{aligned}$$

Thus, noting $f(z) = -1/z + 1/(z-1) + g(z) + h(z)$ *we collect our calculations above to obtain:*

$$f(z) = \frac{-1}{z} - \sum_{j=0}^{\infty}\left(1 + (j+1) + (j+2)(j+1)\right)z^j = \frac{-1}{z} - \sum_{j=0}^{\infty}\left(j^2 + 4j + 4\right)z^j.$$

Neat, $j^2 + 4j + 4 = (j+2)^2$ *hence:*

$$f(z) = \frac{-1}{z} - \sum_{j=0}^{\infty}(j+2)^2 z^j = \frac{-1}{z} + 4 + 9z + 16z^2 + 25z^3 + 36z^4 + \cdots.$$

Term-by-term integration and differentiation allowed us to use geometric series to expand the basic rational functions which appear in the partial fractal decomposition. I hope you see the method I used in the example above allows us a technique to go from a given partial fractal decomposition to the Laurent series about any point we wish. Of course, singular points are most fun.

Chapter 12

The Residue Calculus

In this chapter we collect the essential tools of the residue calculus. Then, we solve a variety of real integrals by relating the integral of interest to the residue of a complex function. The method we present here is not general. Much like second semester calculus, we show some typical examples and hold out hope the reader can generalize to similar examples. These examples date back to the early nineteenth or late eighteenth centuries. Laplace, Poisson and ,of course, Cauchy were able to use complex analysis to solve a myriad of real integrals. That said, according to Remmert [R91] page 395:

> Nevertheless there is no cannonical method of finding, for a given integrand and interval of integration, the best path γ in $\mathbb{C}$ to use.

And if that isn't sobering enough, from Ahlfors:

> even complete mastery does not guarantee success

Ahlfors was a master so this comment is perhaps troubling. Generally, complex integration is an art. For example, if you peruse the answers of Ron Gordon on the *Math Stackexchange Website* you'll see some truly difficult problems solved by one such artist.

Some of the examples solved in this chapter are also solved by techinques of real second semester calculus. I include such examples to illustrate the complex technique with minimal difficulty.

Keep in mind I have additional examples posted in NotesWithE100toE117. I will lecture some from those examples and some from these notes.

12.1 The Residue Theorem

In Theorem 11.1.8 we learned that a function with an isolated singularity has a Laurent expansion: in particular, if $0 \leq \rho < \sigma \leq \infty$, and $f(z)$ is analytic for $\rho < |z - z_o| < \sigma$. Then $f(z)$ can be decomposed as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

where the coefficients a_n are given by:

$$a_n = \frac{1}{2\pi i} \int_{|z-z_o|=r} \frac{f(z)}{(z - z_o)^{n+1}}\, dz$$

for $r > 0$ with $\rho < r < \sigma$. The $n = -1$ coefficient has special significance when we focus on the expansion in a deleted disk about z_o.

Definition 12.1.1. *Suppose $f(z)$ has an isolated singularity z_o and Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

for $0 < |z - z_o| < \rho$ then we define the **residue of f at z_o** *by*

$$Res\,[f(z), z_o] = a_{-1}.$$

Notice, the $n = -1$ coefficient is only the residue when we consider the deleted disk around the singularity. Furthermore, by Theorem 11.1.8, for the Laurent series in the definition above we have

$$a_{-1} = \frac{1}{2\pi i} \oint_{|z-z_o|=r} f(z)\, dz$$

where r is any fixed radius with $0 < r < \rho$.

Example 12.1.2. *Suppose $n \neq 1$,*

$$Res\left[\frac{1}{z - z_o}, z_o\right] = 1 \qquad \& \qquad Res\left[\frac{1}{(z - z_o)^n}, z_o\right] = 0.$$

Example 12.1.3. *In Example 11.4.3 we found*

$$f(z) = \frac{z^3 + z + 1}{z^2 + 1} = z - \frac{1}{2i}\frac{1}{z + i} + \frac{1}{2i}\frac{1}{z - i}.$$

From this partial fractions decomposition we are free to read that

$$Res\,[f(z), i] = \frac{1}{2i} \qquad \& \qquad Res\,[f(z), -i] = \frac{-1}{2i}.$$

Do you understand why there is no hidden $1/(z - i)$ term in $f(z) - \frac{1}{2i}\frac{1}{z-i}$? If you don't then you ought to read $\S VI.4$ again.

Example 12.1.4. *In Example 11.4.4 we derived:*

$$f(z) = \frac{2z+1}{z^2-3z-4} = \frac{1}{5}\left(\frac{5}{z-4} + \frac{1}{z+1}\right)$$

From the above we can read:

$$Res\,[f(z), 4] = 1 \qquad \& \qquad Res\,[f(z), -1] = \frac{1}{5}.$$

Example 12.1.5. *In Example 11.4.5 we derived:*

$$f(z) = \frac{1+z}{z^4-3z^3+3z^2-z} = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}$$

By inspection of the above partial fractal decomposition we find:

$$Res\,[f(z), 0] = -1 \qquad \& \qquad Res\,[f(z), 1] = 1.$$

Example 12.1.6. *Consider* $(\sin z)/z^6$ *observe*

$$\frac{1}{z^6}\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right) = \frac{1}{z^5} - \frac{1}{6z^3} + \frac{1}{120z} + \cdots.$$

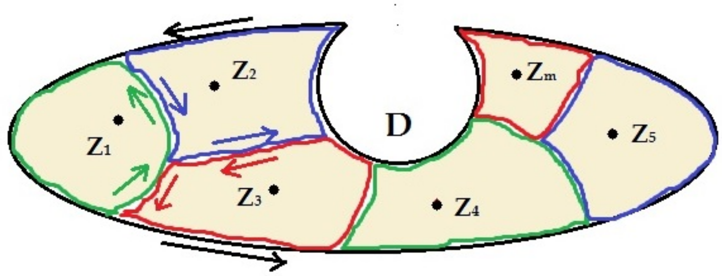
In view of the expansion above, we find:

$$Res\left[\frac{\sin z}{z^6}, 0\right] = \frac{1}{120}$$

Theorem 12.1.7. Cauchy's Residue Theorem: *let D be a bounded domain in the complex plane with a piecewise smooth boundary ∂D. Suppose that f is analytic on $D \cup \partial D$, except for a finite number of isolated singularities $z_1, \dots, z_m$ in D. Then*

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{m} Res\,[f(z), z_j]\,.$$

Proof: this follows immediately from m-applications of Theorem 11.1.8. We simply parse D into m simply connected regions each of which contains just one singular point. The net-integration only gives the boundary as the cross-cuts cancel. The picture below easily generalizes for $m > 3$.



Of course, we could also just envision little circles around each singularity and apply the deformation theorem to reach the ∂D. $\square$

Our focus has shifted from finding the whole Laurent series to just finding the coefficient of the reciprocal term. In the remainder of this section we examine some useful *rules* to find residues.

Proposition 12.1.8. Rule 1: *if $f(z)$ has a simple pole at z_o, then*

$$Res\,[f(z), z_o] = \lim_{z \to z_o} (z - z_o) f(z).$$

Proof: since f has a simple pole at z_o we have:

$$f(z) = \frac{a_{-1}}{z - z_o} + g(z)$$

where $g \in \mathcal{O}(z_o)$. Hence,

$$\lim_{z \to z_o} [(z - z_o) f(z)] = \lim_{z \to z_o} [a_{-1} + (z - z_o) g(z)] = a_{-1}. \quad \square$$

Example 12.1.9.

$$Res\left[\frac{z^3 + z + 1}{z^2 + 1}, i\right] = \lim_{z \to i} (z - i) \frac{z^3 + z + 1}{(z - i)(z + i)} = \lim_{z \to i} \frac{z^3 + z + 1}{z + i} = \frac{-i + i + 1}{i + i} = \frac{1}{2i}.$$

You can contrast the work above with that which was required in Example 12.2.2.

Example 12.1.10. *Following Example 12.1.4, let's see how Rule 1 helps:*

$$Res\left[\frac{2z + 1}{z^2 - 3z - 4}, -1\right] = \lim_{z \to -1} (z + 1) \frac{2z + 1}{(z + 1)(z - 4)} = \frac{2(-1) + 1}{-1 - 4} = \frac{1}{5}.$$

Proposition 12.1.11. Rule 2: *if $f(z)$ has a double pole at z_o, then*

$$Res\,[f(z), z_o] = \lim_{z \to z_o} \frac{d}{dz} \left[(z - z_o)^2 f(z)\right].$$

Proof: since f has a double pole at z_o we have:

$$f(z) = \frac{a_{-2}}{(z - z_o)^2} + \frac{a_{-1}}{z - z_o} + g(z)$$

where $g \in \mathcal{O}(z_o)$. Hence,

$$\begin{aligned} \lim_{z \to z_o} \frac{d}{dz} \left[(z - z_o)^2 f(z)\right] &= \lim_{z \to z_o} \frac{d}{dz} \left[a_{-2} + (z - z_o) a_{-1} + (z - z_o)^2 g(z)\right] \\ &= \lim_{z \to z_o} \left[a_{-1} + 2(z - z_o) g(z) + (z - z_o)^2 g(z)\right] \\ &= a_{-1}. \quad \square \end{aligned}$$

Example 12.1.12.

$$Res\left[\frac{1}{(z^3 + 1) z^2}, 0\right] = \lim_{z \to 0} \frac{d}{dz} \left[\frac{z^2}{(z^3 + 1) z^2}\right] = \lim_{z \to 0} \left[\frac{-3z^2}{(z^3 + 1)^2}\right] = 0.$$

Let me generalize Gamelin's example from page 197. I replace i in Gamelin with a.

Example 12.1.13. *keep in mind $z^2 - a^2 = (z + a)(z - a)$,*

$$Res\left[\frac{1}{(z^2 - a^2)^2}, a\right] = \lim_{z \to a} \frac{d}{dz} \left[\frac{(z - a)^2}{(z^2 - a^2)^2}\right] = \lim_{z \to a} \left[\frac{1}{(z + a)^2}\right] = \frac{2}{(z + a)^3}\bigg|_{z = a} = \frac{-2}{8a^3}.$$

In the classic text of Churchill and Brown, the rule below falls under one of the p, q theorems. See §57 of [C96]. We use the notation of Gamelin here and resist the urge to *mind our p's and q's*.

Proposition 12.1.14. Rule 3: *If $f, g \in \mathcal{O}(z_o)$, and if g has a simple zero at z_o, then*

$$Res\left[\frac{f(z)}{g(z)}, z_o\right] = \frac{f(z_o)}{g'(z_o)}.$$

Proof: if f has a zero of order $N \geq 1$ then $f(z) = (z - z_o)^N h(z)$ and $g(z) = (z - z_o)k(z)$ where $h(z_o), k(z_o) \neq 0$ hence

$$\frac{f(z)}{g(z)} = \frac{(z - z_o)^N h(z)}{(z - z_o)k(z)} = (z - z_o)^{N-1}\frac{h(z)}{k(z)}$$

which shows $\lim_{z\to z_o} \frac{f(z)}{g(z)} = 0$ if $N > 1$ and for $N = 1$ we have $\lim_{z\to z_o} \frac{f(z)}{g(z)} = \frac{h(z_o)}{k(z_o)}$. In either case, for $N \geq 0$ we find $\frac{f(z)}{g(z)}$ has a removable singularity hence the residue is zero which is consistent with the formula of the proposition as $f(z_o) = 0$. Next, suppose $f(z_o) \neq 0$ then by Theorem 11.2.6 we have $f(z)/g(z)$ has a simple pole hence Rule 1 applies:

$$\text{Res}\left[f(z)/g(z), z_o\right] = \lim_{z\to z_o}(z - z_o)\frac{f(z)}{g(z)} = \frac{f(z_o)}{\lim_{z\to z_o}\left(\frac{g(z)-g(z_o)}{z-z_o}\right)} = \frac{f(z_o)}{g'(z_o)}.$$

where in the last step I used that $g(z_o) = 0$ and $g'(z_o), f(z_o) \in \mathbb{C}$ with $g'(z_o) \neq 0$ were given. $\square$

Example 12.1.15. *Observe $g(z) = \sin z$ has simple zero at $z_o = \pi$ since $g(\pi) = \sin\pi = 0$ and $g'(\pi) = \cos\pi = -1 \neq 0$. Rule 3 hence applies as $e^z \in \mathcal{O}(\pi)$,*

$$Res\left[\frac{e^z}{\sin z}, \pi\right] = \frac{e^\pi}{\cos\pi} = -e^\pi.$$

Example 12.1.16. *Notice $g(z) = (z-3)e^z$ has a simple zero at $z_o = 3$. Thus, noting $\cos z \in \mathcal{O}(3)$ we apply Rule 3.*

$$Res\left[\frac{\cos z}{(z-3)e^z}, 3\right] = \frac{\cos(z)}{e^z + (z-3)e^z}\bigg|_{z=3} = \frac{\cos(3)}{e^3}.$$

One more rule to go:

Proposition 12.1.17. Rule 4: *if $g(z)$ has a simple pole at z_o, then*

$$Res\left[\frac{1}{g(z)}, z_o\right] = \frac{1}{g'(z_o)}.$$

Proof: apply Rule 3 with $f(z) = 1$. $\square$

I'll follow Gamelin and offer this example which does clearly show why Rule 4 is so nice to know:

Example 12.1.18. *note that $g(z) = z^2 + 1$ has $g(i) = 0$ and $g'(i) = 2i \neq 0$ hence g has simple zero at $z_o = i$. Apply Rule 4,*

$$Res\left[\frac{1}{z^2+1}, i\right] = \frac{1}{2z}\bigg|_{z=i} = \frac{1}{2i}.$$

12.2 Integrals Featuring Rational Functions

Let $R > 0$. Consider the curve ∂D which is formed by joining the line-segment $[-R, R]$ to the upper-half of the positively oriented circle $|z| = R$. Let us denote the half-circle by C_R hence $\partial D = [-R, R] \cup C_R$. Notice the domain D is a half-disk region of radius R with the diameter along the real axis. If $f(z)$ is a function which is analytic at all but a finite number of isolated singular points $z_1, \dots, z_k$ in D then Cauchy's Residue Theorem yields:

$$\int_{C_R} f(z)\, dz = 2\pi i \sum_{j=1}^{k} \text{Res}\,[f(z), z_j]$$

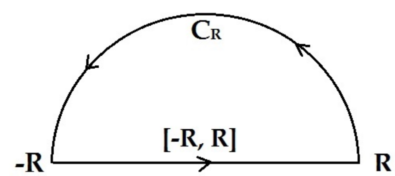
In particular, we find

$$\int_{[-R,R]} f(z)\, dz + \int_{C_R} f(z)\, dz = 2\pi i \sum_{j=1}^{k} \text{Res}\,[f(z), z_j]$$

But, $[-R, R]$ has $z = x$ hence $dz = dx$ and $f(z) = f(x)$ for $-R \leq x \leq R$ and

$$\boxed{\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \sum_{j=1}^{k} \text{Res}\,[f(z), z_j]\,.}$$

The formula above connects integrals in the real domain to residues and the contour integral along a half-circle C_R. We can say something interesting in general for rational functions.



Suppose $f(z) = \frac{p(z)}{q(z)}$ where $\deg(q(z)) \geq \deg(p(z)) + 2$. Let $\deg(q(z)) = n$ and $\deg(p(z)) = m$ hence $n - m \geq 2$. Also, assume $q(x) \neq 0$ for all $x \in \mathbb{R}$ so that no[1] singular points fall on $[-R, R]$. In Problem 44 of the homework, based on an argument from page 131 of [C96], I showed there exists $R > 0$ for which $q(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_o$ is bounded below $|a_n| R^n/2$ for $|z| > R$; that is $|q(z)| \geq \frac{|a_n|}{2} R^n$ for all $|z| > R$. On the other hand, it is easier to argue that $p(z) = b_m z^m + \cdots + b_1 z + b_o$ is bounded for $|z| > R$ by repeated application of the triangle inequality:

$$|p(z)| \leq |b_m z^m| + \cdots + |b_1 z| + |b_o| \leq |b_m| R^m + \cdots + |b_1| R + |b_o|.$$

Therefore, if $|z| > R$ as described above,

$$|f(z)| = \frac{|p(z)|}{|q(z)|} \leq \frac{|b_m| R^m + \cdots + |b_1| R + |b_o|}{\frac{|a_n|}{2} R^n} \leq \frac{M}{R^{n-m}}$$

[1]in §VII.5 we study fractional residues which allows us to treat singularities on the boundary in a natural manner, but, for now, they are forbidden

where M is a constant which depends on the coefficients of $p(z)$ and $q(z)$. Applying the ML-estimate to C_R for $R > 0$ for which the bound applies we obtain:

$$\left| \int_{C_R} f(z)\,dz \right| \leq \frac{M(2\pi R)}{R^{n-m}} = \frac{2M\pi}{R^{n-m-1}}$$

This bound applies for all R beyond some positive value hence we deduce:

$$\lim_{R\to\infty} \left| \int_{C_R} f(z)\,dz \right| \leq \lim_{R\to\infty} \frac{2M\pi}{R^{n-m-1}} = 0 \quad \Rightarrow \quad \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

as $n-m \geq 2$ implies $n-m-1 \geq 1$. Therefore, the boxed formula provides a direct link between the so-called *principal value* of the real integral and the sum of the residues over the upper half-plane of $\mathbb{C}$:

$$\boxed{\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{j=1}^{m} \text{Res}\left[f(z), z_j\right].}$$

Sometimes, for explicit examples, it is expected that you show the details for the construction of M and that you retrace the steps of the general path I sketched above. If I have no interest in that detail then I will tell you to use the Proposition below:

Proposition 12.2.1. *If $f(z)$ is a rational function which has no real-singularities and for which the denominator is of degree at least two higher than the numerator then*

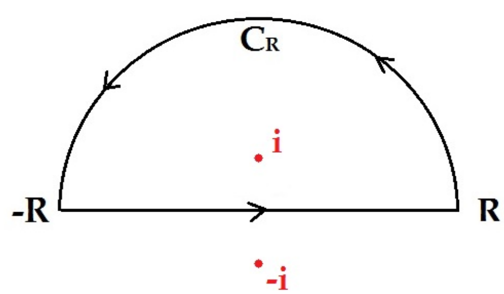
$$\boxed{\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{j=1}^{k} Res\left[f(z), z_j\right].}$$

where $z_1, \dots, z_k$ are singular points of $f(z)$ for which $\mathbf{Im}(z_j) > 0$ for $j = 1, \dots, k$.

Example 12.2.2. *We calculate $\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1}$ by noting the complex extension of the integrand $f(z) = \frac{1}{z^2+1}$ satisfies the conditions of Proposition 12.2.1. Thus,*

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = 2\pi i Res\left[\frac{1}{z^2+1}, i\right] = \left.\frac{2\pi i}{2z}\right|_{z=i} = \frac{2\pi i}{2i} = \pi.$$

Thus[2] $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi.$



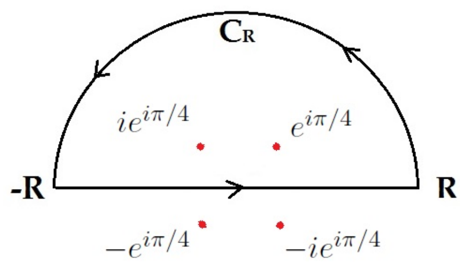
[2] so, technically, the double infinite double integral is defined by distinct parameters tending to ∞ and $-\infty$ independent of one another, however, for this integrand there is no difference between $\int_a^b \frac{dx}{x^2+1}$ with $a \to \infty$ and $b \to -\infty$ verses $a = -b = R$ tending to ∞. Gamelin starts to discuss this issue in $\S VII.6$

You can contrast the way I did the previous example with how Gamelin presents the work.

Example 12.2.3. *Consider $f(z) = \frac{1}{z^4+1}$ notice singularities of this function are the fourth roots of -1; $z^4+1=0$ implies $z \in (-1)^{1/4} = \{e^{i\pi/4}, ie^{i\pi/4}, -e^{i\pi/4}, -ie^{i\pi/4}\}$. Only two of these fall in the upper-half plane. Thus, by Proposition 12.2.1*

$$
\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^4+1} &= 2\pi i Res\left[\frac{1}{z^4+1}, e^{i\pi/4}\right] + 2\pi i Res\left[\frac{1}{z^4+1}, ie^{i\pi/4}\right]. \\
&= \frac{2\pi i}{4z^3}\bigg|_{e^{i\pi/4}} + \frac{2\pi i}{4z^3}\bigg|_{ie^{i\pi/4}} \\
&= \frac{2\pi i}{4e^{i3\pi/4}} + \frac{2\pi i}{4i^3e^{3i\pi/4}} \\
&= \frac{\pi}{2e^{i3\pi/4}}\left[i + \frac{i}{i^3}\right] = \frac{-\pi}{2e^{i3\pi/4}}\left[1-i\right] = \frac{-\pi}{2e^{i3\pi/4}}\sqrt{2}e^{-i\pi/4} = \frac{\pi}{\sqrt{2}}.
\end{aligned}
$$

where we noted $e^{-i\pi/4}/e^{i3\pi/4} = 1/e^{i\pi} = -1$ to cancel the -1. It follows that: $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$.



Wolfram Alpha reveals the antiderivative for the previous example can be directly calculated:

$$\int\frac{dx}{x^4+1} = (-\log(x^2-\sqrt{2}x+1)+\log(x^2+\sqrt{2}x+1)-2\tan^{-1}(1-\sqrt{2}x)+2\tan^{-1}(\sqrt{2}x+1))/(4\sqrt{2})+C.$$

Then to calculate the improper integral you just have to calculate the limit of the expression above at $\pm\infty$ and take the difference. That said, I think I prefer the method which is more *complex*.

The method used to justify Proposition 12.2.1 applies to non-rational examples as well. The key question is how to bound, or more generally capture, the integral along the half-circle as $R \to \infty$. Sometimes the direct complex extension of the real integral is not wise. For example, for $a > 0$, when faced with

$$\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\cos(ax)dx$$

we would not want to use $f(z) = \frac{p(z)\cos(az)}{q(z)}$ since $\cos(aiy) = \cosh(ay)$ is unbounded. Instead, we would consider $f(z) = \frac{p(z)e^{iaz}}{q(z)}$ from which we obtain values for both $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\cos(ax)dx$ and $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\sin(ax)dx$. I will not attempt to derive an analog to Proposition 12.2.1. Instead, I consider the example presented by Gamelin.

Example 12.2.4. *Consider* $f(z) = \frac{e^{iaz}}{z^2+1}$. *Notice* f *has simple poles at* $z = \pm i$, *the picture of Example 12.2.2 applies here. By Rule 3,*

$$Res\left[\frac{e^{iaz}}{z^2+1}, i\right] = \frac{e^{iaz}}{2z}\bigg|_i = \frac{e^{-a}}{2i}.$$

Let D *be the half disk with* $\partial D = [-R, R] \cup C_R$ *then by Cauchy's Residue Theorem*

$$\int_{[-R,R]} \frac{e^{iaz}}{z^2+1}\, dz + \int_{C_R} \frac{e^{iaz}}{z^2+1}\, dz = \frac{2\pi i e^{-a}}{2i} = \pi e^{-a} \quad \star.$$

For C_R *we have* $z = Re^{i\theta}$ *for* $0 \leq \theta \leq \pi$ *hence for* $z \in C_R$ *with* $R > 1$,

$$|f(z)| = \left|\frac{e^{iaz}}{z^2+1}\right| = \frac{1}{|z^2+1|} \leq \frac{1}{||z|^2-1|} = \frac{1}{R^2-1}$$

Thus, by ML*-estimate,*

$$\left|\int_{C_R} \frac{e^{iaz}}{z^2+1}\, dz\right| \leq \frac{2\pi R}{1-R^2} \quad \Rightarrow \quad \lim_{R\to\infty} \int_{C_R} \frac{e^{iaz}}{z^2+1}\, dz = 0.$$

Returning to $\star$ *we find:*

$$\lim_{R\to\infty} \int_{[-R,R]} \frac{e^{iax}}{x^2+1}\, dx = \pi e^{-a} \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}\, dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+1}\, dx = \pi e^{-a}.$$

The real and imaginary parts of the equation above reveal:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}\, dx = \pi e^{-a} \qquad \& \qquad \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+1}\, dx = 0.$$

In §$VII.7$ we learn about Jordan's Lemma which provides an estimate which allows for integration of expressions such as $\frac{\sin x}{x}$.

12.3 Integrals of Trigonometric Functions

The idea of this section is fairly simple once you grasp it:

> Given an integral involving sine or cosine find a way to represent it as the formula for the contour integral around the unit-circle, or some appropriate curve, then use residue theory to calculate the complex integral hence calculating the given real integral.

Let us discuss the main algebraic identities to begin: if $z = e^{i\theta} = \cos\theta + i\sin\theta$ then $\bar{z} = e^{-i\theta} = \cos\theta - i\sin\theta$ hence $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. Of course, we've known these from earlier in the course. But, we also can see these as:

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \& \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

moreover, $dz = ie^{i\theta}d\theta$ hence $d\theta = dz/iz$. It should be emphasized, the formulas above hold for the unit-circle.

Consider a complex-valued rational function $R(z)$ with singular points $z_1, z_2, \dots z_k$ for which $|z_j| \neq 0$ for all $j = 1, 2, \dots, k$. Then, by Cauchy's Residue Theorem

$$\int_{|z|=1} R(z)\, dz = 2\pi i \sum_{|z_j|<1} \text{Res}\,(R(z), z_j)$$

In particular, as $z = e^{i\theta}$ parametrizes $|z| = 1$ for $0 \leq \theta \leq 2\pi$,

$$\int_0^{2\pi} R(\cos\theta + i\sin\theta)\, ie^{i\theta} d\theta = 2\pi i \sum_{|z_j|<1} \text{Res}\,(R(z), z_j)$$

In examples, we often begin with $\int_0^{2\pi} R(\cos\theta + i\sin\theta)\, ie^{i\theta} d\theta$ and work our way back to $\int_{|z|=1} R(z)\, dz$.

Example 12.3.1.

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} &= \int_{|z|=1} \frac{dz/iz}{5 - 4 \cdot \frac{i}{2}\left(z - \frac{1}{z}\right)} \\
&= \int_{|z|=1} \frac{1}{i} \cdot \frac{dz}{5z - 2i\,(z^2 - 1)} \\
&= \int_{|z|=1} \frac{dz}{2z^2 - 2 + 5iz}
\end{aligned}$$

Notice $2z^2 + 5iz - 2 = (2z + i)(z + 2i) = 2(z + i/2)(z + 2i)$ *is zero for* $z_o = -i/2$ *or* $z_1 = -2i$. *Only* z_o *falls inside* $|z| = 1$ *therefore, by Cauchy's Residue Theorem,*

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} &= \int_{|z|=1} \frac{dz}{2z^2 + 5iz - 2} \\
&= 2\pi i\, Res\left[\frac{1}{2z^2 + 5iz - 2}, -i/2\right] \\
&= (2\pi i) \frac{1}{4z + 5i}\bigg|_{z=-i/2} \\
&= \frac{2\pi i}{-2i + 5i} \\
&= \frac{2\pi}{3}.
\end{aligned}$$

The example below is approximately borrowed from Remmert page 397 [R91].

Example 12.3.2. *Suppose* $p \in \mathbb{C}$ *with* $|p| \neq 1$. *We wish to calculate:*

$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta.$$

Converting the integrand and measure to $|z| = 1$ *yields:*

$$\frac{1}{1 - p\left(z + \frac{1}{z}\right) + p^2} \frac{dz}{iz} = \left[\frac{1}{z - pz^2 - p + p^2 z}\right] \frac{dz}{i} = \left[\frac{1}{(z - p)(1 - pz)}\right] \frac{dz}{i}.$$

Hence, if $|p| < 1$ *then* $z = p$ *is in* $|z| \leq 1$ *and it follows* $1 - pz \neq 0$ *for all points* z *on the unit-circle* $|z| = 1$*. Thus, we have only one singular point as we apply the Residue Theorem:*

$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta = \int_{|z|=1} \left[\frac{1}{(z-p)(1-pz)}\right] \frac{dz}{i} = 2\pi Res\left[\frac{1}{(z-p)(1-pz)}, p\right]$$

By Rule 1,

$$Res\left[\frac{1}{(z-p)(1-pz)}, p\right] = \lim_{z \to p}(z-p)\frac{1}{(z-p)(1-pz)} = \frac{1}{1-p^2}$$

and we conclude: if $|p| < 1$ *then*

$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta = \frac{2\pi}{1-p^2}.$$

Suppose $|p| > 1$ *then* $z - p \neq 0$ *for* $|z| = 1$ *and* $1 - pz = 0$ *for* $z_o = 1/p$ *for which* $|z_o| = 1/|p| < 1$*. Thus the Residue Theorem faces just one singularity within* $|z| = 1$ *for the* $|p| > 1$ *case:*

$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta = \int_{|z|=1} \left[\frac{1}{(z-p)(1-pz)}\right] \frac{dz}{i} = 2\pi Res\left[\frac{1}{(z-p)(1-pz)}, 1/p\right]$$

By Rule 1,

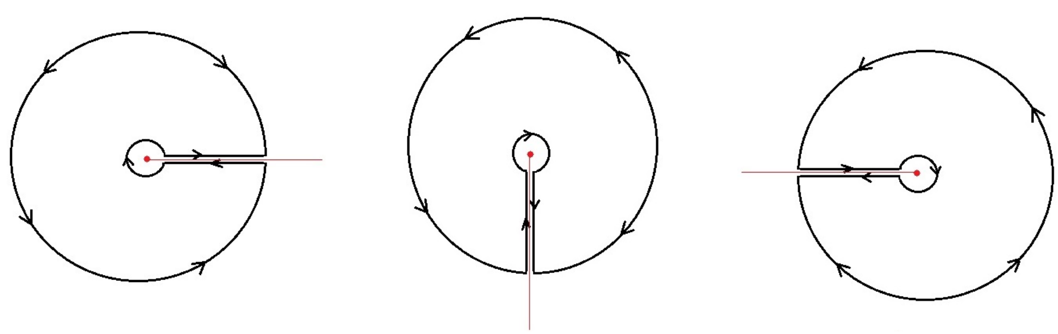
$$Res\left[\frac{1}{(z-p)(1-pz)}, 1/p\right] = \lim_{z \to 1/p}(z-1/p)\frac{1}{(z-p)(z-1/p)(-p)} = \frac{1}{(1/p-p)(-p)} = \frac{1}{p^2-1},$$

neat. Thus, we conclude, for $|p| > 1$*,*

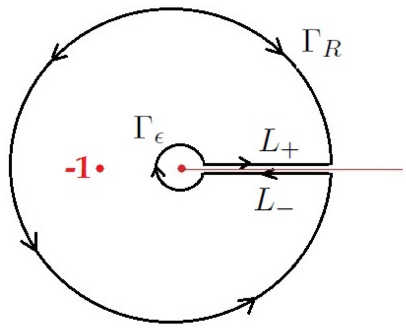
$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta = \frac{2\pi}{p^2-1}.$$

12.4 Integrands with Branch Points

Cauchy's Residue Theorem directly applies to functions with isolated singularities. If we wish to study functions with branch cuts then some additional ingenuity is required. In particular, the **keyhole contour** is often useful. For example, the following template could be used for branch cuts along the positive real, negative imaginary and negative real axis.



Example 12.4.1. *Consider* $\int_0^\infty \frac{x^a}{(1+x)^2}dx$ *where* $a \neq 0$ *and* $-1 < a < 1$. *To capture this integral we study* $f(z) = \frac{z^a}{(1+z)^2}$ *where* $z^a = |z|^a exp(aLog_0(z))$ *is the branch of* z^a *which has a jump-discontinuity along* $\theta = 0$ *which is also at* $\theta = 2\pi$. *Let* Γ_R *be the outside circle in the contour below. Let* Γ_ϵ *be the small circle encircling* $z = 0$. *Furthermore, let* $L_+ = [\epsilon + i\delta, R + i\delta]$ *and* $L_- = [R - i\delta, \epsilon - i\delta]$ *where* δ *is a small positive constant*[3] *for which* $\delta \to 0$ *and* $\epsilon \to 0$. *Notice, in the limits* $\epsilon \to 0$ *and* $R \to \infty$, *we have* $L_+ \to [0, \infty]$ *and* $L_- \to [\infty, 0]$



The singularity $z_o = -1$ *falls within the contour for* $R > 1$ *and* $\epsilon < 1$. *By Rule 2 for residues,*

$$Res\left(\frac{z^a}{(1+z)^2}, -1\right) = \lim_{z \to -1} \frac{d}{dz}[z^a] = \lim_{z \to -1}\left(az^{a-1}\right) = a(-1)^{a-1} = -a(e^{i\pi})^a = -ae^{i\pi a}.$$

Cauchy's Residue Theorem applied to the contour thus yields:

$$\int_{\Gamma_R} f(z)\,dz + \int_{L_-} f(z)\,dz + \int_{\Gamma_\epsilon} f(z)\,dz + \int_{L_+} f(z)\,dz = -2\pi i a e^{i\pi a}$$

If $|z| = R$ *then notice:*

$$\left|\frac{z^a}{(1+z)^2}\right| \leq \frac{R^a}{(R-1)^2}.$$

Also, if $|z| = \epsilon$ *then*

$$\left|\frac{z^a}{(1+z)^2}\right| \leq \frac{\epsilon^a}{(1-\epsilon)^2}.$$

In the limits $\epsilon \to 0$ *and* $R \to \infty$ *we find by the* ML*-estimate*

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \leq \frac{R^a}{(R-1)^2}(2\pi R) = \frac{2\pi R^{a-1}}{(1-1/R)^2} \to 0$$

as $-1 < a < 1$ *implies* $a - 1 < 0$. *Likewise,as* $a + 1 > 0$ *we find:*

$$\left|\int_{\Gamma_\epsilon} f(z)\,dz\right| \leq \frac{\epsilon^a}{(1-\epsilon)^2}(2\pi\epsilon) = \frac{2\pi\epsilon^{a+1}}{(1-\epsilon)^2} \to 0.$$

We now turn to unravel the integrals along $L_\pm$. *For* $z \in L_+$ *we have* $Arg_0(z) = 0$ *whereas* $z \in L_-$ *we have* $Arg_0(z) = 2\pi$. *In the limit* $\epsilon \to 0$ *and* $R \to \infty$ *we have:*

$$\int_{L_+} \frac{z^a}{(1+z)^2}\,dz = \int_0^\infty \frac{x^a}{(1+x)^2}\,dx \qquad \& \qquad -\int_{L_-} \frac{z^a}{(1+z)^2}\,dz = \int_0^\infty \frac{x^a e^{2\pi i a}}{(1+x)^2}\,dx$$

[3]we choose δ as to connect $L_\pm$ and the inner and outer circles

where the phase factor on L_- arises from the definition of z^a by the $Arg_0(z)$ branch of the argument. Bringing it all together,

$$\int_0^\infty \frac{x^a}{(1+x)^2}\,dx - e^{2\pi ia}\int_0^\infty \frac{x^a}{(1+x)^2}\,dx = -2\pi iae^{i\pi a}.$$

Solving for the integral of interest yields:

$$\int_0^\infty \frac{x^a}{(1+x)^2}\,dx = \frac{-2\pi iae^{i\pi a}}{1-e^{2\pi ia}} = \frac{\pi a}{\frac{1}{2i}\left(e^{i\pi a}-e^{-i\pi a}\right)} = \frac{\pi a}{\sin(\pi a)}$$

At this point, Gamein remarks that the function $g(w) = \int_0^\infty \frac{x^w\,dx}{(1+x)^2}$ is analytic on the strip $-1 < \mathbf{Re}(w) < 1$ as is the function $\frac{\pi w}{\sin \pi w}$ thus by the identity princple we find the integral identity holds for $-1 < \mathbf{Re}(w) < 1$.

The following example appears as a homework problem on page 227 of [C96].

Example 12.4.2. *Show that* $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$.

Let $f(z) = \dfrac{z^{-1/2}}{z^2+1}$ *where the root-function has a branch cut along* $[0,\infty]$. *We use the keyhole contour introduced in the previous example. Notice* $z = \pm i$ *are simple poles of* $f(z)$. *We consider* $z^{-1/2} = |z|^{-1/2} exp\left(\frac{-1}{2} Log_0(z)\right)$. *In other words, if* $z = re^{-\theta}$ *for* $0 < \theta \leq 2\pi$ *then* $z^{-1/2} = \frac{1}{\sqrt{r}e^{i\theta/2}}$. *Thus, for* $z = x$ *in* L_+ *we have* $z^{-1/2} = 1/\sqrt{x}$. *On the other hand for* $z = x$ *in* L_- *we have* $z^{-1/2} = -1/\sqrt{x}$ *as* $e^{i(2\pi)/2} = e^{i\pi} = -1$. *Notice,* $z^2+1 = (z-i)(z+i)$ *and apply Rule 3 to see*

$$Res\,(f(z), i) = \frac{i^{-1/2}}{2i} = \frac{e^{-i\pi/4}}{2i} \qquad \& \qquad Res\,(f(z), -i) = \frac{(-i)^{-1/2}}{-2i} = \frac{e^{-3\pi i/4}}{-2i}$$

Consequently, assuming[4] *the integrals along* Γ_R *and* Γ_ϵ *vanish as* $R \to \infty$ *and* $\epsilon \to 0$ *we find:*

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} - \int_0^\infty \frac{dx}{-\sqrt{x}(x^2+1)} = 2\pi i\left(\frac{e^{-i\pi/4}}{2i} + \frac{e^{-3\pi i/4}}{-2i}\right)$$

Notice $-1 = e^{i\pi}$ *and* $e^{i\pi}e^{-3\pi i/4} = e^{\pi i/4}$ *hence:*

$$2\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = 2\pi\left(\frac{e^{-i\pi/4}}{2} + \frac{e^{\pi i/4}}{2}\right) = 2\pi\cos\pi/4 \quad \Rightarrow \quad \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

The key to success is care with the details of the branch cut. It is a critical detail. I should mention that E116 in the handwritten notes is worthy of study. I believe I have assigned a homework problem of a similar nature. There we consider a rectangular path of integration which tends to infinity and uncovers and interesting integral. There are also fascinating examples of wedge-shaped integrations and many other choices I currently have not included in this set of notes.

[4] I leave these details to the reader, but intuitively it is already clear the antiderivative is something like $\sqrt{x}$ at the origin and $1/\sqrt{x}$ for $x \to \infty$.

12.5 Fractional Residues

In general when a singularity falls on a proposed path of integration then there is no simple method of calculation. Generically, you would make a little indentation and then take the limit as the indentation squeezes down to the point. If that limiting process uniquely produces a value then that gives the integral along such a path. In the case of a simple pole there is a nice reformulation of Cauchy's Residue Theorem.

Theorem 12.5.1. *If z_o is a simple pole of f and C_ϵ is an arc of $|z - z_o| = \epsilon$ of angle α then*

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)\, dz = \alpha i Res\,(f(z), z_o)\,.$$

Proof: since f has a simple pole we have:

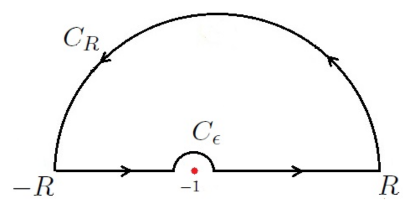
$$f(z) = \frac{A}{z - z_o} + g(z)$$

where, by the definition of residue, $A = \text{Res}\,(f(z), z_o)$. The arc $|z - z_o| = \epsilon$ of angle α is parametrized by $z = z_o + \epsilon e^{i\theta}$ for $\theta_o \leq \theta \leq \theta_o + \alpha$. As the arc is a bounded subset and g is analytic on the arc it follows there exists $M > 0$ for which $|g(z)| < M$ for $|z - z_o| = \epsilon$. Furthermore, the integral of the singular part is calculated:

$$\int_{C_\epsilon} \frac{A dz}{z - z_o} = \int_{\theta_o}^{\theta_o + \alpha} \frac{A i \epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}} = iA \int_{\theta_o}^{\theta_o + \alpha} d\theta = i\alpha A. \qquad \square$$

Of course this result is nicely consistent with the usual residue theorem if we consider $\alpha = 2\pi$ and think about the deformation theorem shrinking a circular path to a point.

Example 12.5.2. *Let $\gamma = C_R \cup [-R, -1 - \epsilon] \cup C_\epsilon \cup [-1 + \epsilon, R]$. This is a half-circular path with an indentation around $z_o = -1$. Here we assume C_ϵ is a half-circle of radius ϵ above the real axis.*



The aperature is π hence the fractional residue theorem yields:

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{dz}{(z+1)(z-i)} = -\pi i Res\left(\frac{1}{(z+1)(z-i)}, -1\right) = -\pi i \left(\frac{1}{-1-i}\right) = \frac{\pi(1+i)}{2}$$

For $|z| = R > 1$ notice $\left|\frac{1}{(z+1)(z-i)}\right| \leq \left|\frac{1}{||z|-|1||\cdot||z|-|i||}\right| = \frac{1}{(R-1)^2} = M$. Thus, $|\int_{C_R} \frac{dz}{(z+1)(z-i)}| \leq \frac{\pi R}{(R-1)^2} \to 0$ as $R \to \infty$. Cauchy's Residue Theorem applied to the region bounded by γ yields:

$$\int_\gamma \frac{dz}{(z+1)(z-i)} = 2\pi i Res\left(\frac{1}{(z+1)(z-i)}, -i\right) = \frac{2\pi i}{-i+1} = \pi(i-1)$$

Hence, in the limit $R \to \infty$ and $\epsilon \to 0$ we find:

$$P.V. \int_{-\infty}^{\infty} \frac{dx}{(x+1)(x-i)} + \frac{\pi(1+i)}{2} = \pi(i-1)$$

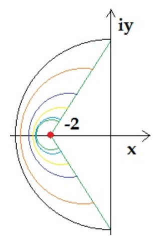
Therefore,

$$P.V. \int_{-\infty}^{\infty} \frac{dx}{(x+1)(x-i)} = \frac{\pi}{2}(i-3).$$

The quantity above is called the principle value for two reasons: first: it approaches $x = \infty$ and $x = -\infty$ symmetrically, second: it approaches the improper point $x = -1$ from the left and right at the same rate. The integral (which is defined in terms of asymmetric limits) itself is divergent in this case. We define the term **principal value** *in the next section.*

Example 12.5.3. *You may recall: Let $\gamma(t) = 2\sqrt{3}e^{it}$ for $\pi/2 \leq t \leq 3\pi/2$. Calculate $\displaystyle\int_\gamma \frac{dz}{z+2}$. A wandering math ninja stumble across your path an mutters $\tan(\pi/3) = \sqrt{3}$.*

Residue Calculus Solution: *if you imagine deforming the given arc from $z = 2i\sqrt{3}$ to $z = -2i\sqrt{3}$ into curves which begin and end along the rays connecting $z = -2$ to $z = \pm 2i\sqrt{3}$ then eventually we reach tiny arcs C_ϵ centered about $z = -2$ each subtending $4\pi/3$ of arc.*



Now, there must be some reason that this deformation leaves the integral unchanged since the fractional residue theorem applied to the limiting case of the small circles yields:

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{dz}{z+2} = \frac{4\pi}{3} i Res\left(\frac{1}{z+2}, -2\right) = \frac{4\pi i}{3}.$$

Of course, direct calculation by the complex FTC yields the same:

$$\begin{aligned}
\int_\gamma \frac{dz}{z+2} &= Log_0(z+2)\Big|_{2i\sqrt{3}}^{-2i\sqrt{3}} \\
&= Log_0(-2i\sqrt{3}+2) - Log_0(2i\sqrt{3}+2) \\
&= Log_0(2(1-i\sqrt{3})) - Log_0(2(1+i\sqrt{3})) \\
&= \ln|2(1-i\sqrt{3}| + iArg_0(4\exp(5\pi i/3)) - \ln|2(1+i\sqrt{3}| + iArg_0(4\exp(\pi i/3)) \\
&= \frac{5\pi i}{3} - \frac{\pi i}{3} \\
&= \frac{4\pi i}{3}
\end{aligned}$$

It must be that the integral along the line-segments is either zero or cancels. Notice $z = -2 + t(2 \pm 2i\sqrt{3})$ for $\epsilon \leq t \leq 1$ parametrizes the rays $(-2, \pm 2i\sqrt{3}]$ in the limit $\epsilon \to 0$ and $dz = (2 \pm 2i\sqrt{3})dt$ thus

$$\int_{(-2,\pm 2i\sqrt{3}]} \frac{dz}{z+2} = \int_\epsilon^1 \frac{dt}{t} = \ln 1 - \ln \epsilon = -\ln \epsilon.$$

However, the direction of the rays differs to complete the path in a consistent CCW direction. We go from -2 to $2i\sqrt{3}$, but, the lower ray goes from $2i\sqrt{3}$ to -2. Apparently these infinities cancel (gulp). I think the idea of this example is a dangerous game.

I covered the example on page 210 of Gamelin in lecture. There we derive the identity:

$$\int_0^\infty \frac{\ln(x)}{x^2-1}\,dx = \frac{\pi^2}{4}.$$

by examining a half-circular path with indentations about $z=0$ and $z=-1$.

12.6 Principal Values

If $\int_{-\infty}^{\infty} f(x)\,dx$ diverges or $\int_a^b f(x)dx$ diverges due to a singularity for $f(x)$ at $c \in [a,b]$ then it may still be the case that the corresponding *principal values* exist. When the integrals converge absolutely then the principal value agrees with the integral. These have mathematical application as Gamelin describes briefly at the conclusion of the section.

Definition 12.6.1. *We define P.V.* $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$*. Likewise, if f is continuous on $[a,c)$ and $(c,b]$ then we define*

$$P.V.\int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+}\left(\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx\right)$$

In retrospect, this section is out of place. We would do better to introduce the concept of principal value towards the beginning. For example, in [C96] this is put forth at the outset. Thus I am inspired to present the following example stolen from [C96].

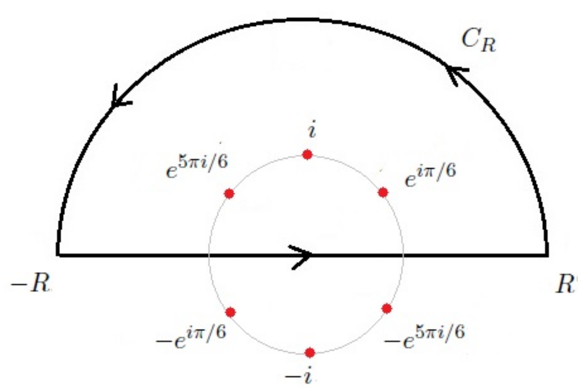
Example 12.6.2. *We wish to calculate $\int_0^\infty \frac{x^2}{x^6+1}\,dx$. The integral can be argued to exist by comparison with other convergent integrals and, as the integrand is non-negative, it converges absolutely. Thus we may find P.V. $\int_0^\infty \frac{x^2}{x^6+1}\,dx$ to calculate $\int_{-\infty}^\infty \frac{x^2}{x^6+1}\,dx$. The integrand is even thus:*

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^6+1}\,dx = \frac{1}{2}P.V.\int_{-\infty}^\infty \frac{x^2}{x^6+1}\,dx.$$

Observe $f(z)=\frac{z^2}{z^6+1}$ has singularities at solutions of $z^6+1=0$. In particular, $z\in(-1)^{1/6}$.

$$\begin{aligned}(-1)^{1/6} &= e^{i\pi/6}\{1, e^{2\pi i/6}, e^{4\pi i/6}, -1, -e^{2\pi i/6}, -e^{4\pi i/6}\}\\ &= \{e^{i\pi/6}, e^{3\pi i/6}, e^{5\pi i/6}, -e^{i\pi/6}, -e^{3\pi i/6}, -e^{5\pi i/6}\}\\ &= \{e^{i\pi/6}, i, e^{5\pi i/6}, -e^{i\pi/6}, -i, -e^{5\pi i/6}\}\end{aligned}$$

We use the half-circle path $\partial D = C_R \cup [-R,R]$ as illustrated below:



Application of Cauchy's residue theorem requires we calculate the residue of $\frac{z^2}{1+z^6}$ at $w = e^{i\pi/6}, i$ and $e^{5\pi i/6}$. In each case we have a simple pole and Rule 3 applies:

$$Res\left(\frac{z^2}{1+z^6}, w\right) = \frac{w^2}{6w^5}.$$

Hence,

$$Res\left(\frac{z^2}{1+z^6}, e^{i\pi/6}\right) = \frac{(e^{i\pi/6})^2}{6(e^{i\pi/6})^5} = \frac{1}{6e^{3i\pi/6}} = \frac{1}{6i},$$

and

$$Res\left(\frac{z^2}{1+z^6}, i\right) = \frac{(i)^2}{6(i)^5} = -\frac{1}{6i},$$

and

$$Res\left(\frac{z^2}{1+z^6}, e^{5i\pi/6}\right) = \frac{(e^{5i\pi/6})^2}{6(e^{5i\pi/6})^5} = \frac{1}{6e^{15i\pi/6}} = \frac{1}{6i}.$$

Therefore,

$$\int_{\partial D} \frac{z^2}{z^6+1}\,dz = 2\pi i\left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right) = \frac{\pi}{3}.$$

Notice if $|z| = R > 1$ then $\left|\frac{z^2}{z^6+1}\right| \leq \frac{R^2}{R^6-1}$ hence the ML-estimate provides:

$$\left|\int_{C_R} \frac{z^2}{z^6+1}\,dz\right| \leq \frac{R^2}{R^6-1}(\pi R) \to 0$$

as $R \to \infty$. If $z \in [-R, R]$ then $z = x$ for $-R \leq x \leq R$ and $dz = dx$ hence

$$\int_{[-R,R]} \frac{z^2}{z^6+1}\,dz = \int_{-R}^{R} \frac{x^2}{x^6+1}\,dx.$$

Thus, noting $\partial D = C_R \cup [-R, R]$ we have:

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{3} \quad\Rightarrow\quad P.V.\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{3} \quad\Rightarrow\quad \int_{0}^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{6}.$$

12.7 Jordan's Lemma

Lemma 12.7.1. Jordan's Lemma: *if C_R is the semi-circular contour $z(\theta) = Re^{i\theta}$ for $0 \leq \theta \leq \pi$, in the upper half plane, then $\displaystyle\int_{C_R} |e^{iz}||dz| < \pi$.*

Proof: note $|e^{iz}| = \exp(\mathbf{Re}(iz)) = \exp(\mathbf{Re}(iRe^{i\theta})) = e^{-R\sin\theta}$ and $|dz| = |iRe^{i\theta}d\theta| = Rd\theta$ hence the Lemma is equivalent to the claim:

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}.$$

By definition, a concave down function has a graph that resides above its secant line. Notice $y = \sin\theta$ has $y'' = -\sin\theta < 0$ for $0 \leq \theta \leq \pi/2$. The secant line from $(0,0)$ to $(\pi/2, 1)$ is $y = 2\theta/\pi$.

Therefore, it is geometrically (and analytically) evident that $\sin\theta \geq 2\theta/\pi$. Consequently, following Gamelin page 216,

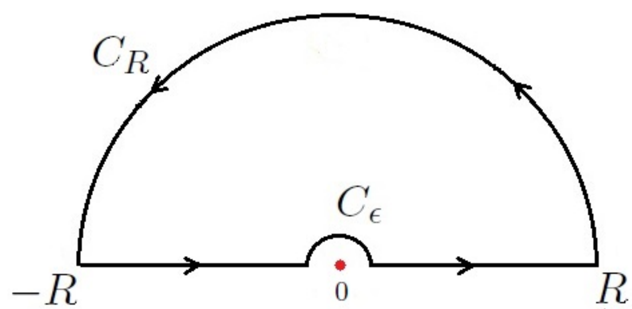
$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \leq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta$$

make a $t = 2R\theta/\pi$ substitution to find:

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}\int_0^{1/R} e^{-t}dt < \frac{\pi}{R}\int_0^{\infty} e^{-t}\,dt = \frac{\pi}{R}. \qquad \square$$

Jordan's Lemma allows us to treat integrals of rational functions multiplied by sine or cosine where the rational function has a denominator function with just one higher degree than the numerator. Previously we needed two degrees higher to make the ML-estimate go through nicely. For instance, see Example 12.2.4.

Example 12.7.2. *To show $\int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}$ we calculate the integral of $f(z) = \frac{e^{iz}}{z}$ along an indented semi-circular path pictured below:*



Notice, for $|z| = R$ we have:

$$\left|\int_{C_R}\frac{e^{iz}}{z}dz\right| \leq \int_{C_R}\left|\frac{e^{iz}}{z}\right||dz| = \frac{1}{R}\int_{C_R}\left|e^{iz}\right||dz| < \frac{\pi}{R}$$

where in the last step we used Jordan's Lemma. Thus as $R\to\infty$ we see the integral of $f(z)$ along C_R vanishes. Suppose $R\to\infty$ and $\epsilon\to 0$ then Cauchy's residue and fractional residue theorems combine to yield:

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ix}}{x}\,dx - \pi i Res\left(\frac{e^{iz}}{z},0\right) + \lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z}dz = 0$$

hence, noting the residue is 1,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ix}}{x}\,dx = i\pi \quad\Rightarrow\quad \lim_{R\to\infty}\int_{-R}^{R}\left(\frac{\cos x}{x} + i\frac{\sin x}{x}\right)dx = i\pi.$$

Note, $\frac{\cos x}{x}$ is an odd function hence the principal value of that term vanishes. Thus,

$$\lim_{R\to\infty} i\int_{-R}^{R}\frac{\sin x}{x}\,dx = i\pi \quad\Rightarrow\quad P.V.\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \pi \quad\Rightarrow\quad \int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Example 12.7.3. *We can calculate* $\displaystyle\int_0^\infty \frac{x\sin(2x)}{x^2+3}$ *by studying the integral of* $f(z) = \dfrac{ze^{2iz}}{z^2+3}$ *around the curve* $\gamma = C_R \cup [-R, R]$ *where* C_R *is the half-circular path in the CCW-direction. Notice* $z = \pm i\sqrt{3}$ *are simple poles of* f*, but, only* $z = i\sqrt{3}$ *falls within* γ*. Notice, by Rule 3,*

$$Res\left(\frac{ze^{2iz}}{z^2+3}, i\sqrt{3}\right) = \frac{i\sqrt{3}e^{-2\sqrt{3}}}{2i\sqrt{3}} = \frac{e^{-2\sqrt{3}}}{2}.$$

Next, we consider $|z| = R$*, in particular notice:*

$$\left|\int_{C_R} \frac{ze^{2iz}}{z^2+3}\,dz\right| \leq \int_{C_R} \left|\frac{ze^{2iz}}{z^2+3}\right| |dz| \leq \frac{R}{R^2-3}\int_{C_R} \left|e^{2iz}\right| |dz| \leq \frac{R}{R^2-3}\int_{C_R} \left|e^{iz}\right| \left|e^{iz}\right| |dz|$$

Notice, Jordan's Lemma gives

$$\int_{C_R} \left|e^{iz}\right| |dz| < \pi = \pi\cdot\frac{1}{\pi R}\int_{C_R} |dz| = \int_{C_R} \frac{1}{R}|dz|$$

hence,

$$\frac{R}{R^2-3}\int_{C_R} \left|e^{iz}\right| \left|e^{iz}\right| |dz| \leq \frac{R}{R^2-3}\int_{C_R} \left|e^{iz}\right| \frac{1}{R}|dz| = \frac{1}{R^2-3}\int_{C_R} \left|e^{iz}\right| |dz| < \frac{\pi^2}{R^2-3}.$$

Clearly as $R \to \infty$ *the integral of* $f(z)$ *along* C_R *vanishes. We find the integral along* $[-R, R]$ *where* $z = x$ *and* $dz = dx$ *must match the product of* $2\pi i$ *and the residue by Cauchy's residue theorem*

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{xe^{2ix}}{x^2+3}\,dx = (2\pi i)\frac{e^{-2\sqrt{3}}}{2} = \pi i e^{-2\sqrt{3}}.$$

Of course, $e^{2ix} = \cos(2x) + i\sin(2x)$ *and the integral of* $\frac{x\cos(2x)}{x^2+3}$ *vanishes as it is an odd function. Cancelling the factor of* i *we derive:*

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x\sin(2x)}{x^2+3}\,dx = \pi e^{-2\sqrt{3}} \quad \Rightarrow \quad \int_0^\infty \frac{x\sin(2x)}{x^2+3}\,dx = \frac{\pi}{2}e^{-2\sqrt{3}}$$

We have shown the solution of Problem 4 on page 214 of [C96]. The reader will find more useful practice problems there as is often the case.

12.8 Exterior Domains

Exterior domains are interesting. Basically this is Cauchy's residue theorem turned inside out. Interestingly a term appears to account for the residue at ∞. We decided to move on to the next chapter this semester. If you are interested in further reading on this topic, you might look at: this MSE exchange or this MSE exchange or this nice Wikipedia example or this lecture from Michael VanValkenburgh at UC Berkeley. Enjoy.

Chapter 13

The Logarithmic Integral

We just cover the basic part of Gamelin's exposition in this chapter. It is interesting that he provides a proof of the Jordan curve theorem in the smooth case. In addition, there is a nice couple pages on simply connected and equivalent conditions in view of complex analysis. All of these are interesting, but our interests take us elsewhere this semester.

The *argument principle* is yet another interesting application of the residue calculus. In short, it allows us to count the number of zeros and poles of a given complex function in terms of the logarithmic integral of the function. Then, Rouché's Theorem provides a technique for counting zeros of a given function which has been extended by a small *perturbation*. Both of these sections give us tools to analyze zeros of functions in surprising new ways.

13.1 The Argument Principle

Let us begin by defining the main tool for our analysis in this section:

Definition 13.1.1. *Suppose f is analytic on a domain D. For a curve γ in D such that $f(z) \neq 0$ on γ we say:*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma d\log f(z)$$

is the **logarithmic integral of $f(z)$ along γ**.

Essentially, the logarithmic integral measures the change of $\log f(z)$ along γ.

Example 13.1.2. *Consider $f(z) = (z - z_o)^n$ where $n \in \mathbb{Z}$. Let $\gamma(z) = z_o + Re^{i\theta}$ for $0 \leq \theta \leq 2\pi k$. Calculate,*

$$\frac{f'(z)}{f(z)} = \frac{n(z-z_o)^{n-1}}{(z-z_o)^n} = \frac{n}{z-z_o}$$

thus,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{n\,dz}{z-z_o} = \frac{n}{2\pi i}\int_0^{2\pi k}\frac{Rie^{i\theta}d\theta}{Re^{i\theta}} = \frac{n}{2\pi}\int_0^{2\pi k} d\theta = nk.$$

The number $k \in \mathbb{Z}$ is the **winding number** *of the curve and n is either ($n > 0$) the* **number of zeros** *or ($n < 0$) $-n$ is the* **number of poles** *inside γ. In the case $n = 0$ then there are neither zeros nor poles inside γ. Our counting here is that a pole of order 5 counts as 5 poles and a zero repeated counts as two zeros etc..*

The example above generalizes to the theorem below:

Theorem 13.1.3. argument principle I: *Let D be a bounded domain with a piecewise smooth boundary ∂D, and let f be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then*

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}\,dz = N_0 - N_\infty,$$

where N_0 is the number of zeros of $f(z)$ in D and N_∞ is the number of poles of $f(z)$ in D, counting multiplicities.

Proof: Let z_o be a zero of order N for $f(z)$ then $f(z) = (z-z_o)^N h(z)$ where $h(z_o) \neq 0$. Calculate:

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{N(z-z_o)^{N-1}h(z) + (z-z_o)^N h'(z)}{(z-z_o)^N h(z)}\\ &= \frac{N}{z-z_o} + \frac{h'(z)}{h(z)}\end{aligned}$$

likewise, if z_o is a pole of order N then $f(z) = \dfrac{h(z)}{(z-z_o)^N} = (z-z_o)^{-N}h(z)$ hence

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{-N(z-z_o)^{-N-1}h(z) + (z-z_o)^{-N} h'(z)}{(z-z_o)^{-N} h(z)}\\ &= \frac{-N}{z-z_o} + \frac{h'(z)}{h(z)}\end{aligned}$$

Thus,

$$\text{Res}\left(\frac{f'(z)}{f(z)}, z_o\right) = \pm N$$

where $(+)$ is for a zero of order N and $(-)$ is for a pole of order N. Let $z_1, \dots, z_j$ be the zeros and poles of f, which are finite in number as we assumed f was meromorphic. Cauchy's residue theorem yields:

$$\int_{\partial D}\frac{f'(z)}{f(z)}\,dz = 2\pi i\sum_{k=1}^{j}\text{Res}\left(\frac{f'(z)}{f(z)}, z_o\right) = 2\pi i\sum_{k=1}^{j}N_k = 2\pi i(N_0 - N_\infty). \qquad \square$$

To better understand the theorem is it useful to break down the logarithmic integral. The calculations below are a shorthand for the local selection of a branch of the logarithm

$$\log(f(z)) = \ln|f(z)| + i\arg(f(z)),$$

hence

$$d\log(f(z)) = d\ln|f(z)| + i d\arg(f(z))$$

for a curve with $f(z) \neq 0$ along the curve it is clear that $ln|f(z)|$ is well-defined along the curve and if $z : [a,b] \to \gamma$ then

$$\int_\gamma d\ln|f(z)| = \ln|f(b)| - \ln|f(a)|.$$

If the curve γ is closed then $f(a) = f(b)$ and clearly

$$\int_\gamma d\ln|f(z)| = 0.$$

However, the argument cannot be defined on an entire circle because we must face the 2π-jump somewhere. The logarithmic integral does not measure the argument of γ directly, rather, the arguments of the image of γ under f:

$$\int_\gamma d\arg(f(z)) = \arg(f(\gamma(b))) - \arg(f(\gamma(a))).$$

For a piecewise smooth curve we simply repeat this calculation along each piece and obtain the net-change in the argument of f as we trace out the curve.

Theorem 13.1.4. argument principle II: *Let D be a bounded domain with a piecewise smooth boundary ∂D, and let f be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then the increase in the argument of $f(z)$ around the boundary of D is 2π times the number of zeros minus the number of poles in D,*

$$\int_{\partial D} d\arg(f(z)) = 2\pi(N_0 - N_\infty).$$

We have shown this is reasonable by our study of $d\log(f(z)) = d\ln|f(z)| + i\,d\arg(f(z))$. Note,

$$\frac{d}{dz}\log(f(z)) = \frac{f'(z)}{f(z)} \quad \Rightarrow \quad d\log(f(z)) = \frac{f'(z)}{f(z)}\,dz.$$

Thus the Theorem 13.1.4 is a just a reformulation of Theorem 13.1.3.

Gamelin's example on page 227-228 is fascinating. I will provide a less sophisticated example of the theorem above in action.

Example 13.1.5. *Consider $f(z) = z^3 + 1$. Let $\gamma(t) = z_o + Re^{it}$ for $R > 0$ and $0 \leq t \leq 2\pi$. Thus $[\gamma]$ is $|z - z_o| = R$ given the positive orientation. If $R = 2$ and $z_o = 0$ then*

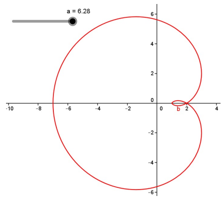
$$f(\gamma(t)) = 8e^{3it} + 1$$

The points traced out by $f(\gamma(t))$ above cover a circle centered at 1 with radius 8 three times. It follows the argument of $f(z)$ has increased by 6π along γ thus revealing $N_0 - N_\infty = 3$ and as f is entire we know $N_\infty = 0$ hence $N_0 = 3$. Of course, this is not surprising, we can solve $z^3 + 1 = 0$ to obtain $z \in (-1)^{1/3}$. All of these zeros fall within the circle $|z| = 2$.

Consider $R = 1$ and $z_o = -1$. Then $\gamma(t) = -1 + e^{it}$ hence

$$f(\gamma(t)) = \left(e^{it} - 1\right)^3 + 1 = e^{3it} - 3e^{2it} + 3e^{it} - 1 + 1$$

If we plot the path above in the complex plane we find:



Which shows $f(\gamma(t))$ increases its argument by 2π hence just one zero falls within $[\gamma]$ in this case. I used Geogebra to create the image above. Notice the slider allows you to animate the path which helps as we study the dynamics of the argument for examples such as this. To plot, as far as I currently know, you'll need to find $\mathbf{Re}(\gamma(t))$ and $\mathbf{Im}(\gamma(t))$ then its pretty straightforward.

13.2 Rouché's Theorem

This is certainly one of my top ten favorite theorems:

Theorem 13.2.1. Rouché's Theorem: *Let D be a bounded domain with a piecewise smooth boundary ∂D. Let f and h be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ and $f(z) + h(z)$ have the same number of zeros in D, counting multiplicities.*

Proof: by assumption $|h(z)| < |f(z)|$ we cannot have a zero of f on the boundary of D hence $f(z) \neq 0$ for $z \in \partial D$. Moreover, it follows $f(z) + h(z) \neq 0$ on ∂D. Observe, for $z \in \partial D$,

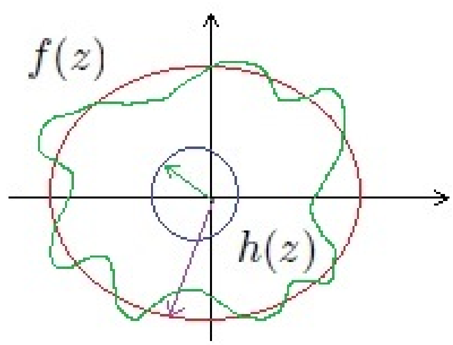
$$f(z) + h(z) = f(z)\left[1 + \frac{h(z)}{f(z)}\right],$$

We are given $|h(z)| < |f(z)|$ thus $\left|\frac{h(z)}{f(z)}\right| < 1$ and we find $\mathbf{Re}\left(1 + \frac{h(z)}{f(z)}\right) > 0$. Thus all the values of $1 + \frac{h(z)}{f(z)}$ on ∂D fall into a half plane which permits a single-valued argument function throughout hence any closed curve gives no gain in argument from $1 + \frac{h(z)}{f(z)}$. Moreover,

$$\arg\left(f(z) + h(z)\right) = \arg\left(f(z)\right) + \arg\left[1 + \frac{h(z)}{f(z)}\right]$$

hence the change in $\arg\left(f(z) + h(z)\right)$ is matched by the change in $\arg\left(f(z)\right)$ and by Theorem 13.1.4, and the observation that there are no poles by assumption, we conclude the number of zeros for f and $f + h$ are the same counting multiplicities. $\square$

Once you understand the picture below it offers a convincing reason to believe:



The red curve we can think of as the image of $f(z)$ for $z \in \partial D$. Note, ∂D is not pictured. Continuing, the green curve is a *perturbation* or *deformation* of the red curve by the blue curve which is the graph of $h(z)$ for $z \in \partial D$. In order for $f(z) + h(z) = 0$ we need for $f(z)$ to be cancelled by $h(z)$. But, that is clearly impossible given the geometry.

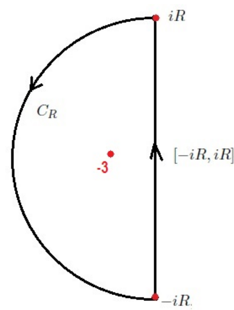
Often the following story is offered: suppose you walk a dog on a path which is between R_1 and R_2 feet from a pole. If your leash is less than R_1 feet then there is no way the dog can get caught on the pole. The function $h(z)$ is like the leash, the path which doesn't cross the origin is the red curve and the green path is formed by the dog wandering about the path while being restricted by the leash.

Example 13.2.2. *Find the number of zeros for $p(z) = z^{11}+12z^7-3z^2+z+2$ within the unit circle. Let $f(z) = 12z^7$ and $h(z) = z^{11}-3z^2+z+2$ observe for $|z| = 1$ we have $|h(z)| \leq 1+3+1+2 = 7$ and $|f(z)| = 12|z|^7 = 12$ hence $|h(z)| \leq f(z)$ for all z with $|z| = 1$. Observe $f(z) = 12z^7$ has a zero of multiplicity 7 at $z = 0$ hence by Rouché's Theorem $p(z) = f(z)+h(z) = z^{11}+12z^7-3z^2+z+2$ also has seven zeros within the unit-circle.*

Rouché's Theorem also has great application beyond polynomial problems:

Example 13.2.3. *Prove that the equation $z+3+2e^z = 0$ has precisely one solution in the left-half-plane. The idea here is to view $f(z) = z+3$ as being perturbed by $h(z) = 2e^z$. Clearly $f(-3) = 0$ hence if we can find a curve γ which bounds $\mathbf{Re}(z) < 0$ and for which $|h(\gamma(t))| \leq |f(\gamma(t))|$ for all $t \in dom(\gamma)$ then Rouché's Theorem will provide the conclusion we desire.*

Therefore, consider $\gamma = C_R \cup [-iR, iR]$ where C_R has $z = Re^{it}$ for $\pi/2 \leq t \leq 3\pi/2$.



Consider $z \in [-iR, iR]$ then $z = iy$ for $-R \leq y \leq R$ observe:

$$|f(z)| = |iy+3| = \sqrt{9+y^2} \qquad \& \qquad |h(z)| = |2e^{iy}| = 2$$

thus $|h(z)| < |f(z)|$ for all $z \in [-iR, iR]$. Next, suppose $z = x+iy \in C_R$ hence $-R \leq x \leq 0$ and $-R \leq y \leq R$ with $x^2+y^2 = R^2$. In particular, assume $R > 5$. Note:

$$|f(z)| = |x+iy+3| \quad \Rightarrow \quad R-3 \leq |f(z)| \leq \sqrt{9+R^2}.$$

the claim above is easy to see geometrically as $|z+3|$ is simply the distance from z to -3 which is smallest when $y = 0$ and largest when $x = 0$. Furthermore, as $-R \leq x \leq 0$ and e^x is a strictly increasing function,

$$|h(z)| = \left|2e^x e^{iy}\right| = 2e^x < 2 < R-3 < |f(z)|$$

where you now hopefully appreciate why we assumed $R > 5$. Consequently $|h(z)| \leq |f(z)|$ for all $z \in C_R$ with $R > 5$. We find by Rouché's Theorem $f(z)$ and $f(z)+h(z) = z+3+2e^z$ has only one zero in γ for $R > 5$. Thus, suppose $R \to \infty$ and observe γ serves as the boundary of $\mathbf{Re}(z) < 0$ and so the equation $z+3+2e^z = 0$ has just one solution in the left-half plane.

Notice, Rouché's Theorem does not tell us what the solution of $z+3+2e^z = 0$ with $\mathbf{Re}(z) < 0$ is. The theorem merely tells us that the solution uniquely exists.

Example 13.2.4. *Consider $p(z) = a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_o$ where $a_n \neq 0$. Let $f(z) = a_nz^n$ and $h(z) = a_{n-1}z^{n-1}+\cdots+a_1z+a_o$ then $p(z) = f(z)+h(z)$. Moreover, if we choose $R > 0$ sufficiently large then $|h(z)| \leq |a_{n-1}|R^{n-1}+\cdots+|a_1|R+|a_o| < |a_n|R^n = |f(z)|$ for $|z| = R$ hence Rouché's Theorem tells us that there are n-zeros for $p(z)$ inside $|z| = R$ as it is clear that $z = 0$ is a zero of multiplicity n for $f(z) = a_nz^n$. Thus every $p(z) \in \mathbb{C}[z]$ has n-zeros, counting multiplicity, on the complex plane.*

The proof of the Fundamental Theorem of Algebra above is nicely direct in contrast to other proofs by contradiction we saw in previous parts of this course.

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